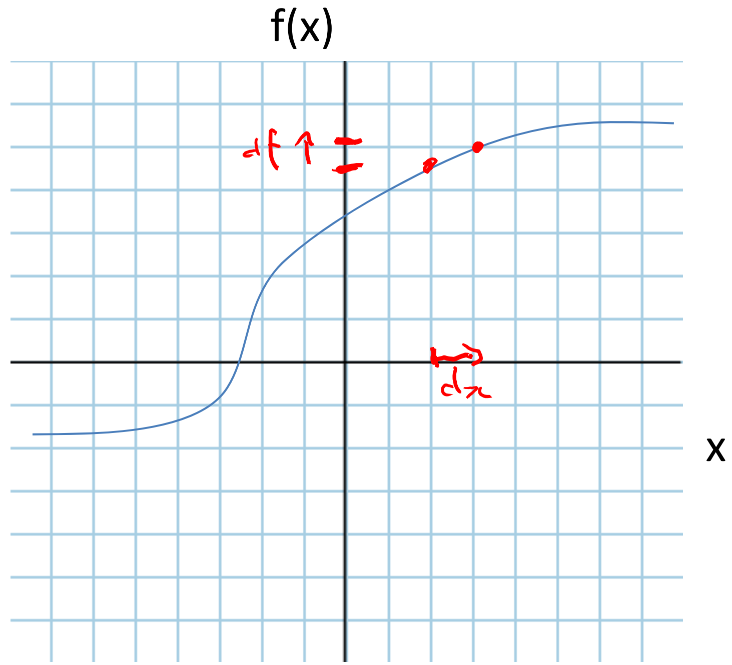
Differentials, the chain rule,
and changing coordinate systems

Differentials

For a single variable function f(x), the *differential* df is the change in the function for a small change in x:

$$df = \frac{\partial f}{\partial x} dx$$

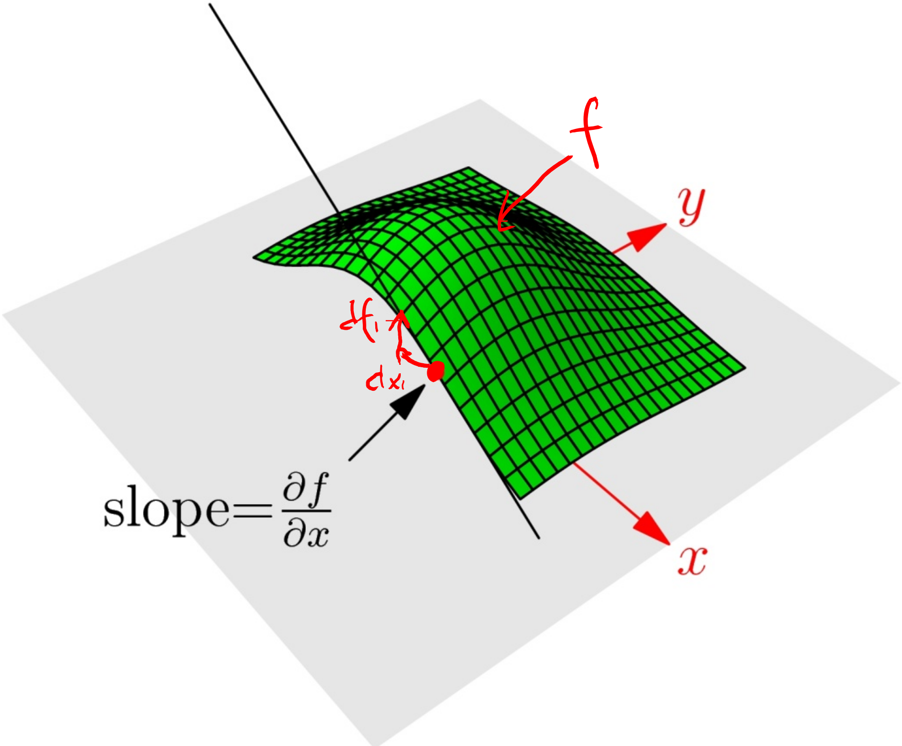


- The differentials df and dx can be thought of as changes in f and x taken *in the limit that both become very small*

- The differentials are not “proper numbers” and only really make sense when they appear in an integral.

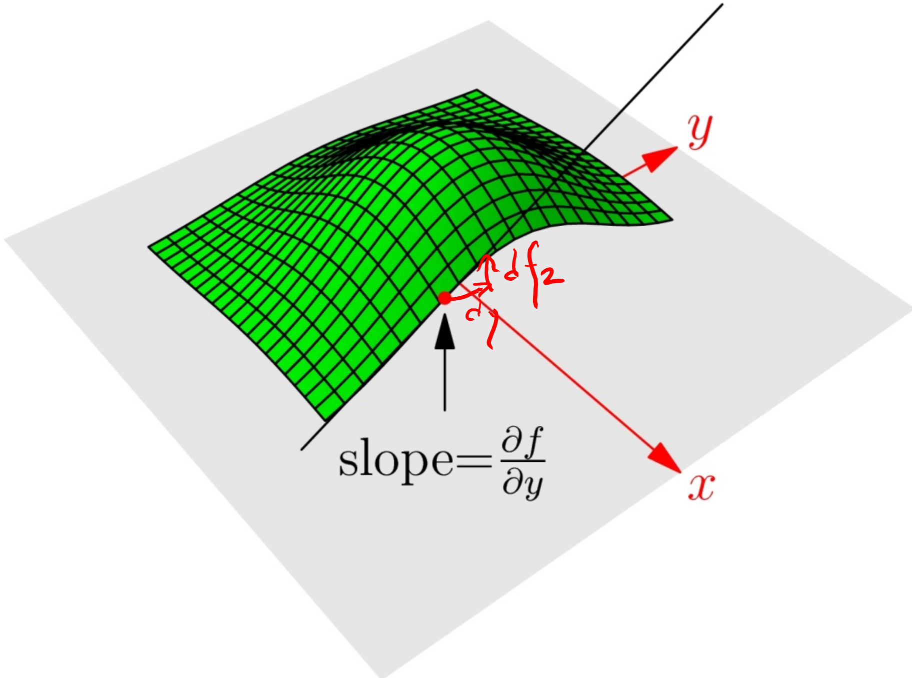
- They are very useful for *changing coordinates.*

Can we do the same thing for functions of two variables?



For a small step dx in the x direction
the function changes by

$$df_1 = \frac{\partial f}{\partial x} dx$$



For a small step dy in the y direction
the function changes by

$$df_2 = \frac{\partial f}{\partial y} dy$$

Therefore for a change in position
in *both* dx *and* dy the function changes by

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

Definition:

For a differentiable function $f\,(x,y)$, we define the *total differential* df as

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

where the quantities dx and dy represent infinitessimal changes in x and y.

In 3D, the differential is:

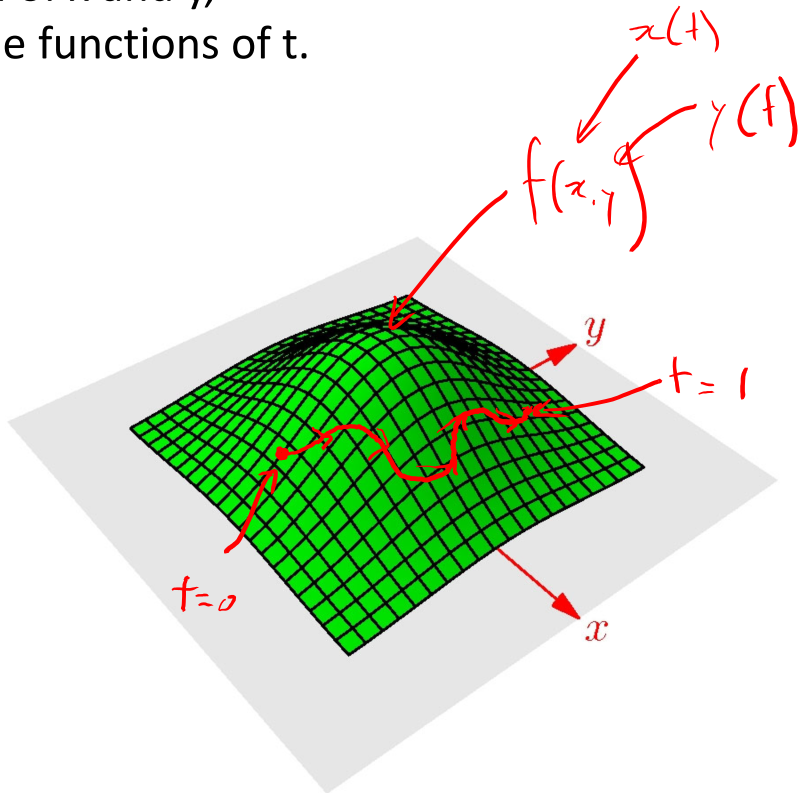
$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

$$= \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle \cdot \langle dx, dy, dz \rangle$$

$$= \nabla f \cdot d\underset{\sim}{r}$$

The chain rule with one independent variable

Suppose that $z = f(x,y)$ is a differentiable function of x and y, and that $x = x(t)$ and $y = y(t)$ are both differentiable functions of t.



From the differential

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

we obtain the *chain rule*

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

(x,y) are called *intermediate variables.*

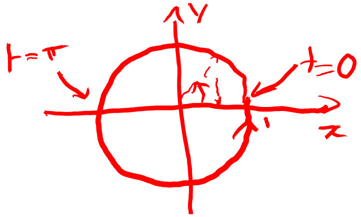
t is the only *independent variable.*

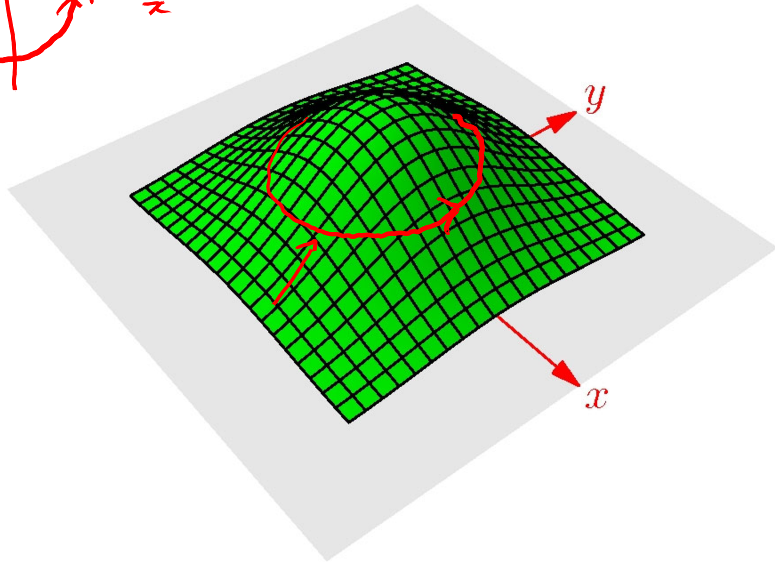
Note that df/dt is a ***full derivative*** because f is a function only of t.

Example: Let

$$z = f(x,y) = e^{-x^2-y^2}$$

and let x(t) = cos t, y(t) = sin t. Compute dz/dt.





$$\frac{dz}{dt} = \frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$$= \left(-2x e^{-x^2-y^2}\right)\frac{d}{dt}(\cos t) + \left(-2y e^{-x^2-y^2}\right)\frac{d}{dt}(\sin t)$$

$$= +2x e^{-x^2-y^2}\sin t - 2y e^{-x^2-y^2}\cos t$$

$$= 2\cos t \sin t\, e^{-1} - 2\sin t \cos t\, e^{-1}$$

$$= 0$$

We can remember the chain rule using a tree diagram:

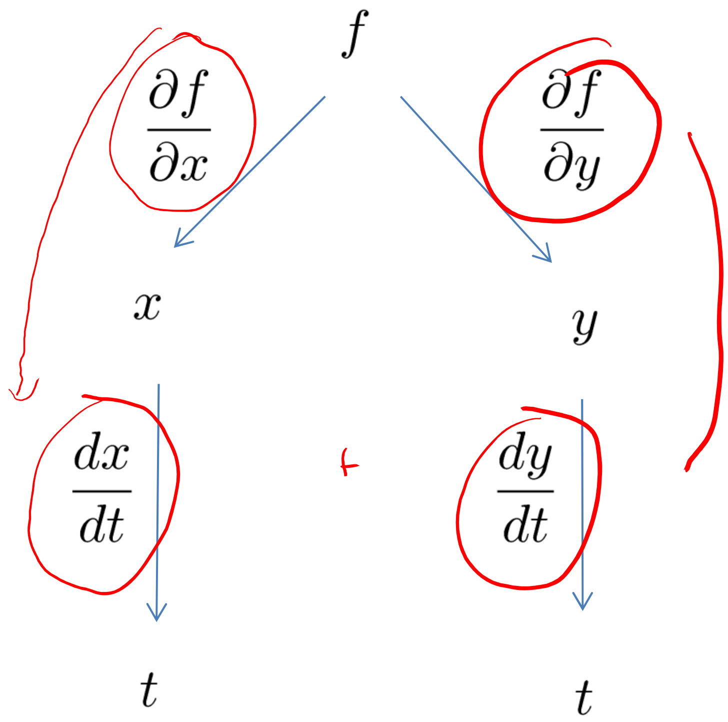
f

is a function of

x and y

which are functions of

t

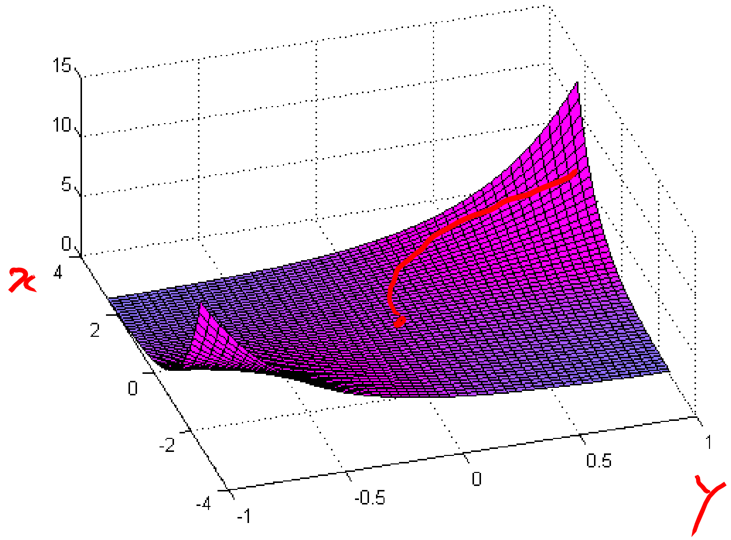


$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

Example: Let

$$z = f(x, y) = e^{xy}$$

and let x(t) = t^2, y(t) = t^3. Compute dz/dt.



$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$$= ye^{xy}2t + xe^{xy}3t^2$$

$$= t^3e^{t^5} \times 2t + t^2e^{t^5} \times 3t^2$$

$$= 2t^4e^{t^5} + 3t^4e^{t^5}$$

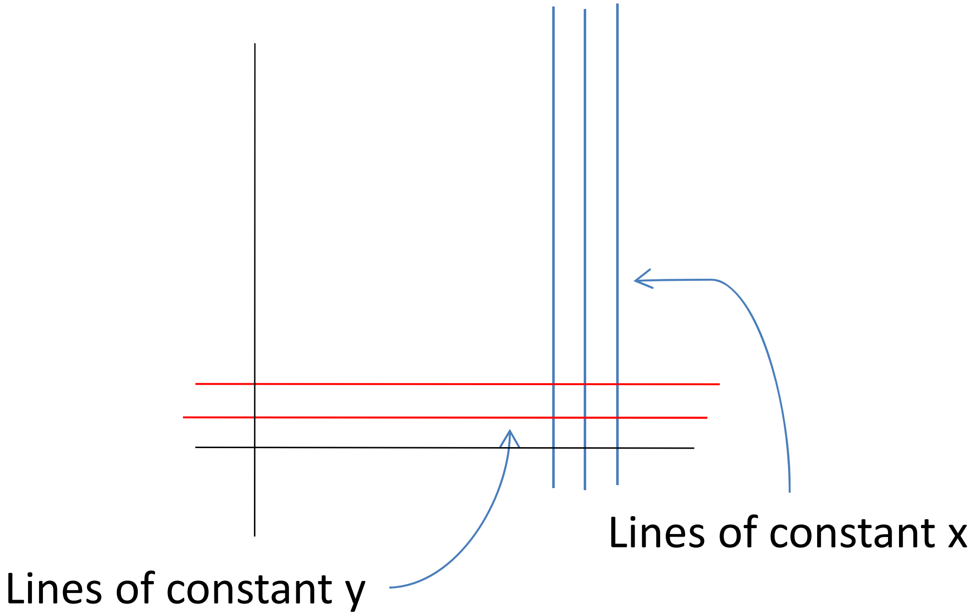
$$= 5t^4e^{t^5}$$

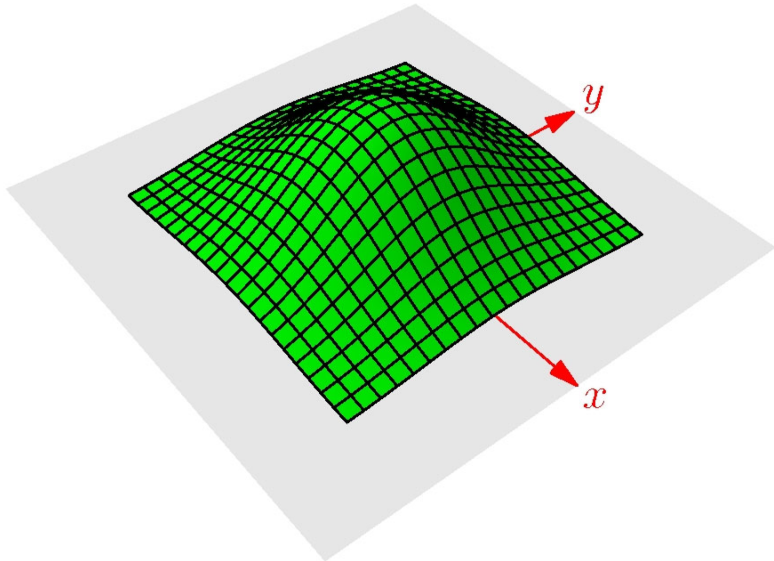
Ans: $(2+3t)t^3e^{t^5}$

Changing coordinate systems

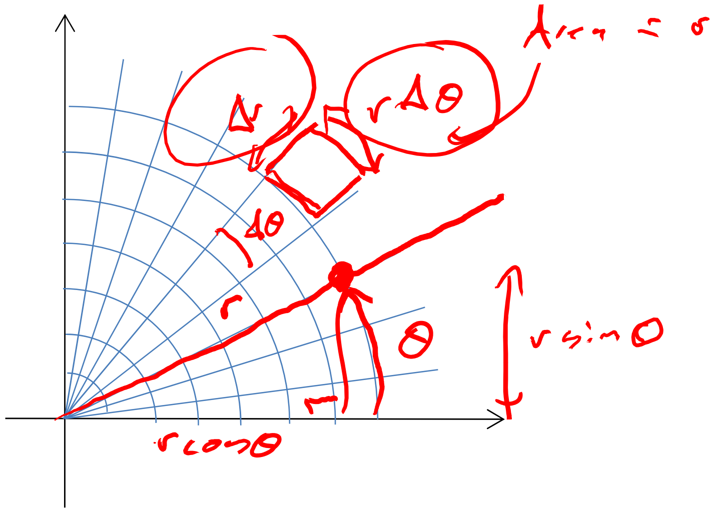
We usually represent a point in two dimensions by an ordered pair (x,y)

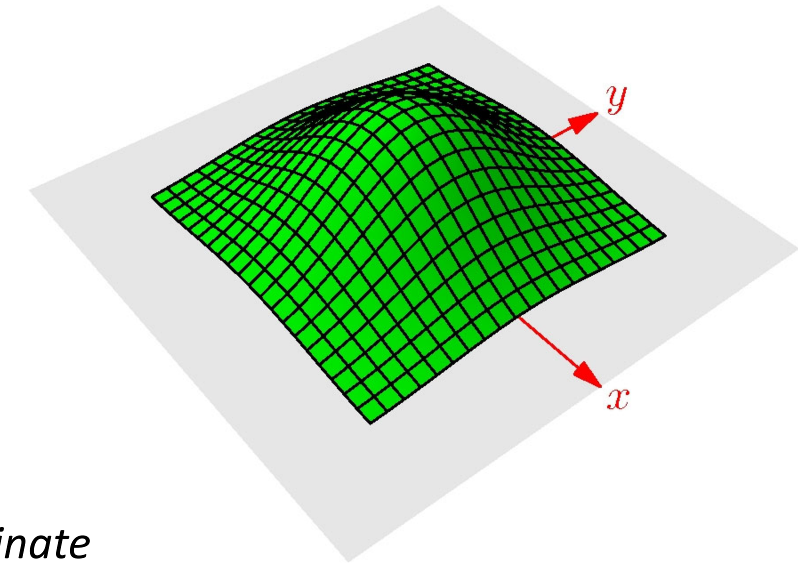
Cartesian Coordinates





We can represent *the same point* using polar coordinates:





These two coordinate systems are related by *coordinate transformations*

Polar to Cartesian:

$$x = r\cos\theta$$

$$y = r\sin\theta$$

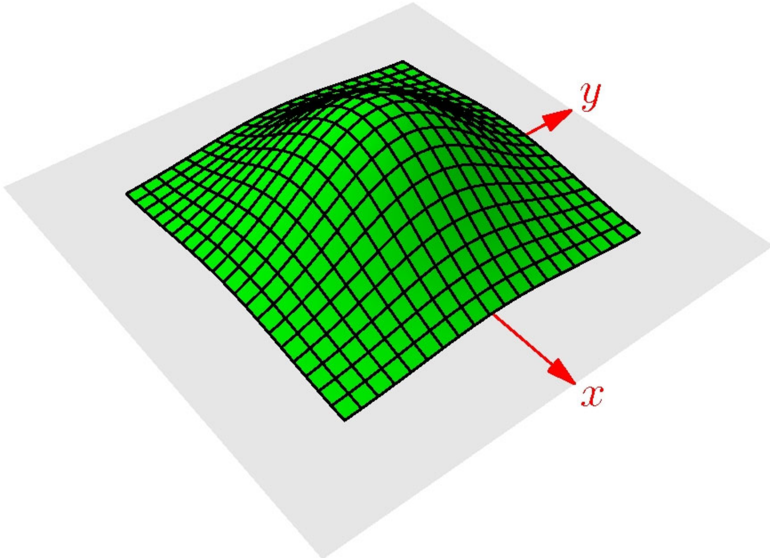
Cartesian to Polar:

$$r = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1}\frac{y}{x}$$

A function in one coordinate system can be changed to another by substituting:

$$\begin{aligned} f_{\text{cart}}(x,y) &= f(r\cos\theta, r\sin\theta) \\ &= f_{\text{pol}}(r,\theta) \end{aligned}$$



e.g.

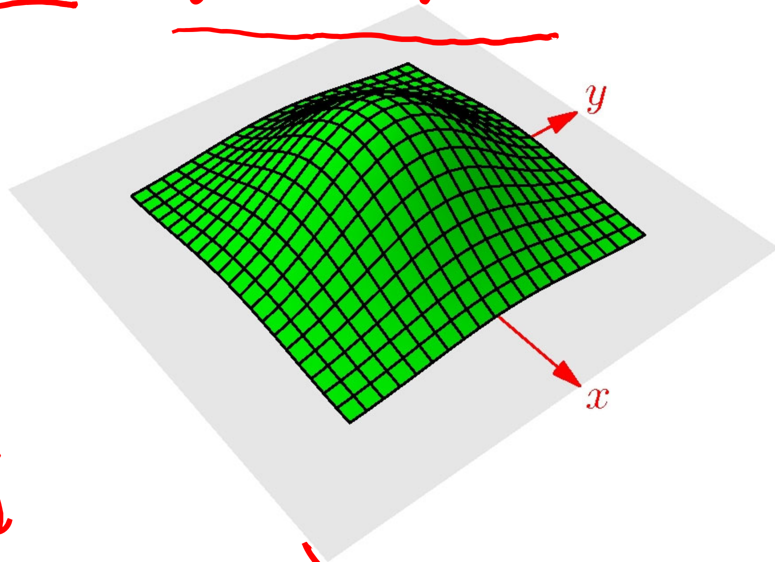
$$f(x,y) = \frac{1}{x^2+y^2}$$

$$= \frac{1}{r^2} = f(r,\theta)$$

How do derivatives transform? That is, if we know the slope of f in x and y, Can we work out the slope in r and θ?

We now find expressions for $\frac{\partial f}{\partial r}$ and $\frac{\partial f}{\partial \theta}$.

Start with the *differential* of *f*: $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = \frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial \theta}d\theta$



Now x and y are *also* functions of r and θ, so:

$$dx = \frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial \theta}d\theta$$

$$dy = \frac{\partial y}{\partial r}dr + \frac{\partial y}{\partial \theta}d\theta$$

$$df = \frac{\partial f}{\partial x}\left(\frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial \theta}d\theta\right) + \frac{\partial f}{\partial y}\left(\frac{\partial y}{\partial r}dr + \frac{\partial y}{\partial \theta}d\theta\right)$$

$$= \underbrace{\left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r}\right)}_{\frac{\partial f}{\partial r}}dr + \underbrace{\left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta}\right)}_{\frac{\partial f}{\partial \theta}}d\theta$$

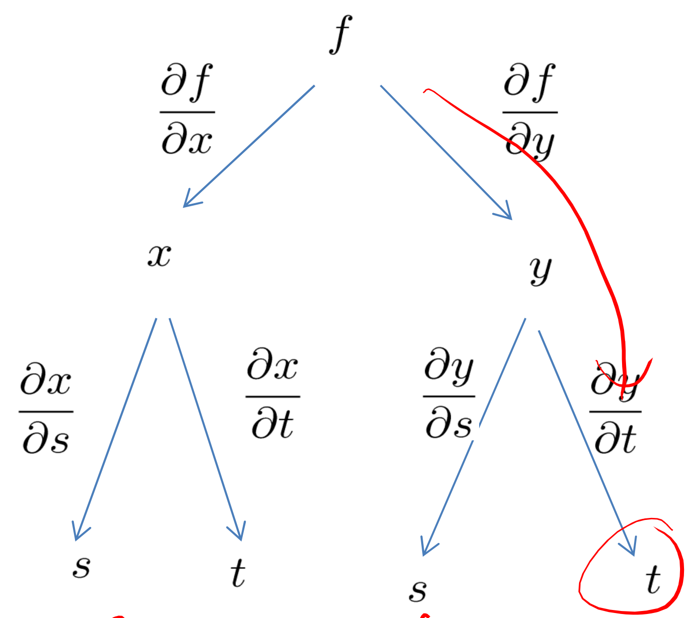
This is the *chain rule* for functions of two variables:

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r}$$
$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta}$$

How to remember it:

1. It's like the chain rule for 1D, *but you have to add an additional term because it's a function of two variables*

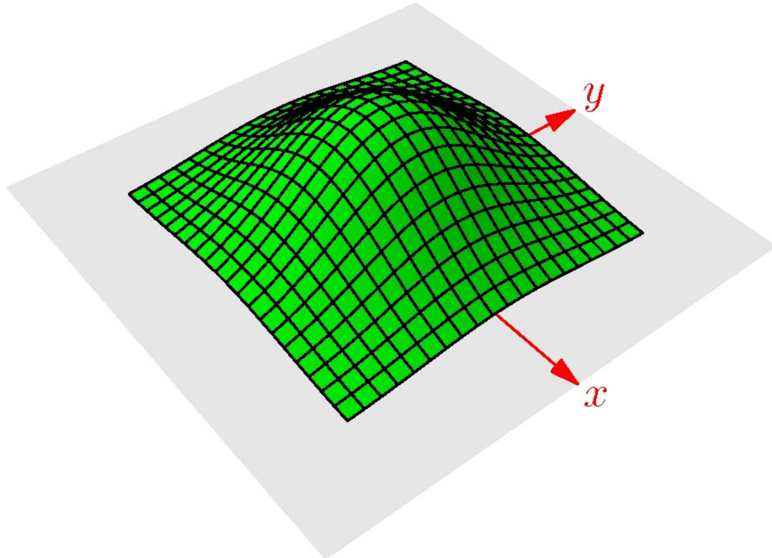
2. Graphically:



$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}.$$

Example 1:

For the function $f(x,y) = e^{-x^2-y^2}$
compute ∂ f/ ∂ r, where x = r cos „, y = r sin „



Example 2: For the function $f(r, \theta) = 1/r$, find $\partial f/\partial x$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial x}.$$

$= 0$ because $\partial f/\partial \theta = 0$.

$$= \frac{-1}{r^2} \cdot \frac{\partial r}{\partial x}$$

But

$$r = \sqrt{x^2+y^2} = (x^2+y^2)^{\frac{1}{2}}$$

$$\frac{\partial r}{\partial x} = \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}} \times \frac{\partial}{\partial x}(x^2)$$

$$= \frac{x}{\sqrt{x^2+y^2}} = \frac{r\cos\theta}{r} = \cos\theta.$$

$$\frac{\partial f}{\partial x} = \frac{-1}{r^2}\cos\theta.$$

$$\theta = \tan^{-1}\frac{y}{x}$$

$$\tan\theta = \frac{y}{x}$$

$$\Rightarrow \frac{\partial}{\partial x}\tan\theta = \frac{-y}{x^2}$$

$$\frac{\partial}{\partial \theta}\tan\theta \cdot \frac{\partial \theta}{\partial x} = \frac{-y}{x^2}.$$

$$\sec^2\theta \frac{\partial \theta}{\partial x} = \frac{-y}{x^2}$$

$$\Rightarrow \frac{\partial \theta}{\partial x} = \frac{-y}{x^2}\cos^2\theta = \frac{-r\sin\theta\cos^2\theta}{r^2\cos^2\theta}$$

$$= \frac{-1}{r}\sin\theta.$$

(c) (i) Write out the tree diagram for the chain rule for

$$g = f(x, y)$$

where

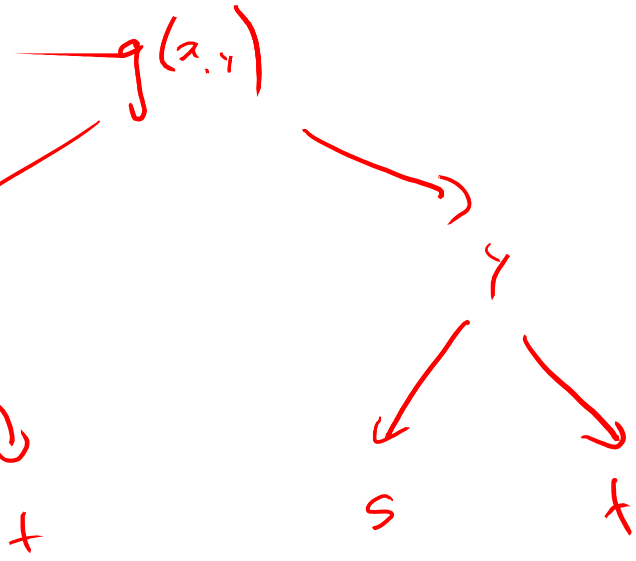
$$x = x(s, t)$$
$$y = y(s, t)$$

(ii) Use the Chain to calculate the $\partial f/\partial s$ and $\partial f/\partial t$ for the function

$$f(x, y) = (x + y)^4$$

where

$$x(s, t) = s^3 t$$
$$y(s, t) = st^3$$



$$\Rightarrow \quad \frac{\partial g}{\partial s} = \frac{\partial g}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial g}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial g}{\partial t} = \frac{\partial g}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial g}{\partial y}\frac{\partial y}{\partial t}.$$

(ii) Use the Chain to calculate the $\partial f/\partial s$ and $\partial f/\partial t$ for the function

$$f(x,y) = (x+y)^4$$

where

$$x(s,t) = s^3 t$$
$$y(s,t) = st^3$$

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$

$$= 4(x+y)^3 3ts^2 + 4(x+y)^3 t^3$$

$$= 12(s^3t + t^3s)^3 ts^2 + 4t^3(s^3t + st^3)^3.$$

$$\frac{\partial f}{\partial t} = 4s^3(s^3t + st^3)^3 + 12st^2(s^3t + st^3)^3.$$

General formulation of the chain rule:

Suppose that u is a differentiable function of the n variables $x_1, x_2, x_3, \ldots, x_n$
And each x_j is a differentiable function of the m variables $t_1, t_2, \ldots t_m$.

Then the derivative of u with respect to each of the t_i variables is

$$\frac{\partial f}{\partial t_i} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial f}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

Note that we can write this in matrix form:

$$\begin{bmatrix} \frac{\partial f}{\partial t_1} \\ \frac{\partial f}{\partial t_2} \\ \\ \\ \frac{\partial f}{\partial t_m} \end{bmatrix} = \begin{bmatrix} \frac{\partial x_1}{\partial t_1} & \frac{\partial x_2}{\partial t_2} & \cdots \\ \\ \\ \\ \\ \end{bmatrix} \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \\ \\ \frac{\partial f}{\partial x_n} \end{bmatrix}$$

Derivatives and differentials transform according to the following rules, which become important in higher-level physics:

$$dr = \frac{dr}{dx}dx + \frac{dr}{dy}dy$$

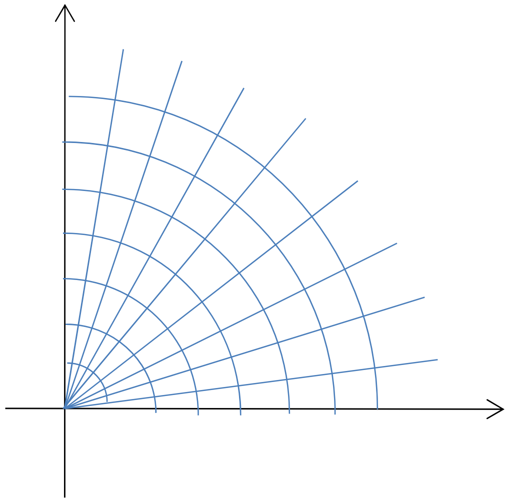
Contravariant transformation

$$\frac{\partial}{\partial r} = \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y}$$

Covariant transformation

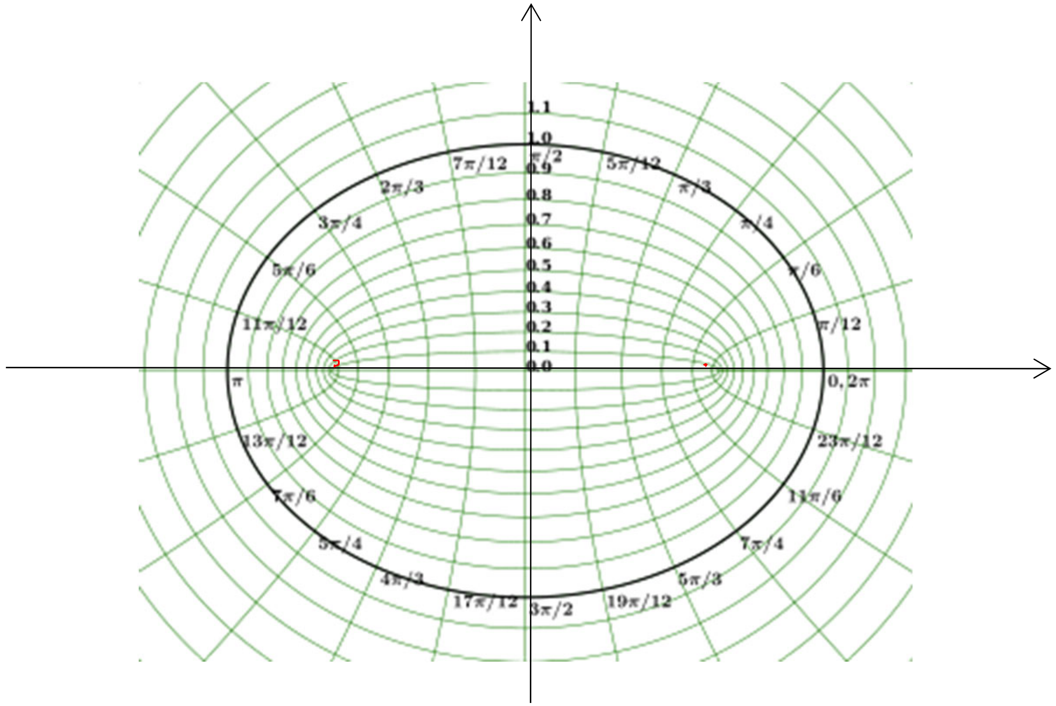
Different Coordinate systems

The Cartesian and polar are the most important systems in 2D, but there are an infinite number of possibilities:



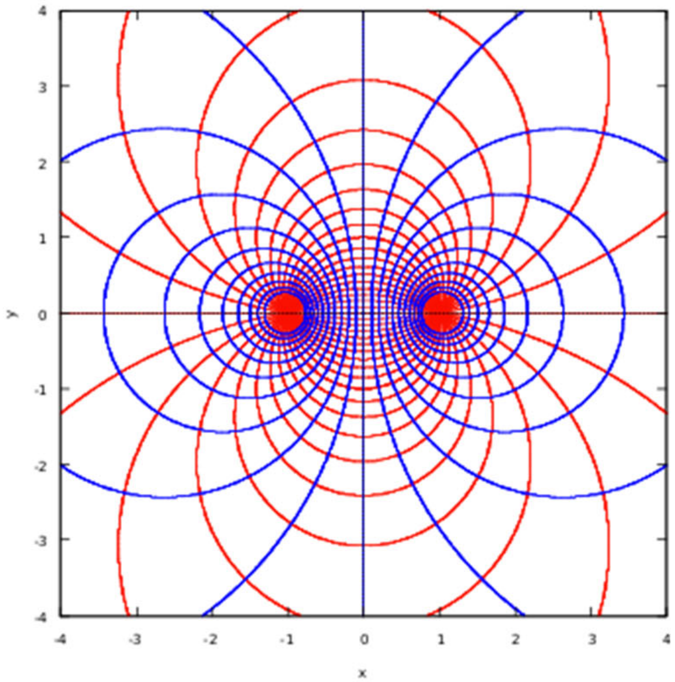
Polar coordinates

$$x = r\cos\theta$$
$$y = r\sin\theta$$



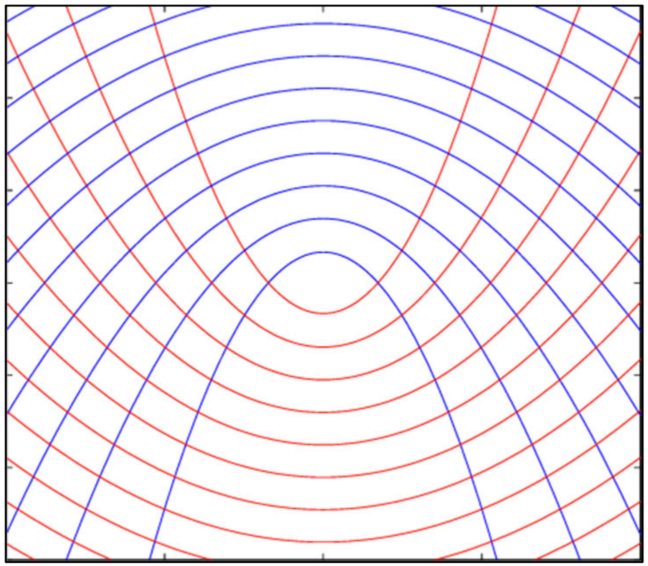
Elliptic coordinates

$$x = a\cosh\mu\cos\nu$$
$$y = a\sinh\mu\sin\nu$$



Bipolar coordinates

$$x = a\,\frac{\sinh\tau}{\cosh\tau - \cos\sigma}, \qquad y = a\,\frac{\sin\sigma}{\cosh\tau - \cos\sigma}.$$



Parabolic coordinates

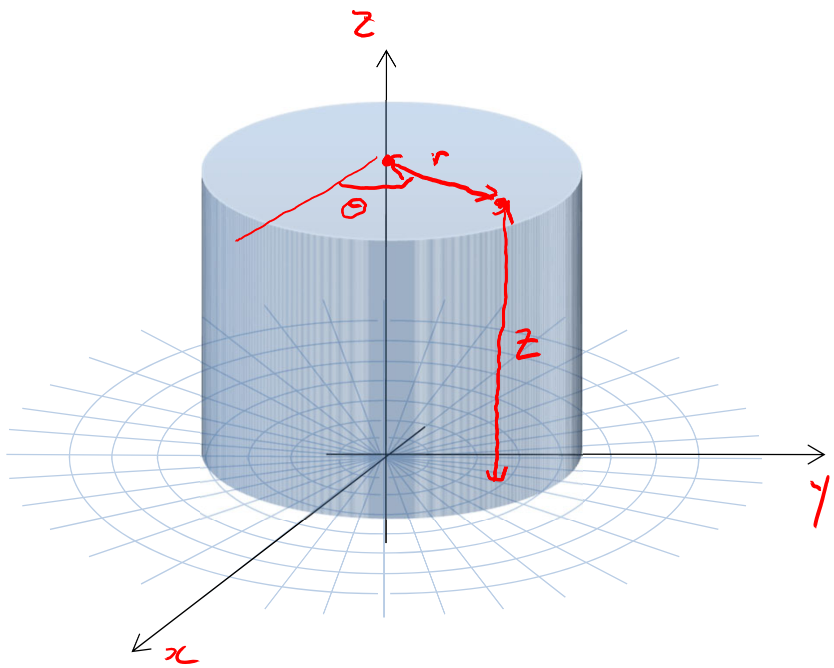
$$\begin{aligned} x &= \sigma\tau \\ y &= \frac{1}{2}\left(\tau^2 - \sigma^2\right) \end{aligned}$$

Coordinate systems in three dimensions

The most important coordinate systems in 3D are the Cartesian, the Cylindrical and the Spherical polar coordinates:

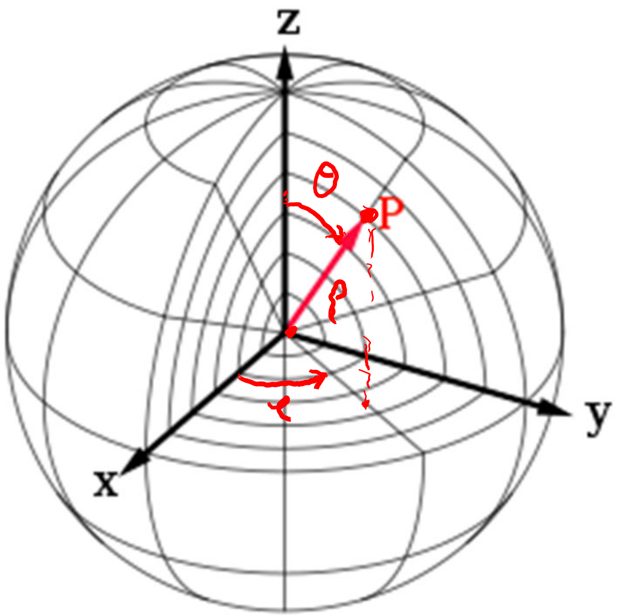
Cylindrical coordinates:

$$(r, \theta, z)$$



$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}$$

Spherical coordinates:



$$(\rho, \theta, \varphi)$$

$\equiv \phi$

$$
\begin{aligned}
x &= \rho \cos\varphi \sin\theta \\
y &= \rho \sin\varphi \sin\theta \\
z &= \rho \cos\theta
\end{aligned}
$$

The inverse transformations are more complicated
(keyword search: "spherical coordinates"); the important one is Pythagorus' theorem

$$\rho = \sqrt{x^2 + y^2 + z^2}$$

Example: Convert the function

$$f(x,y,z) = e^{-x^2-y^2+iz}$$

into a) cylindrical, and b) spherical coordinates.

a) $x = r\cos\theta$

$y = r\sin\theta$

$z = z$

$$\Rightarrow f = e^{-r^2\cos^2\theta - r^2\sin^2\theta + iz}$$

$$= e^{-r^2+iz}.$$

b) $x^2+y^2 = (\rho\cos\varphi\sin\theta)^2 + (\rho\sin\varphi\sin\theta)^2$

$= \rho^2\cos^2\varphi\sin^2\theta + \rho^2\sin^2\varphi\sin^2\theta$

$= \rho^2\sin^2\theta(\cos^2\varphi + \sin^2\varphi)$

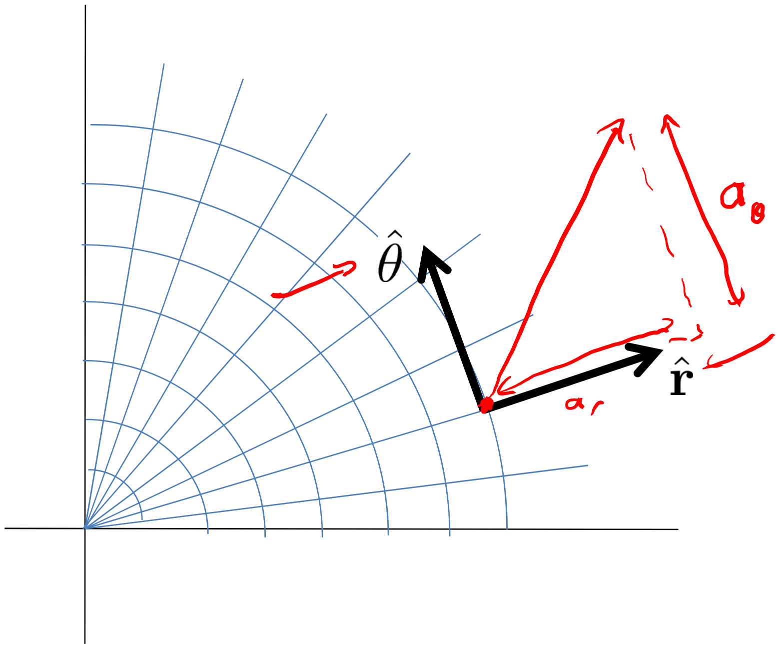
$= \rho^2\sin^2\theta$

So, $f(\rho,\varphi,\theta) = e^{-\rho^2\sin^2\theta + i\rho\cos\theta}$.

$$\begin{aligned} x &= \rho\cos\varphi\sin\theta \\ y &= \rho\sin\varphi\sin\theta \\ z &= \rho\cos\theta \end{aligned}$$

Changing coordinates of vectors

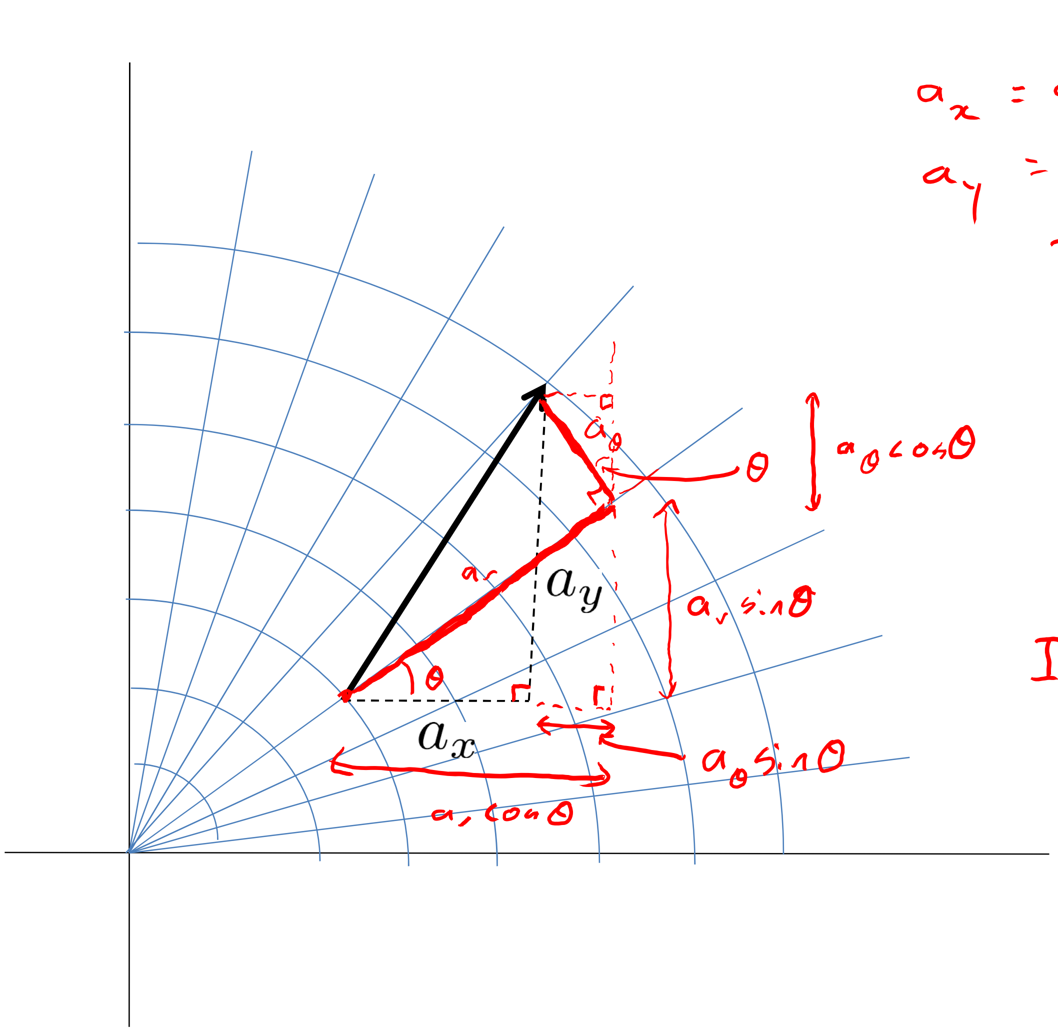
We can define unit vectors for polar coordinates: $\hat{\mathbf{r}}$ and $\hat{\theta}$



Any vector $\mathbf{a}$ can be expanded in terms of $\hat{\mathbf{r}}$ and $\hat{\theta}$: $\underset{\sim}{a} = a_r \hat{\underset{\sim}{r}} + a_\theta \hat{\underset{\sim}{\theta}}$

Note that these unit vectors *depend on position*.

To convert a vector from Polar to Cartesian coordinates we use Trigonometry:



$$a_x = a_r \cos\theta - a_\theta \sin\theta$$

$$a_y = a_r \sin\theta + a_\theta \cos\theta$$

In matrix form:

$$\begin{pmatrix} a_x \\ a_y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a_r \\ a_\theta \end{pmatrix}$$

Inverting:

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}^{-1} \begin{pmatrix} a_x \\ a_y \end{pmatrix} = \begin{pmatrix} a_r \\ a_\theta \end{pmatrix}$$

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a_x \\ a_y \end{pmatrix} = \begin{pmatrix} a_r \\ a_\theta \end{pmatrix}.$$

In Cylindrical Coordinates, the transformations of vectors follow from 2D:

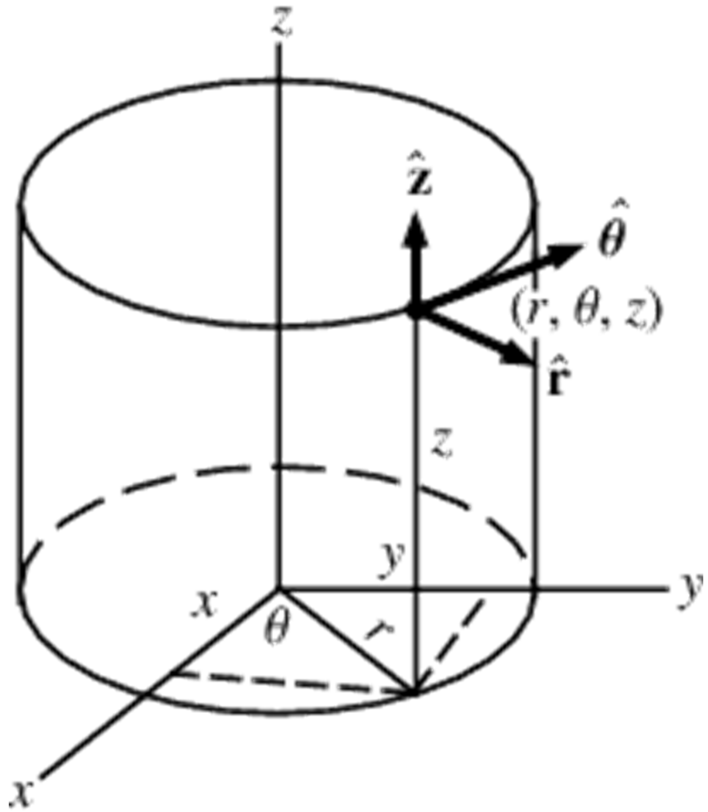
New coordinates:

$$(r, \theta, z)$$

New unit vectors:

$$\hat{\mathbf{r}},\ \hat{\theta},\ \text{and}\ \hat{\mathbf{z}}$$

$$\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_r \\ v_\theta \\ v_z \end{pmatrix}$$



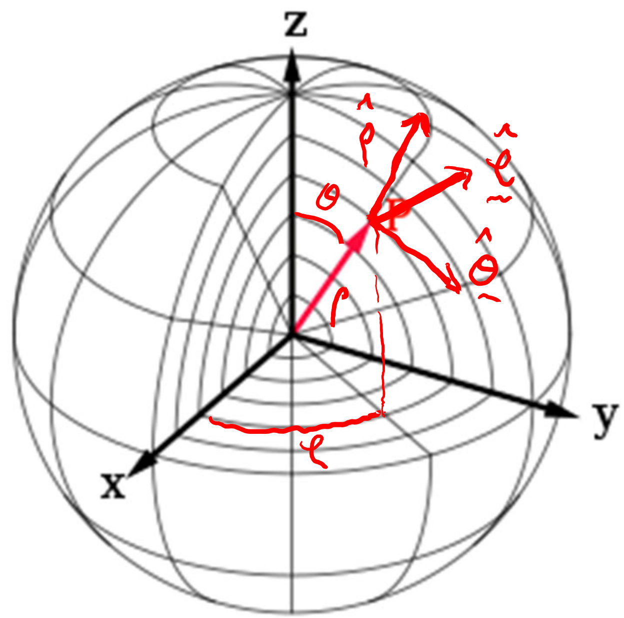
In Spherical Coordinates, the transformations get more complicated:

New coordinates:

$$(r, \theta, \varphi)$$

New unit vectors:

$$\hat{\mathbf{r}},\ \ \hat{\theta},\ \ \text{and}\ \ \hat{\varphi}$$



$$\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\varphi & \cos\theta\cos\varphi & -\sin\varphi \\ \sin\theta\sin\varphi & \cos\theta\sin\varphi & \cos\varphi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} v_\rho \\ v_\varphi \\ v_\theta \end{bmatrix}$$

3d rotation matrix

Divergence, Gradient and Curl in different coordinate systems

The form of the gradient, divergence and curl *changes* in different coordinate systems

In cylindrical cords the gradient is:

$$\nabla f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{\partial f}{\partial z}\hat{\mathbf{z}}$$

$$\underset{\sim}{A} = A_r \hat{r} + A_\theta \hat{\theta} + A_\phi \hat{\phi}.$$

The *divergence* is:

$$\nabla \cdot \mathbf{A} = \frac{1}{r}\frac{\partial (rA_r)}{\partial r} + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z}$$

And the *curl* is:

$$\nabla \times \mathbf{A} = \left(\frac{1}{r}\frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z}\right)\hat{\mathbf{r}} + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\hat{\theta} + \frac{1}{r}\left(\frac{\partial (rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta}\right)\hat{\mathbf{z}}$$

We can derive these using the *transformations of partial derivatives* that we did a few slides ago.

In spherical cords the *gradient* is:

$$\nabla f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \varphi}\hat{\boldsymbol{\varphi}}$$

The *divergence* is

$$\nabla \cdot \underset{\sim}{A} = \frac{1}{r^2}\frac{\partial\left(r^2 A_r\right)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(A_\theta \sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial\varphi}$$

And the *curl* is:

$$\nabla \times \underset{\sim}{A} = \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}\left(A_\varphi\sin\theta\right) - \frac{\partial A_\theta}{\partial\varphi}\right)\hat{\mathbf{r}} + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\varphi} - \frac{\partial}{\partial r}\left(rA_\varphi\right)\right)\hat{\boldsymbol{\theta}} + \frac{1}{r}\left(\frac{\partial}{\partial r}\left(rA_\theta\right) - \frac{\partial A_r}{\partial\theta}\right)\hat{\boldsymbol{\varphi}}$$

In this subject you will not be tested on how to derive these, but you will need to know how to use them.

Coordinate transformations of divergence, gradient and curl
are usually tabulated for you (here the internet is particularly useful):

Del formula [edit]

Table with the del operator in cartesian, cylindrical and spherical coordinates

Operation	Cartesian coordinates (x, y, z)	Cylindrical coordinates (ρ, φ, z)	Spherical coordinates (r, θ, φ), where θ is the polar angle and φ is the azimuthal angle[α]
Vector field $\mathbf{A}$	$A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}$	$A_\rho\hat{\boldsymbol{\rho}} + A_\varphi\hat{\boldsymbol{\varphi}} + A_z\hat{\mathbf{z}}$	$A_r\hat{\mathbf{r}} + A_\theta\hat{\boldsymbol{\theta}} + A_\varphi\hat{\boldsymbol{\varphi}}$
Gradient ∇f[1]	$\frac{\partial f}{\partial x}\hat{\mathbf{x}} + \frac{\partial f}{\partial y}\hat{\mathbf{y}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}}$	$\frac{\partial f}{\partial \rho}\hat{\boldsymbol{\rho}} + \frac{1}{\rho}\frac{\partial f}{\partial \varphi}\hat{\boldsymbol{\varphi}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}}$	$\frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \varphi}\hat{\boldsymbol{\varphi}}$
Divergence $\nabla \cdot \mathbf{A}$[1]	$\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$	$\frac{1}{\rho}\frac{\partial(\rho A_\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial A_\varphi}{\partial \varphi} + \frac{\partial A_z}{\partial z}$	$\frac{1}{r^2}\frac{\partial(r^2 A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial\varphi}$
Curl $\nabla \times \mathbf{A}$[1]	$\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{\mathbf{z}}$	$\left(\frac{1}{\rho}\frac{\partial A_z}{\partial \varphi} - \frac{\partial A_\varphi}{\partial z}\right)\hat{\boldsymbol{\rho}} + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right)\hat{\boldsymbol{\varphi}} + \frac{1}{\rho}\left(\frac{\partial(\rho A_\varphi)}{\partial \rho} - \frac{\partial A_\rho}{\partial \varphi}\right)\hat{\mathbf{z}}$	$\frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(A_\varphi\sin\theta) - \frac{\partial A_\theta}{\partial\varphi}\right)\hat{\mathbf{r}} + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\varphi} - \frac{\partial}{\partial r}(rA_\varphi)\right)\hat{\boldsymbol{\theta}} + \frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right)\hat{\boldsymbol{\varphi}}$
Laplace operator $\nabla^2 f \equiv \Delta f$[1]	$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$	$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial f}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial\varphi^2} + \frac{\partial^2 f}{\partial z^2}$	$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\varphi^2}$
Vector Laplacian $\nabla^2\mathbf{A} \equiv \Delta\mathbf{A}$[2]	$\nabla^2 A_x\hat{\mathbf{x}} + \nabla^2 A_y\hat{\mathbf{y}} + \nabla^2 A_z\hat{\mathbf{z}}$	$\left(\nabla^2 A_\rho - \frac{A_\rho}{\rho^2} - \frac{2}{\rho^2}\frac{\partial A_\varphi}{\partial\varphi}\right)\hat{\boldsymbol{\rho}} + \left(\nabla^2 A_\varphi - \frac{A_\varphi}{\rho^2} + \frac{2}{\rho^2}\frac{\partial A_\rho}{\partial\varphi}\right)\hat{\boldsymbol{\varphi}} + \nabla^2 A_z\hat{\mathbf{z}}$	$\left(\nabla^2 A_r - \frac{2A_r}{r^2} - \frac{2}{r^2\sin\theta}\frac{\partial(A_\theta\sin\theta)}{\partial\theta} - \frac{2}{r^2\sin\theta}\frac{\partial A_\varphi}{\partial\varphi}\right)\hat{\mathbf{r}} + \left(\nabla^2 A_\theta - \frac{A_\theta}{r^2\sin^2\theta} + \frac{2}{r^2}\frac{\partial A_r}{\partial\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial A_\varphi}{\partial\varphi}\right)\hat{\boldsymbol{\theta}} + \left(\nabla^2 A_\varphi - \frac{A_\varphi}{r^2\sin^2\theta} + \frac{2}{r^2\sin\theta}\frac{\partial A_r}{\partial\varphi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial A_\theta}{\partial\varphi}\right)\hat{\boldsymbol{\varphi}}$

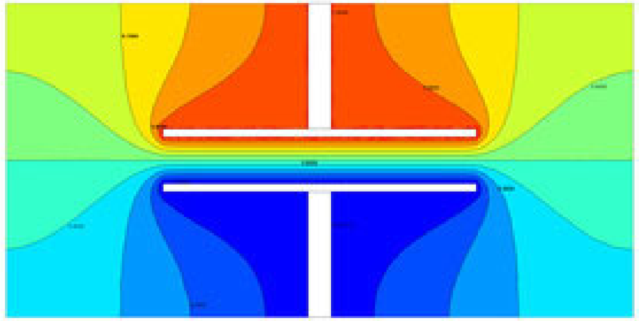
Search term e.g.: "Vector calculus identities spherical coordinates"

Complicated, yet important example:
the Laplacian of f is the quantity

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

In 2D polar coordinates, this transforms to

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial f}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \theta^2}$$



Example: Find the divergence in cylindrical coordinates of the Vector function

$$\mathbf{E} = \frac{\sin\theta}{r^2}\hat{\mathbf{r}} + z\hat{\mathbf{z}}$$

Del formula [edit]

Table with the del operator in cartesian, cylindrical

Operation	Cartesian coordinates (x, y, z)	Cylindrical coordinates (ρ, φ, z)
Vector field $\mathbf{A}$	$A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}$	$A_\rho\hat{\boldsymbol{\rho}} + A_\varphi\hat{\boldsymbol{\varphi}} + A_z\hat{\mathbf{z}}$
Gradient ∇f[1]	$\frac{\partial f}{\partial x}\hat{\mathbf{x}} + \frac{\partial f}{\partial y}\hat{\mathbf{y}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}}$	$\frac{\partial f}{\partial \rho}\hat{\boldsymbol{\rho}} + \frac{1}{\rho}\frac{\partial f}{\partial \varphi}\hat{\boldsymbol{\varphi}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}}$
Divergence $\nabla \cdot \mathbf{A}$[1]	$\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$	$\frac{1}{\rho}\frac{\partial(\rho A_\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial A_\varphi}{\partial \varphi} + \frac{\partial A_z}{\partial z}$
Curl $\nabla \times \mathbf{A}$[1]	$\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{\mathbf{z}}$	$\left(\frac{1}{\rho}\frac{\partial A_z}{\partial \varphi} - \frac{\partial A_\varphi}{\partial z}\right)\hat{\boldsymbol{\rho}} + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right)\hat{\boldsymbol{\varphi}} + \frac{1}{\rho}\left(\frac{\partial(\rho A_\varphi)}{\partial \rho} - \frac{\partial A_\rho}{\partial \varphi}\right)\hat{\mathbf{z}}$
Laplace operator $\nabla^2 f \equiv \Delta f$[1]	$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$	$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2 f}{\partial z^2}$
Vector Laplacian $\nabla^2\mathbf{A} \equiv \Delta\mathbf{A}$[2]	$\nabla^2 A_x\hat{\mathbf{x}} + \nabla^2 A_y\hat{\mathbf{y}} + \nabla^2 A_z\hat{\mathbf{z}}$	$\left(\nabla^2 A_\rho - \frac{A_\rho}{\rho^2} - \frac{2}{\rho^2}\frac{\partial A_\varphi}{\partial \varphi}\right)\hat{\boldsymbol{\rho}} + \left(\nabla^2 A_\varphi - \frac{A_\varphi}{\rho^2} + \frac{2}{\rho^2}\frac{\partial A_\rho}{\partial \varphi}\right)\hat{\boldsymbol{\varphi}} + \nabla^2 A_z\hat{\mathbf{z}}$

$$\nabla \cdot \mathbf{E} = \frac{1}{r}\frac{\partial}{\partial r}(r E_r) + \frac{1}{r}\frac{\partial E_\theta}{\partial \theta} + \frac{\partial E_z}{\partial z}$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\sin\theta}{r^2}\right) + \frac{\partial}{\partial z}z$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\sin\theta}{r}\right) + 1$$

$$= \frac{1}{r}\sin\theta\left(\frac{-1}{r^2}\right) + 1 = \frac{-\sin\theta}{r^3} + 1$$

Exercise: Find the Laplacian ∇^2 in spherical coordinates of the scalar function

$$f(r,\theta,\varphi) = e^{ikr}$$

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right)$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r} e^{ikr}\right)$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\left[ik e^{ikr}\right]\right)$$

$$= \frac{1}{r^2} ik \frac{\partial}{\partial r}\left(r^2 e^{ikr}\right) = \frac{ik}{r^2}\left(r^2 ik e^{ikr} + e^{ikr} 2r\right)$$

$$= -k^2 e^{ikr} + \frac{2ik}{r} e^{ikr}.$$