So far we've looked at PDEs in Cartesian coordinates:

The heat equation:

$$\kappa \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

Laplace's equation

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

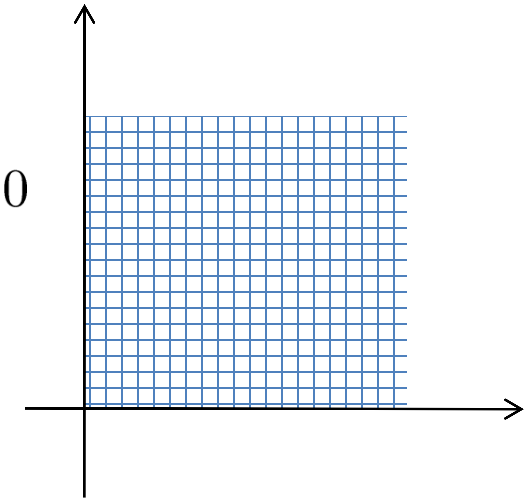
The wave equation:

$$\frac{\partial^2 \psi}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} = 0$$

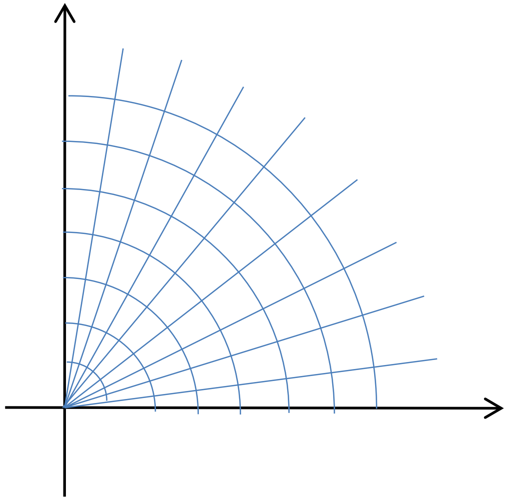
We would also like to solve equations in different coordinate systems, and in 3 dimensions.

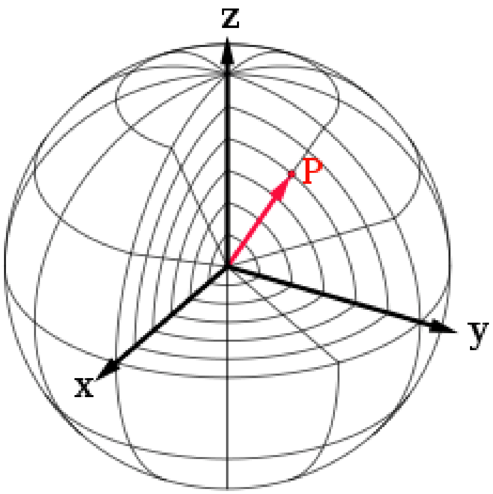
$$\nabla^2 \psi = 0$$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$



$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} = 0$$





The steps for solving using separation of variables remain the same

1. Separate variables

2. Identify the Sturm-Liouville problem and compute the eigenfunctions

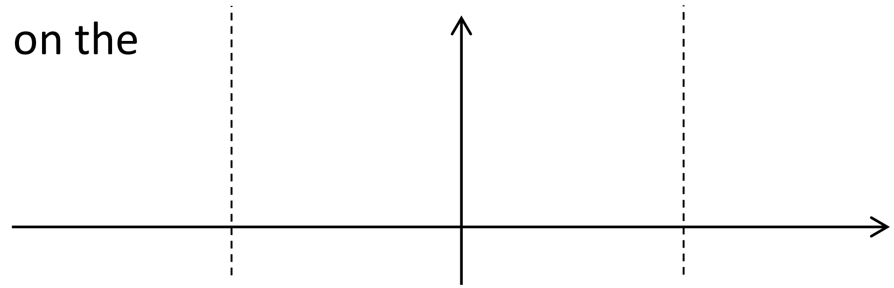
3. Use an infinite series to match the boundary conditions.

<u>However</u> we often have to introduce *Special Functions* to express the solution.

Before we begin, we will need to look at
Periodic boundary conditions for S-L problems

Consider the 1-D eigenvalue problem defined on the domain $D = \{x \mid 0 \leq x \leq d\}$

$$\mathcal{L}\phi = \lambda\phi \quad \text{on } D$$



with boundary conditions

$$\phi(0) = \phi(d)$$

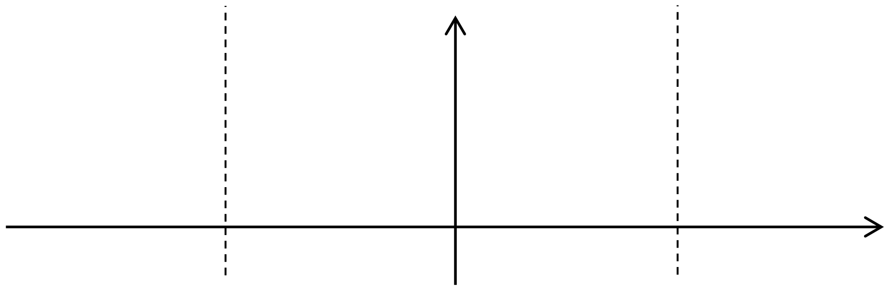
$$\phi'(0) = \phi'(d)$$

This problem has *all the properties of a Sturm-Liouville problem.*

We find a *countably infinite set* of eigenfunctions and eigenvalues where the λ_n are all real and positive, and the eigenfunctions are all orthogonal.

Example: Consider the periodic S-L problem

$$-\frac{d^2}{dx^2}\phi = \lambda\phi$$



defined on the domain |x|< d, with

$$\phi(0) = \phi(2\pi)$$

$$\phi'(0) = \phi'(2\pi)$$

This has eigenfunctions:

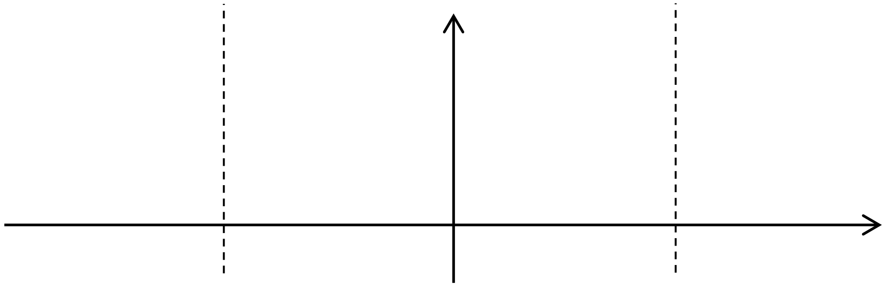
$$\phi_m = e^{\pm imx}$$

With eigenvalues:

$$\lambda_m = m^2$$

We have two sets of eigenfunctions:

$$\phi_m = e^{imx} \qquad \phi_m = e^{-imx}$$



where $m = 0,1,2,3, \ldots$

By letting m take negative values as well as positive values, we can write the general solution instead as

$$\phi_m = e^{imx}$$ where $m = \ldots -2, -1, 0, 1, 2, 3, \ldots$

This allows the solutions to be written more compactly.

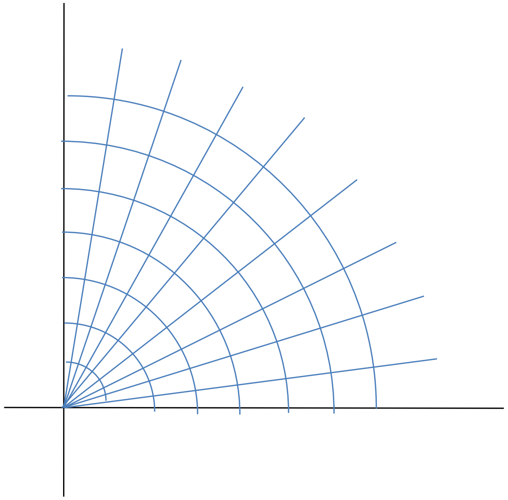
It is straightforward to check orthogonality:

The 2D Laplacian in polar coordinates

Consider the Laplacian operator ∇^2

In polar coordinates, this is

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$$



We would like to find the solution of problems of the type

$$\nabla^2\phi = 0$$

in polar coordinates, in some 2D domain.

We first try to find the general solution to

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right]\psi(r,\theta) = 0$$

We use the *separation Ansatz*:

So we have separated the problem into

$$r^2 \frac{R''}{R} + r \frac{R'}{R} = -\frac{\Theta''}{\Theta} = \pm\lambda$$

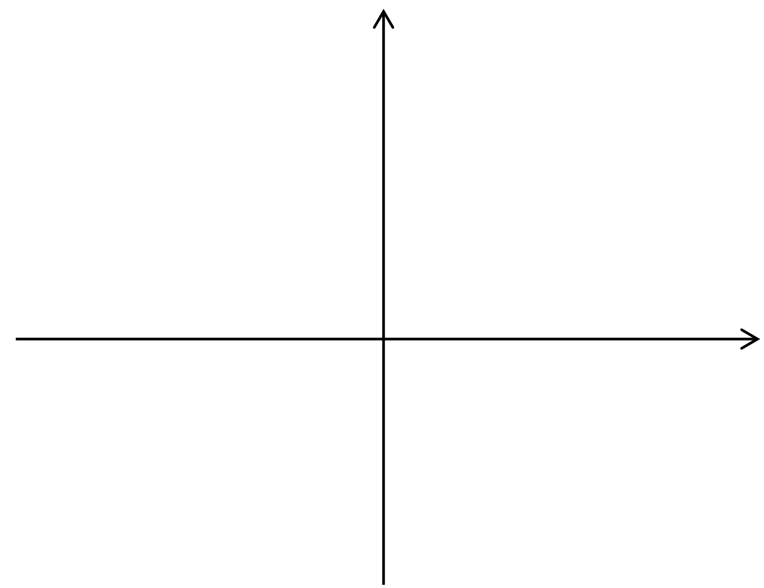
We can put the problem for θ in the form:

We also have that

$$\Theta(0) = \Theta(2\pi)$$

$$\Theta'(0) = \Theta'(2\pi)$$

Because Θ must be single-valued if we go around a full circle.



The eigenfunctions are

$$\Theta_m = e^{im\theta}$$

with eigenvalues:

$$\lambda_m = m^2$$

The problem for $R(r)$ is then

$$r^2 \frac{R''}{R} + r\frac{R'}{R} = m^2$$

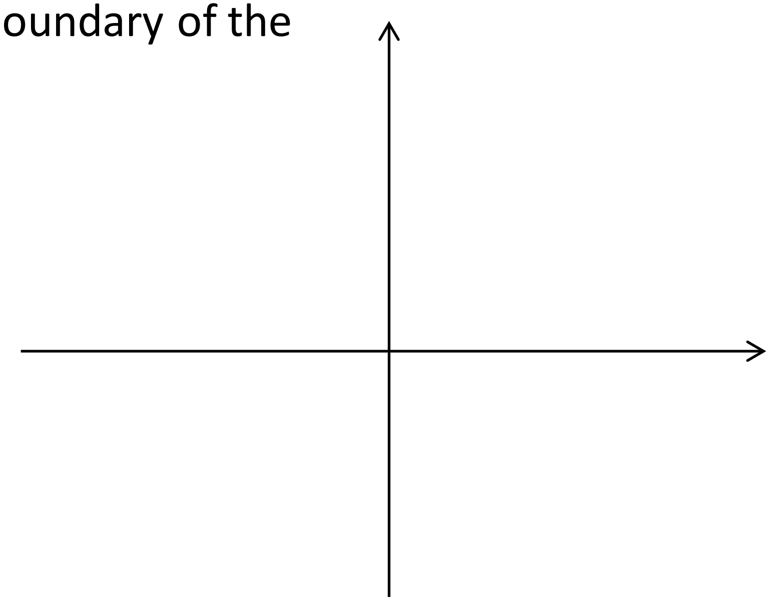
So we have found:

$$\Theta_m(\theta) = e^{im\theta}$$

$$R_m(r) = \begin{cases} A_0 + B_0 \log|\mathrm{r}| & \text{for } m = 0 \\ Ar^{|m|} + Br^{-|m|} & \text{for } m \neq 0 \end{cases}$$

The general solution to Laplace's equation in polar coordinates is therefore

Example: Solve Laplace's equation on the domain r ≤ a, with a Dirichlet condition $\psi = \sin(2\theta)$ defined on the boundary of the domain.



The Helmholtz equation

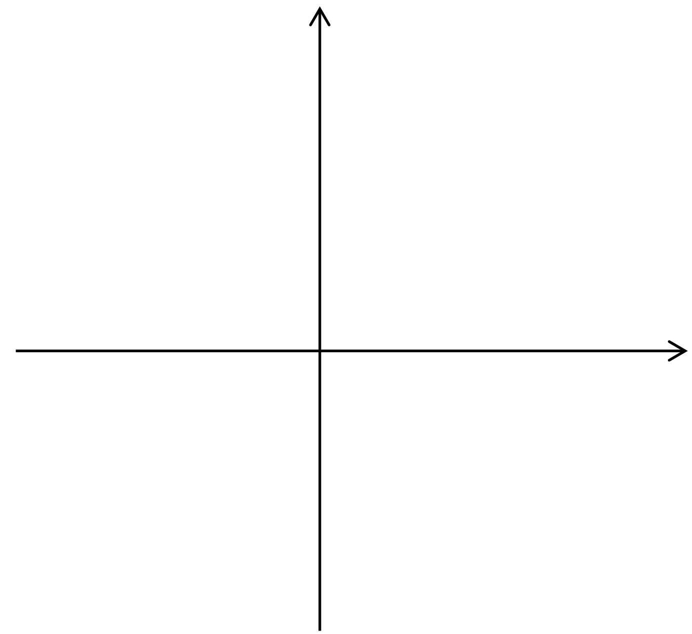
Consider the 2D problem with the 2D Laplacian as the differential operator:

$$-\nabla^2\phi = \lambda\phi$$

on the domain D, where D is a finite domain in 2D, and with homogeneous Dirichlet conditions

$$\phi = 0$$

on the boundary of D.

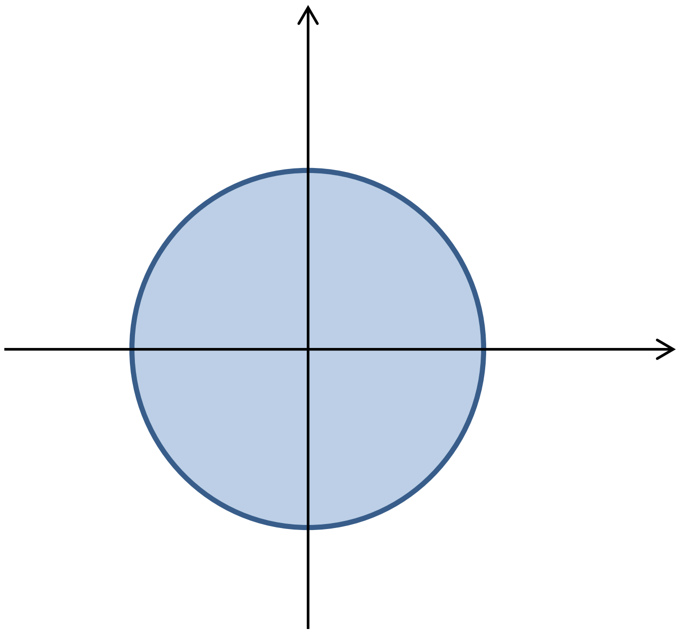


Although the operator is 2D, we can show that this is also a Sturm-Liouville problem.

This form of the PDE is often known as the Helmholtz equation.

We now attempt to solve this problem using separation of variables:

$$\begin{cases} -\nabla^2\phi = \lambda\phi & \text{on } D \\ \phi = 0 & \text{on } \partial D \end{cases}$$



The problem for $R(r)$ is

This is a special differential equation known as Bessel's equation.

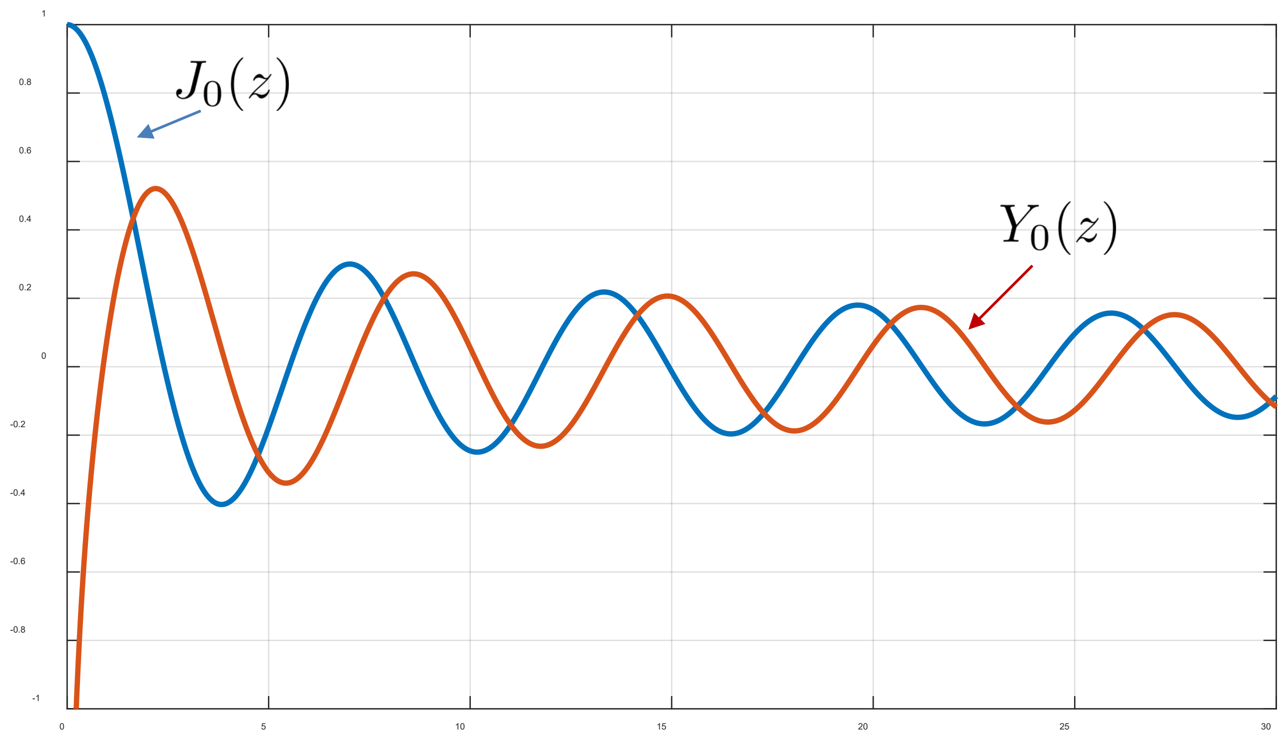
The solutions to this equation are known as *Bessel functions*, and
There are two types:

$$J_m(z) \qquad\qquad Y_m(z)$$

These functions are analogous to sine and cosine for polar coordinates.

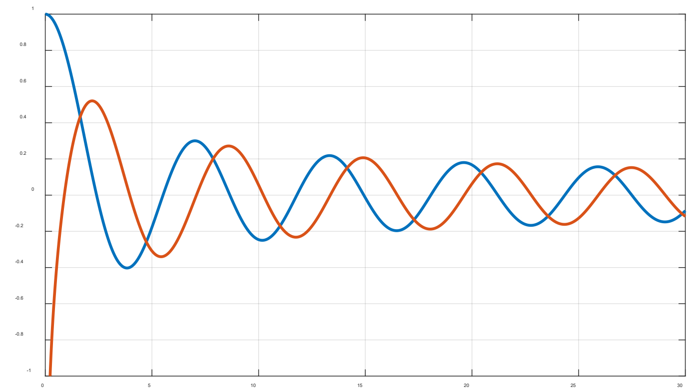
Properties of Bessel functions:

- They are normally expressed as *infinite series*
- They oscillate, but decay slowly to zero
- J's are finite at the origin, Y's are singular



Bessel's equation is:

$$z^2 \frac{d^2 J}{dz^2} + z \frac{dJ}{dz} + (z^2 - m^2) = 0$$



And this has solutions

$$J_m(z) \qquad\qquad Y_m(z)$$

And our problem for R is

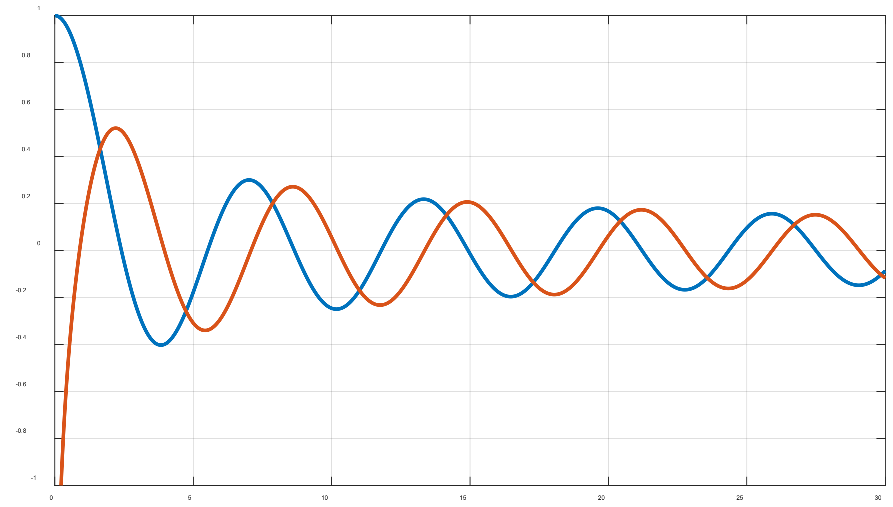
$$r^2 \frac{d^2 R}{dr^2} + r \frac{dR}{dr} + (\lambda r^2 - m^2) R = 0.$$

The general solution is therefore

We would like the solution to be a) non-singular at the origin, and b) have $R(a) = 0$.

So the general expression for the eigenfunctions is

$$\phi_m(r,\theta) = J_m(\sqrt{\lambda}r)e^{im\theta}$$



By choosing $a\sqrt{\lambda}$ to be a zero of $J_m(z)$, we can satisfy $\phi(a,\theta) = 0$.

Letting $j_{m,n}$ denote the nth zero of the mth order Bessel function, we then have

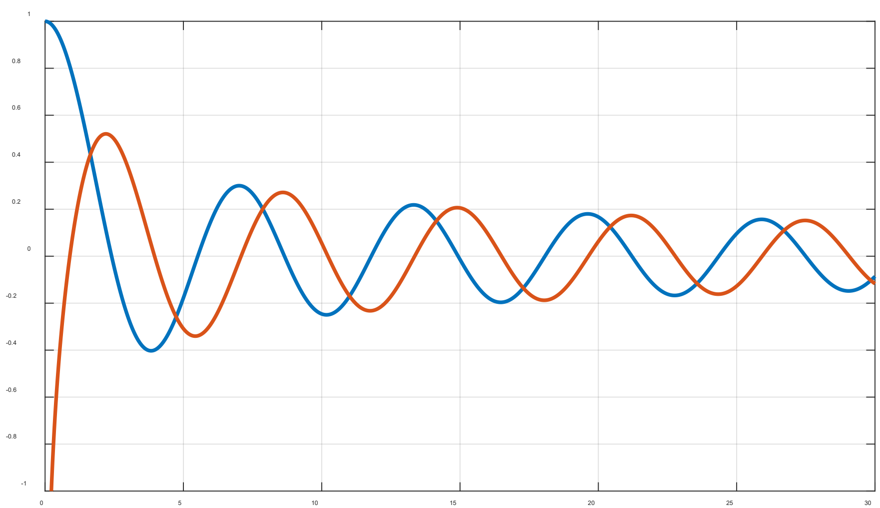
$$\phi_{m,n}(r,\theta) = J_m(j_{m,n}\frac{r}{a})e^{im\theta}$$

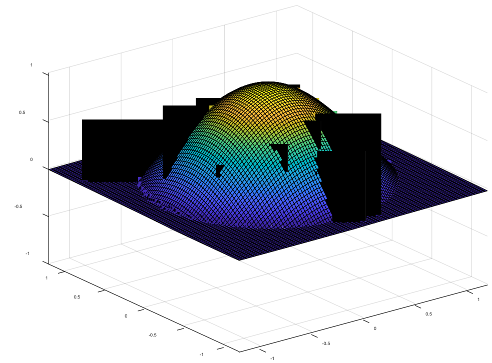
We then obtain an infinite number of eigenfunctions for each m and n:

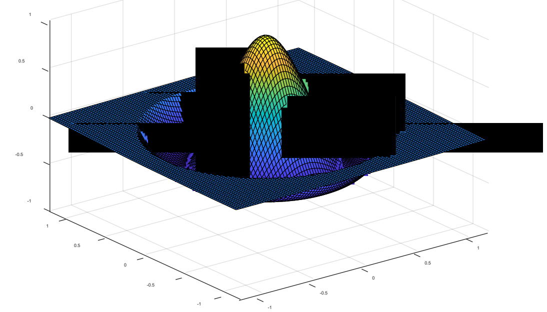
$$\lambda_{m,n} = \left(\frac{j_{m,n}}{a}\right)^2$$

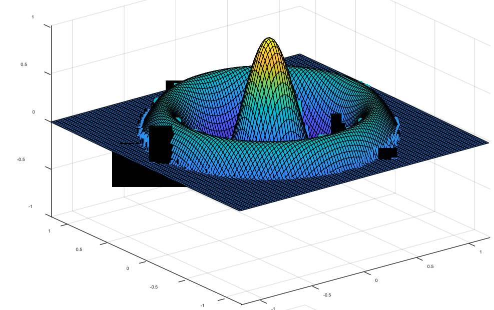
All of these functions satisfy the SL problem.

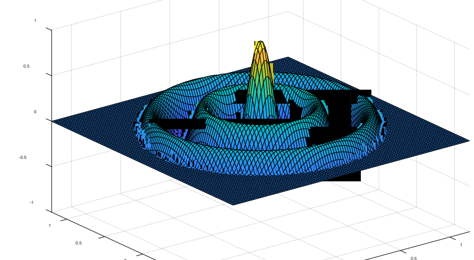
$$\begin{cases} -\nabla^2\phi = \lambda\phi & \text{on D} \\ \phi = 0 & \text{on } \partial\text{D} \end{cases}$$











Waves and the wave equation

The wave equation for a function Ã(x,y,z) is

$$\nabla^2 \psi - \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} = 0$$

where c is the *phase velocity* of the wave.

The wave equation can be *separated* using the ansatz

Problem for time T(t):

$$\frac{dT^2}{dt^2} = -\omega^2 T$$

Problem for u(x,y,z):

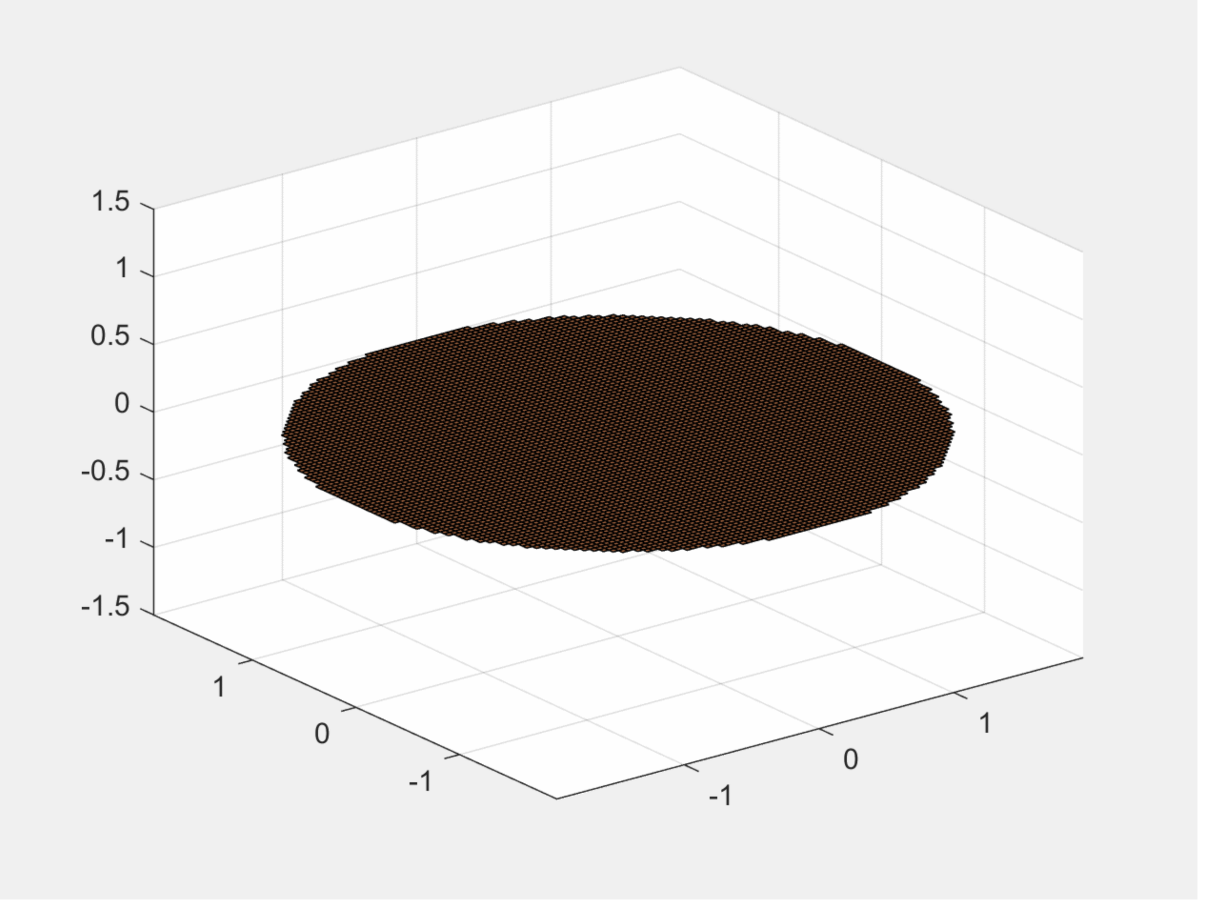
$$c^2 \nabla^2 u = -\omega^2 u$$

Example: Solve the wave equation

$$\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = 0$$

defined on the domain r<a, together with homogeneous Dirichlet boundary conditions Ã=0 on the boundary r = a and initial conditions ∂Ã / ∂t = 0 and

$$\psi(r,\theta) = e^{-r^2}$$



1.5
1
0.5
0
-0.5
-1
-1.5
1
0
-1
-1
0
1