So far we've looked at PDEs in Cartesian coordinates:

The heat equation:

$$\kappa \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

Laplace's equation

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

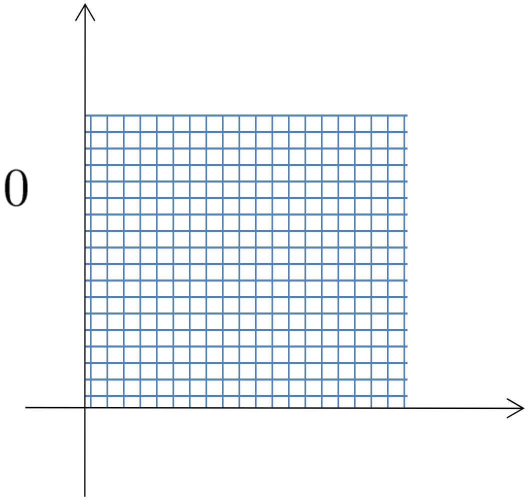
The wave equation:

$$\frac{\partial^2 \psi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} = 0$$

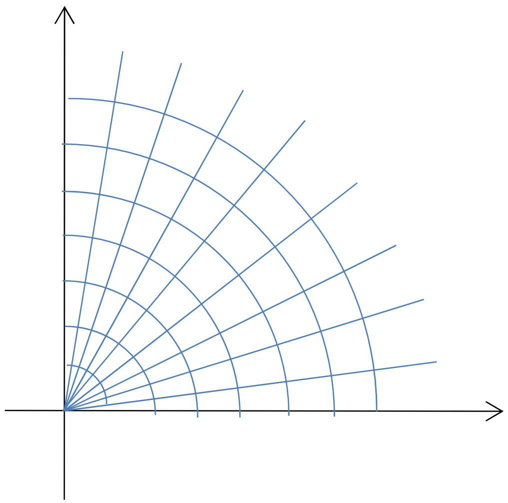
We would also like to solve equations in different coordinate systems, and in 3 dimensions.

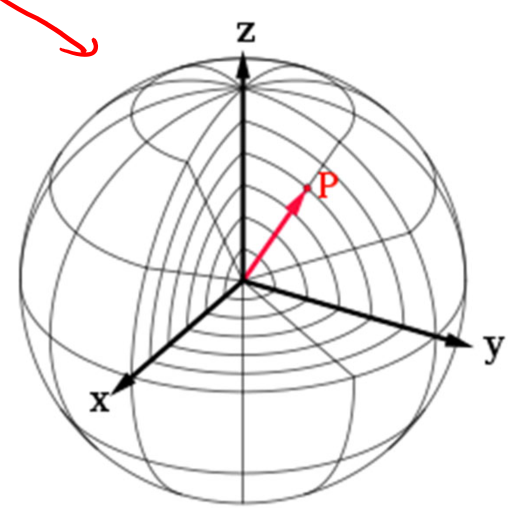
$$\nabla^2\psi = 0$$

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0$$



$$\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} = 0$$





The steps for solving using separation of variables remain the same

1. Separate variables

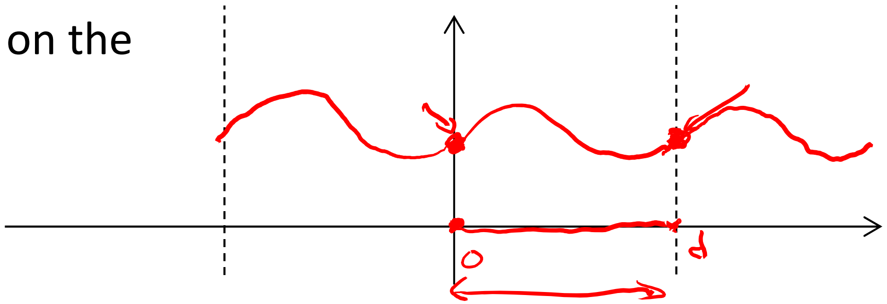
2. Identify the Sturm-Liouville problem and compute the eigenfunctions

3. Use an infinite series to match the boundary conditions.

However we often have to introduce *Special Functions* to express the solution.

Before we begin, we will need to look at
<u>Periodic boundary conditions for S-L problems</u>

Consider the 1-D eigenvalue problem defined on the domain $D = \{x \mid 0 \leq x \leq d\}$



$$\mathcal{L}\phi = \lambda\phi \quad \text{on } D$$

with boundary conditions

$$\phi(0) = \phi(d)$$

$$\phi'(0) = \phi'(d)$$

This problem has *all the properties of a Sturm-Liouville problem.*

We find a *countably infinite set* of eigenfunctions and eigenvalues where the … are all real and positive, and the eigenfunctions are all orthogonal.

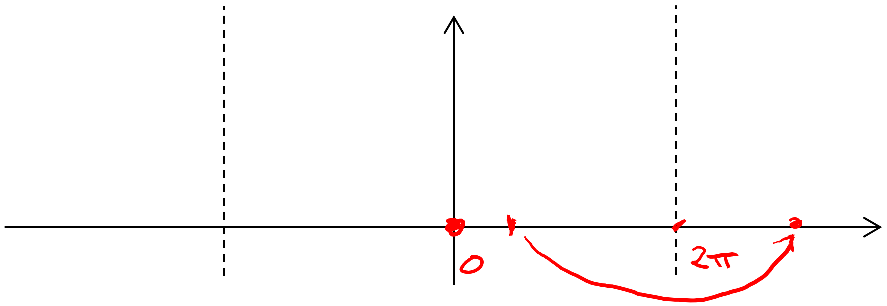
Example: Consider the periodic S-L problem

$$-\frac{d^2}{dx^2}\phi = \lambda\phi$$

defined on the domain |x|< a (corrected by hand to 2π), with

$$\phi(0) = \phi(2\pi)$$

$$\phi'(0) = \phi'(2\pi)$$



This has eigenfunctions:

$$\phi_m = e^{\pm imx}$$

With eigenvalues:

$$\lambda_m = m^2$$

Why? Subbing in:

$$-\frac{d^2}{dx^2}\left(e^{\pm imx}\right) = -\frac{d}{dx}\left(\pm im\, e^{\pm imx}\right)$$

$$= -(\pm im)(\pm im)e^{imx}$$

$$= -i^2 m^2 e^{\pm imx} = m^2 e^{\pm imx}$$

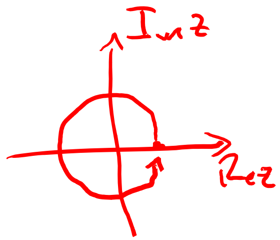
$$= \lambda\phi.$$

Also

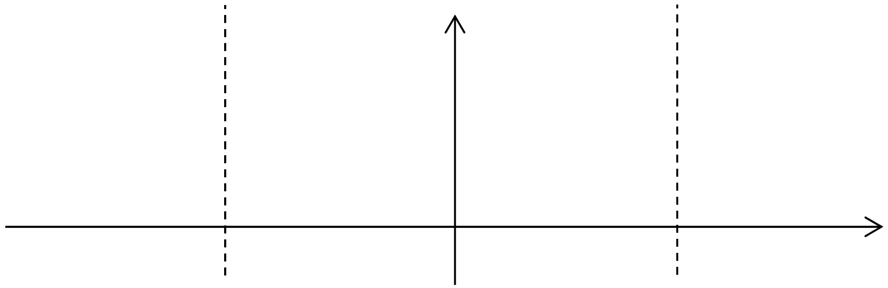
$$\phi_m(x+2\pi) = e^{\pm im(x+2\pi)}$$

$$= e^{\pm imx} e^{\pm im2\pi} = e^{\pm imx} = \phi(x).$$

arg = m2π



We have two sets of eigenfunctions:



$$\phi_m = e^{imx} \qquad \phi_m = e^{-imx}$$

where $m = 0,1,2,3, \ldots$

By letting m take negative values as well as positive values, we can write the general solution instead as

$$\phi_m = e^{imx} \qquad \text{where } m = \ldots -2, -1, 0, 1, 2, 3, \ldots$$

This allows the solutions to be written more compactly.

It is straightforward to check orthogonality:

$$\langle \phi_n, \phi_m \rangle = \int_0^{2\pi} \phi_n^*(x)\, \phi_m(x)\, dx$$

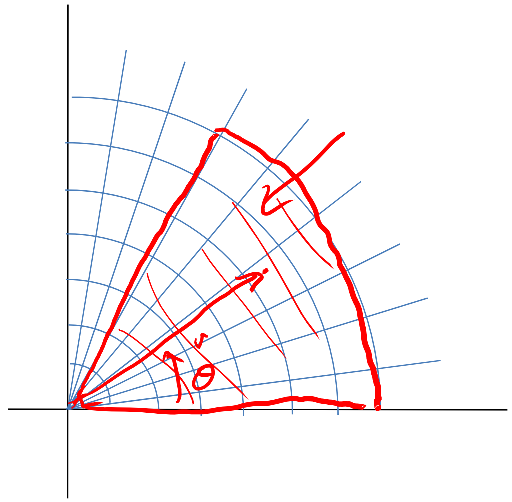
$$= \int_0^{2\pi} e^{-inx} e^{+imx}\, dx$$

$$= \int_0^{2\pi} e^{i(m-n)x}\, dx = \int_0^{2\pi} e^0\, dx = 2\pi \quad \text{if } m=n.$$

$$= \left[\frac{e^{i(m-n)x}}{m-n} \right]_0^{2\pi} \quad \text{if } m \neq n.$$

$$= \frac{1}{m-n} \Big(\underbrace{e^{i(m-n)2\pi}}_{=1} - \underbrace{e^{-i(m-n)2\pi}}_{=1} \Big)$$

$$= 0$$



The 2D Laplacian in polar coordinates

Consider the Laplacian operator $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

In polar coordinates, this is

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$$

We would like to find the solution of problems of the type

$$\nabla^2\phi = 0$$

in polar coordinates, in some 2D domain.

We first try to find the general solution to

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right]\psi(r,\theta) = 0$$

We use the *separation Ansatz*:

capital theta

$$\psi(r,\theta) = R(r)\,\Theta(\theta)$$

Subbing into the PDE:

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right) R\Theta = 0$$

$$\Theta \frac{\partial^2 R}{\partial r^2} + \frac{1}{r}\Theta\frac{\partial R}{\partial r} + \frac{1}{r^2} R \frac{\partial^2 \Theta}{\partial \theta^2} = 0$$

Or

$$\Theta R'' + \frac{1}{r}\Theta R' + \frac{1}{r^2} R\Theta'' = 0.$$

Divide by ΘR, multiply by r^2:

So

$$\frac{r^2 \Theta R''}{\Theta R} + \frac{r\,\Theta R'}{\Theta R} + \frac{R\Theta''}{\Theta R} = 0$$

$$\Rightarrow \quad r^2\frac{R''}{R} + r\frac{R'}{R} + \frac{\Theta''}{\Theta} = 0.$$

depends on r — depends on θ.

So we have separated the problem into

$$r^2 \frac{R''}{R} + r\frac{R'}{R} = -\frac{\Theta''}{\Theta} = \pm\lambda$$

We can put the problem for θ in the form:

$$-\frac{\Theta''}{\Theta} = \lambda \Rightarrow -\Theta'' = \lambda\Theta$$

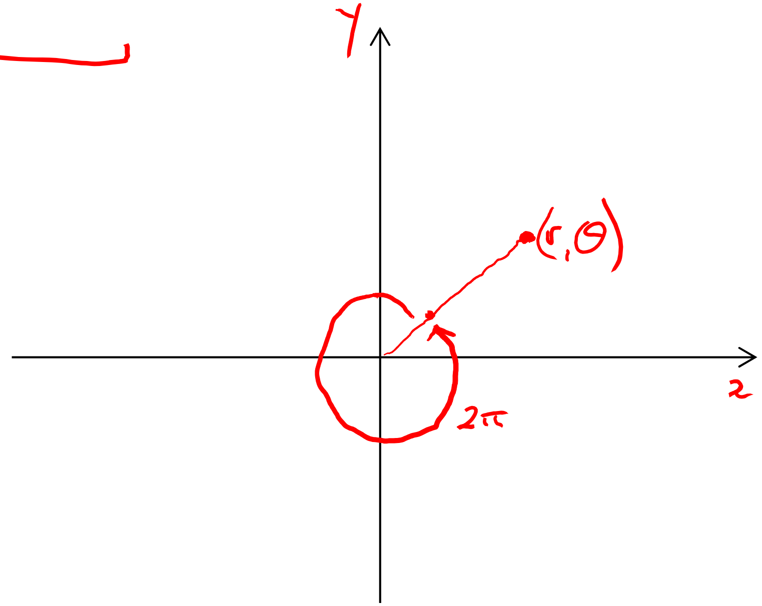
We also have that

$$-\frac{d^2}{dx^2}\Theta = \lambda\Theta$$

$$\Theta(0) = \Theta(2\pi)$$

$$\Theta'(0) = \Theta'(2\pi)$$

Because Θ must be single-valued if we go around a full circle.



The eigenfunctions are

$$\Theta_m = e^{im\theta}$$

where $m = \ldots -2, -1, 0, 1, 2, \ldots$

$m \in \mathbb{Z}$.

with eigenvalues:

$$\lambda_m = m^2$$

The problem for $R(r)$ is then

$$r^2 \frac{R''}{R} + r\frac{R'}{R} = m^2$$

λ

case $m = 0$:

$$\frac{r^2 R''}{R} + r\frac{R'}{R} = 0$$

Multiply by R:

$$r^2 R'' + rR' = 0.$$

divide by r:

$$r\frac{d^2}{dr^2}R + \frac{dR}{dr} = 0.$$

$$\frac{d}{dr}\left[r\frac{dR}{dr}\right] = 0.$$

$$= r\frac{d^2R}{dr^2} + \frac{dR}{dr}\frac{dr}{dr}$$ ✓

So

$$r\frac{dR}{dr} = B_0 \leftarrow \text{constant}$$

Then

$$\frac{dR}{dr} = \frac{B_0}{r}$$

$$\Rightarrow R = \int \frac{B_0}{r}dr$$

$$= B_0 \ln|r| + A_0.$$

Case $m \neq 0$:

$$r^2 R'' + r R' - m^2 R = 0$$

Solutions are

$$R(r) = A r^{|m|}$$

$$R(r) = B r^{-|m|}.$$

Try the Ansatz

$$R(r) = r^\alpha$$

$$r R'(r) = \alpha r^\alpha$$

$$r^2 R''(r) = \alpha(\alpha - 1) r^\alpha.$$

$$\alpha(\alpha-1) r^\alpha + \alpha r^\alpha - m^2 r^\alpha = 0$$

$$\left[\alpha(\alpha-1) + \alpha - m^2\right] r^\alpha = 0$$

$$\left[\alpha^2 - \alpha + \alpha - m^2\right] r^\alpha = 0$$

$$\left[\alpha^2 - m^2\right] r^\alpha = 0$$

$$\alpha^2 = m^2$$

$$\alpha = \pm m = \pm |m|.$$

So

$$R(r) = A r^{\pm|m|}.$$

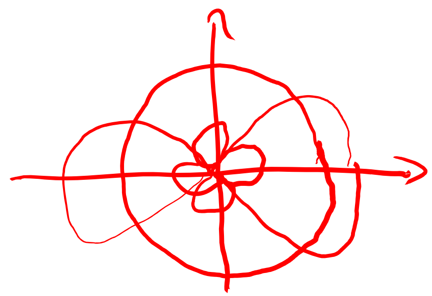
So we have found:

$$\Theta_m(\theta) = e^{im\theta}$$

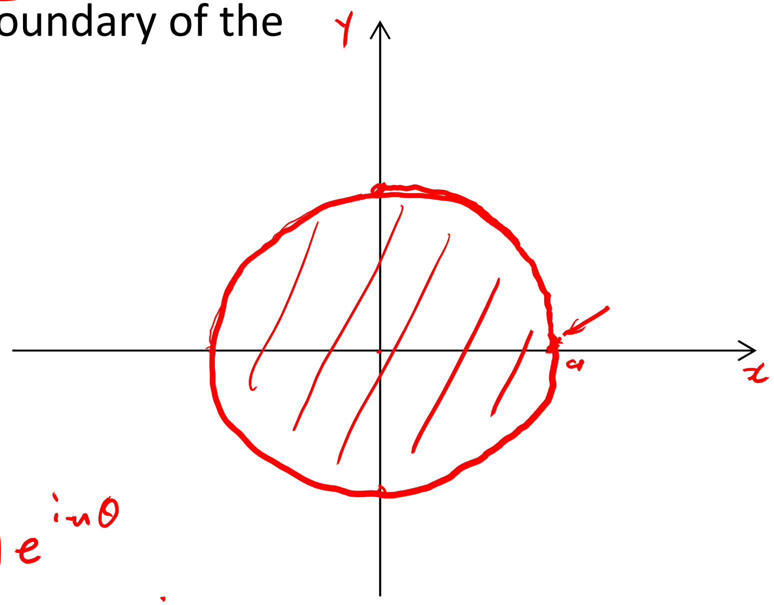
$$R_m(r) = \begin{cases} A_0 + B_0 \log|\mathrm{r}| & \text{for } m = 0 \\ A r^{|m|} + B r^{-|m|} & \text{for } m \neq 0 \end{cases}$$

The general solution to Laplace's equation in polar coordinates is therefore

$$\psi(r,\theta) = A_0 + B_0 \ln|r| + \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \left(A_m r^{|m|} + B_m r^{-|m|} \right) e^{im\theta}.$$



Example: Solve Laplace's equation on the domain r ≤ a, with a Dirichlet condition $\psi = \sin(2\theta)$ defined on the boundary of the domain.



$\nabla^2 \psi = 0$

The general solution is

$\psi(r,\theta) = B_0 \ln r + A_0 + \sum_{m=-\infty}^{\infty} \left(A_m r^{|m|} + B_m r^{-|m|} \right) e^{im\theta}$

The solution has to be finite at r=0, so $B_m = 0 \ \forall m$. So

$\psi(r,\theta) = A_0 + \sum_{m=-\infty}^{\infty} A_m r^{|m|} e^{im\theta}$

$= \sum_{m=-\infty}^{\infty} A_m r^{|m|} e^{im\theta}$

$\psi(r,\theta) = A_2 r^2 e^{2i\theta} + A_{-2} r^2 e^{-2i\theta}$

We want $\psi(a,\theta) = \underbrace{\sin(2\theta)}_{f(\theta)} = \sum_{m=-\infty}^{\infty} A_m a^{|m|} e^{im\theta}$.

$$f(\theta) = \sum_{m=-\infty}^{\infty} A_m a^{|m|} \underbrace{e^{im\theta}}_{\phi_m(\theta)}.$$

Take the inner product with $\phi_n = e^{in\theta}$:

$$\langle \phi_n, f \rangle = \sum_{m=-\infty}^{\infty} A_m a^{|m|} \langle \phi_n, \phi_m \rangle$$

$$= A_n a^{|n|} \int_0^{2\pi} \underbrace{e^{-in\theta} e^{in\theta}}_{=1} d\theta$$

$$= A_n a^{|n|} 2\pi$$

So

$$A_n = \frac{1}{2\pi a^{|n|}} \langle \phi_n, f \rangle$$

$$= \frac{1}{2\pi a^{|n|}} \int_0^{2\pi} e^{-in\theta} \sin(2\theta)\, d\theta.$$

$$= \frac{1}{2\pi a^{|n|}} \int_0^{2\pi} e^{-in\theta} \frac{1}{2i}\left(e^{i2\theta} - e^{-i2\theta}\right) d\theta$$

$$= \frac{1}{2\pi a^{|n|} 2i} \int_0^{2\pi} e^{-i(n-2)\theta}\, d\theta$$

$$- \frac{1}{2\pi a^{|n|}} \frac{1}{2i} \int_0^{2\pi} e^{-i(n+2)\theta}\, d\theta.$$

$$A_2 = \frac{1}{2\pi a^2 2\pi} \ldots \frac{1}{2ia^2}.$$

Recall

$$\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right).$$

$$\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right).$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

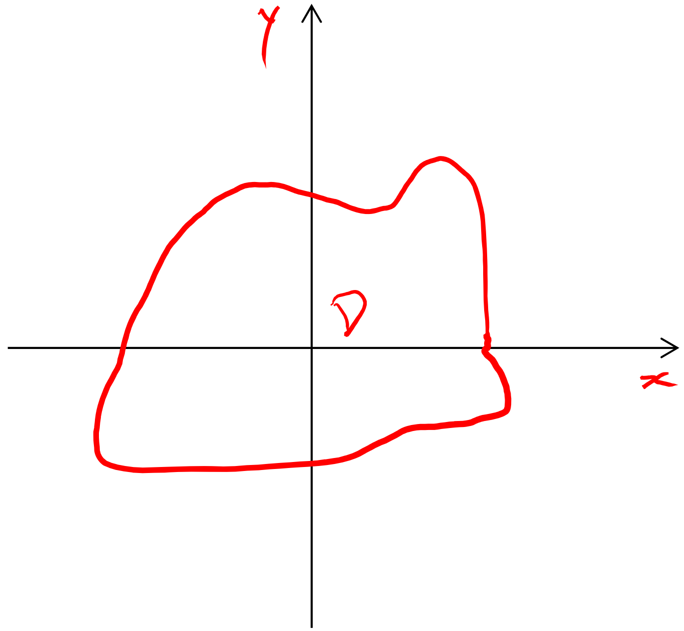
$$e^{-i\theta} = \cos\theta - i\sin\theta$$

The Helmholtz equation

Consider the 2D problem with the 2D Laplacian as the differential operator:

$$-\nabla^2\phi = \lambda\phi$$

$-\nabla^2\phi = -\nabla\cdot\nabla\phi$ $\left(-\frac{d^2}{dx^2}\phi\right)$



on the domain D, where D is a finite domain in 2D, and with homogeneous Dirichlet conditions

$$\phi = 0$$

on the boundary of D.

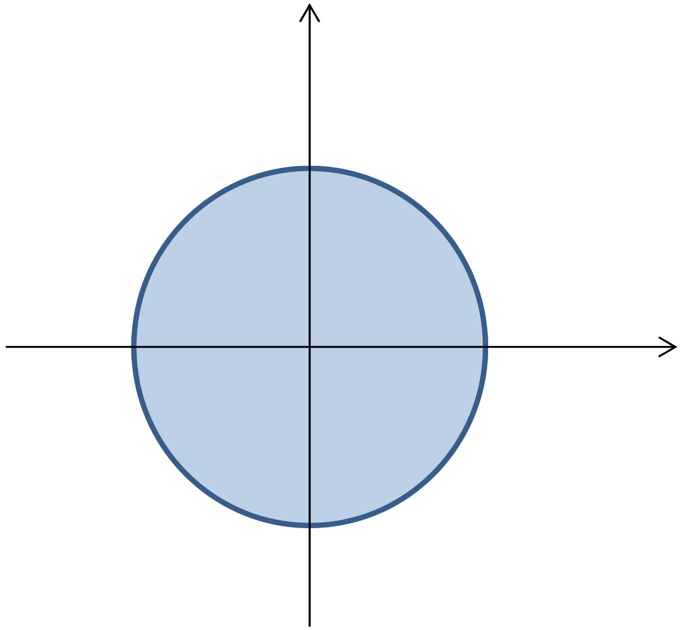
Although the operator is 2D, we can show that this is also a Sturm-Liouville problem.

This form of the PDE is often known as the Helmholtz equation.

$$\nabla^2\phi + k^2\phi = 0$$

We now attempt to solve this problem using separation of variables:

$$\begin{cases} -\nabla^2\phi = \lambda\phi & \text{on D} \\ \phi = 0 & \text{on } \partial\text{D} \end{cases}$$



Use the separation ansatz

$$\phi(r,\theta) = R(r)\Theta(\theta)$$

Subbing in, we find

$$-\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right) R\Theta = \lambda R\Theta$$

$$+R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = -\lambda R\Theta$$

Divide by $R\Theta$:

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} = -\lambda$$

multiply by r^2:

$$r^2\frac{R''}{R} + r\frac{R'}{R} + \lambda r^2 + \frac{\Theta''}{\Theta} = 0$$

$$-\frac{\Theta''}{\Theta} = \mu \quad \leftarrow \text{SL-problem}$$

$$r^2\frac{R''}{R} + r\frac{R'}{R} + \lambda r^2 - \mu = 0.$$

The problem for Θ has solution

$$\Theta = e^{im\theta}$$

$$\mu_m = m^2.$$

The problem for $R(r)$ is

$$r^2 R'' + rR' + (\lambda r^2 - m^2)R = 0.$$

$$r^2 \frac{d^2R}{dr^2} + r\frac{dR}{dr} + (\lambda r^2 - m^2)R = 0.$$

This is a special differential equation known as Bessel's equation.

The solutions to this equation are known as *Bessel functions*, and
There are two types:

$$\frac{d^2y}{dx^2} = -y.$$

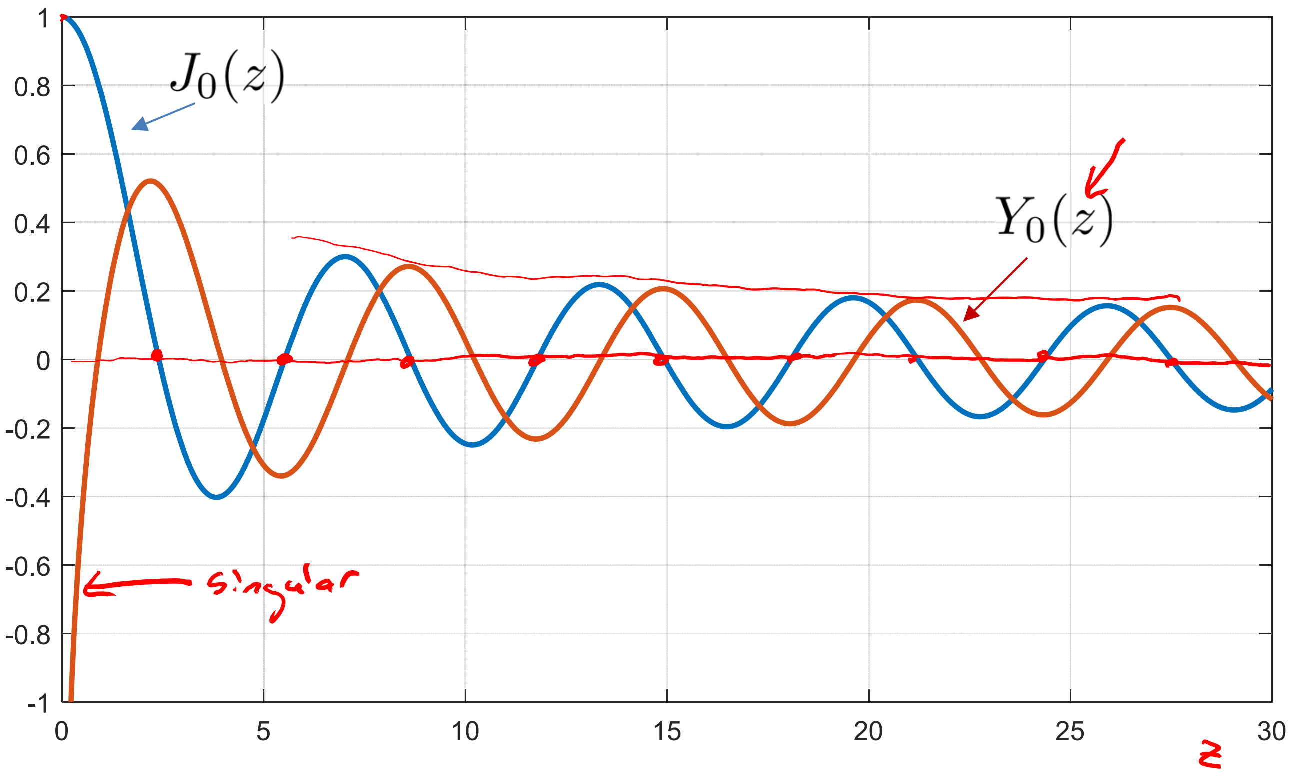
$J_m(z)$ — 1st kind

$Y_m(z)$ — 2nd kind

These functions are analogous to sine and cosine for polar coordinates.

<u>Properties of Bessel functions:</u>

- They are normally expressed as *infinite series*
- They oscillate, but decay slowly to zero
- J's are finite at the origin, Y's are singular

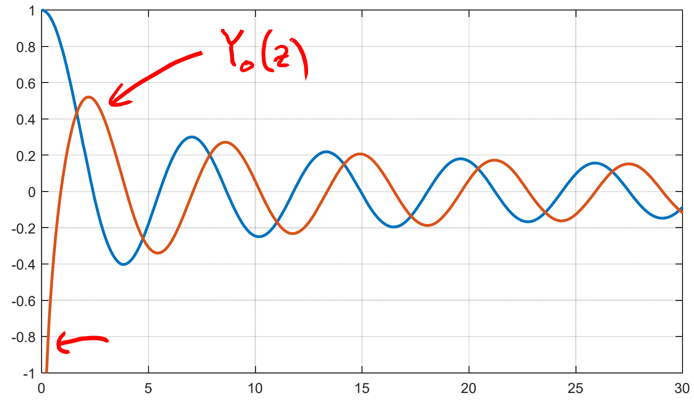


Bessel's equation is:

$$z^2 \frac{d^2 J}{dz^2} + z \frac{dJ}{dz} + (z^2 - m^2) = 0$$

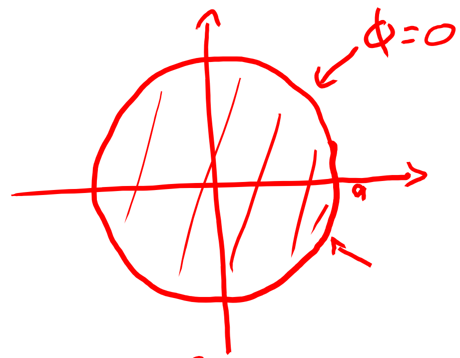
And this has solutions

$$J_m(z) \qquad Y_m(z)$$



And our problem for R is

$$r^2 \frac{d^2 R}{dr^2} + r \frac{dR}{dr} + (\lambda r^2 - m^2) R = 0.$$



$z^2 = \lambda r^2 \qquad z = \sqrt{\lambda} r \qquad \nabla^2 \phi = \lambda \phi$

The general solution is therefore

$$J_m(\sqrt{\lambda} r) \qquad Y_m(\sqrt{\lambda} r)$$

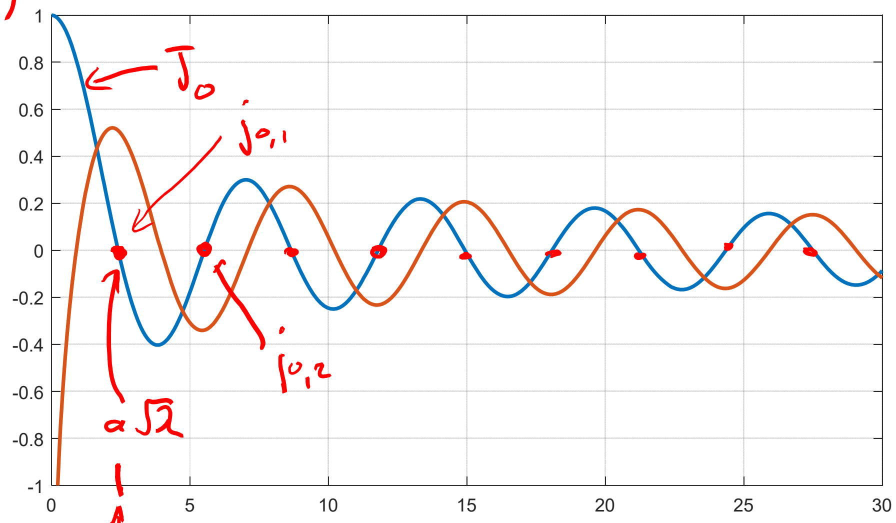
singular

We would like the solution to be a) non-singular at the origin, and b) have $R(a) = 0$.

So the general expression for the eigenfunctions is

$$\phi_m(r,\theta) = J_m(\sqrt{\lambda}r)e^{im\theta}$$

By choosing $a\sqrt{\lambda}$ to be a zero of $J_m(z)$, we can satisfy $\phi(a,\theta) = 0$.



Letting $j_{m,n}$ denote the nth zero of the mth order Bessel function, we then have

$$a\sqrt{\lambda} = j_{m,n} \Rightarrow \lambda = \left(\frac{j_{m,n}}{a}\right)^2$$

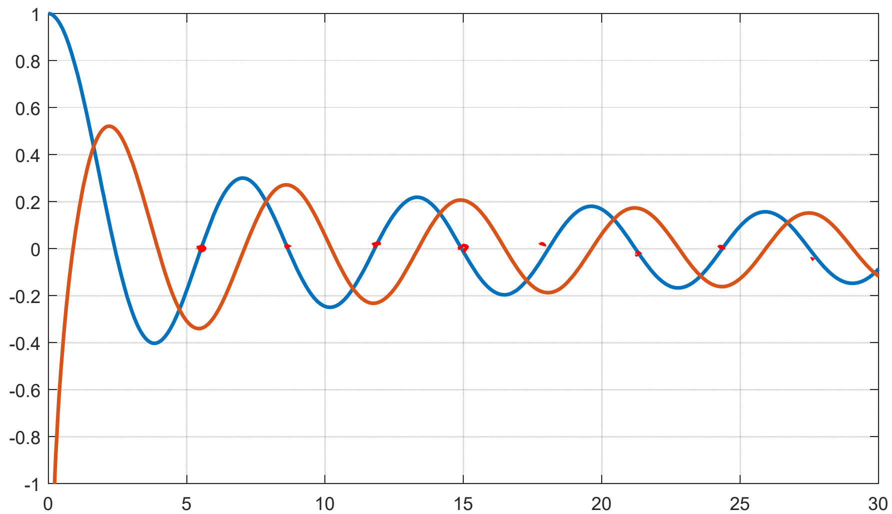
We then obtain an infinite number of eigenfunctions for each m and n:

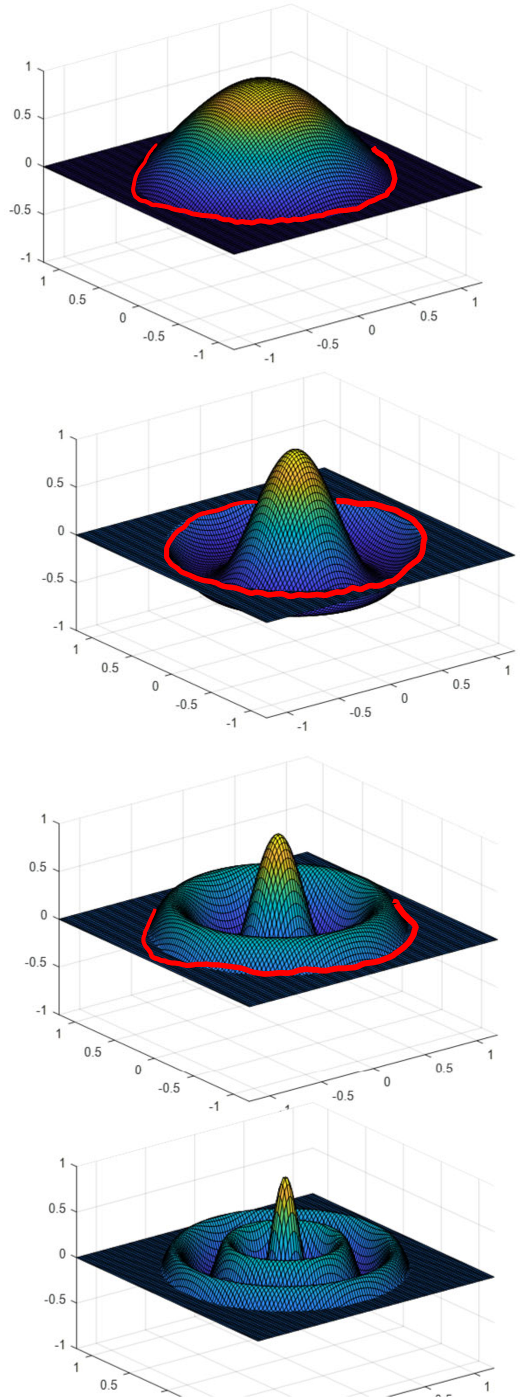
$$\phi_{m,n}(r,\theta) = J_m(j_{m,n}\frac{r}{a})e^{im\theta}$$

$$\lambda_{m,n} = \left(\frac{j_{m,n}}{a}\right)^2$$

All of these functions satisfy the SL problem.

$$\begin{cases} -\nabla^2 \phi = \lambda \phi & \text{on D} \\ \phi = 0 & \text{on } \partial\text{D} \end{cases}$$





Waves and the wave equation

The wave equation for a function ° (x,y,z) is

$$\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = 0$$

where c is the *phase velocity* of the wave.

The wave equation can be *separated* using the ansatz

$$\psi(x,y,z,t) = U(x,y,z)\,T(t)$$

Subbing in

$$\nabla^2 U T - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}T(t)U = 0$$

$$T\nabla^2 U - \frac{1}{c^2}U T'' = 0$$

Divide

$$c^2\frac{\nabla^2 U}{U} = \frac{T''}{T} = -\omega^2$$

separation constant $\omega = 2\pi f$.

Problem for time T(t):

$$\frac{dT^2}{dt^2} = -\omega^2 T$$

$T(t) = e^{\pm i\omega t}$

$= \cos(\omega t) \pm i \sin(\omega t)$

Problem for u(x,y,z):

$$c^2 \nabla^2 u = -\omega^2 u$$

$-\nabla^2 u = \frac{\omega^2}{c^2} u$

$= \lambda u$ (confusing)

$= k^2 u \qquad k = \frac{\omega}{c}$

$\nabla^2 u + k^2 u = 0$.

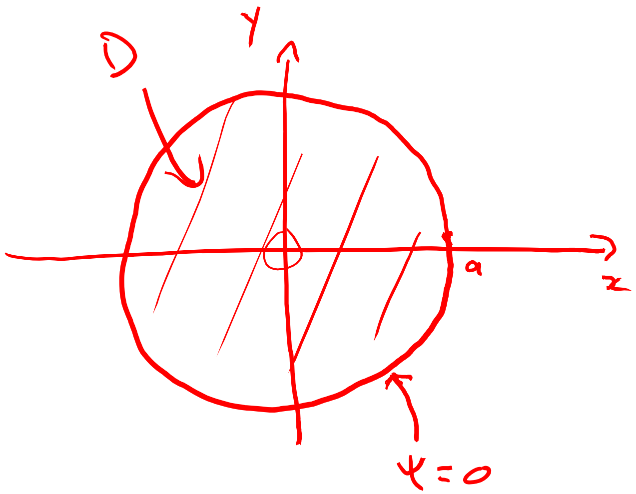
Helmholtz equation.

Example: Solve the wave equation

$$\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = 0$$

defined on the domain r<a, together with homogeneous Dirichlet boundary conditions ˚=0 on the boundary r = a and initial conditions ∂˚ / ∂t = 0 and

$$\psi(r,\theta) = e^{-r^2} \quad \text{at } t = 0.$$



Now let $\psi(r,\theta,t) = u(r,\theta)\,T(t)$

where $T(t) = Ae^{i\omega t} + Be^{-i\omega t}$ and

$u(r,\theta)$ solve the problem

S-L problem → $\begin{cases} -\nabla^2 u = k^2 u & \text{on } D \\ u = 0 & \text{on the boundary} \end{cases}$

This we solved:

The eigenfunctions are

$$\phi_{m,n}(r,\theta) = J_m\left(j_{m,n}\frac{r}{a}\right)e^{im\theta}$$

So the general solution for ψ can be written

$$\psi(r,\theta,t) = \sum_{m,n} \phi_{m,n} e^{i\omega t} a_{m,n} + \sum_{m,n} \phi_{m,n} e^{-i\omega t} b_{m,n}.$$

Now $\partial\psi/\partial t = 0$ at $t=0$, so

$$\partial\psi/\partial t = \sum_{m,n}\left(a_{m,n}\phi_{m,n} i\omega e^{i\omega t} + b_{m,n}\phi_{m,n}(-i\omega)e^{-i\omega t}\right)$$

At $t=0$,

$$\frac{\partial\psi}{\partial t} = \sum_{m,n}(a_{m,n} - b_{m,n})\phi_{m,n} i\omega.$$

This condition is satisfied if $a_{m,n} = b_{m,n}$.

So

$$\psi(r,\theta,t) = \sum_{m,n} a_{m,n}\, \phi_{m,n}(r,\theta)\left(e^{i\omega t} + e^{-i\omega t}\right)$$

$$\underbrace{e^{i\omega t} + e^{-i\omega t}}_{2\cos(\omega t)}$$

$$= \sum_{m,n} a_{m,n}\, \phi_{m,n}\, 2\cos(\omega t).$$

We want that, at $t=0$, $\psi(r,\theta,0) = e^{-r^2} = f(r)$.

That is,

$$f(r) = 2\sum_{m,n} a_{m,n}\, \phi_{m,n}$$

Take the inner product with $\phi_{m',n'}$

$$\langle \phi_{m',n'}, f \rangle = 2\sum_{m,n} a_{m,n} \langle \phi_{m',n'}, \phi_{m,n} \rangle$$

$$= 2\, a_{m',n'} \|\phi_{m,n}\|^2.$$

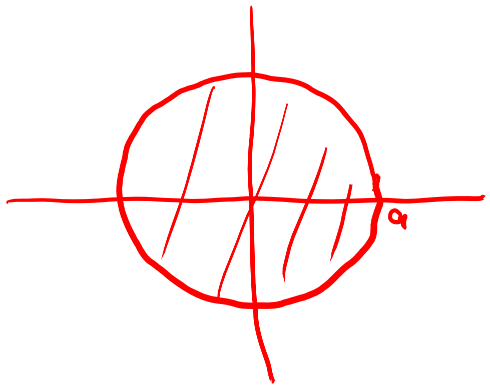
So

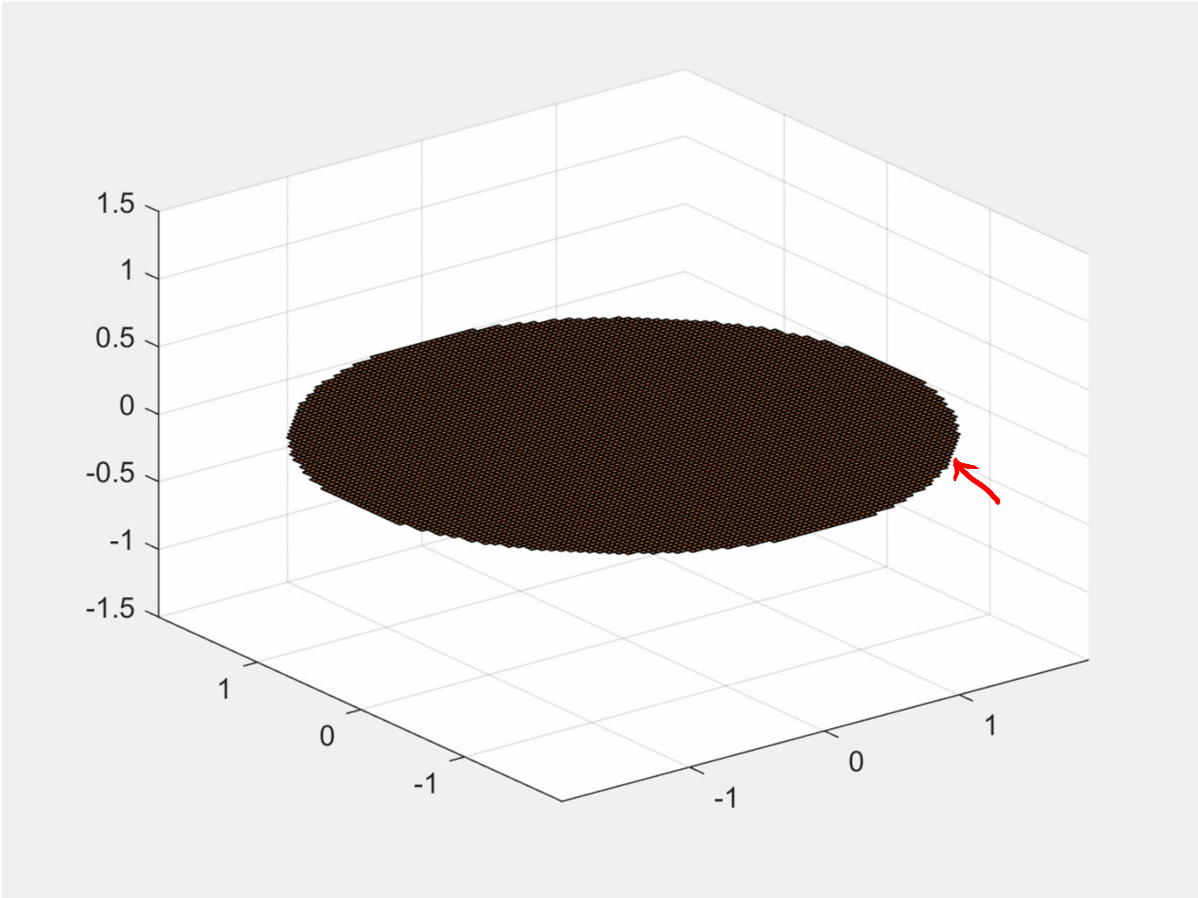
$$a_{m,n} = \frac{1}{2}\, \frac{\langle \phi_{m,n}, f \rangle}{\|\phi_{m,n}\|^2}$$

So

$$\psi(r,\theta,t) = \sum_{m,n} \frac{\langle \phi_{m,n}, f \rangle}{\|\phi_{m,n}\|^2} \phi_{m,n}(r,\theta) \cos(\omega t)$$

$$\langle \phi_{m,n}, f \rangle = \int_0^{2\pi}\int_0^{a} \phi^*_{m,n}(r,\theta) f(r)\, r\, dr\, d\theta$$





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-1
-1.5
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0
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