An integral is extremely useful for computing *aggregate* quantities

Examples:

Average over an area:
$$P_{av} = \frac{1}{A(D)} \int\int_D P(x,y)\mathrm{dx}\ \mathrm{dy}$$

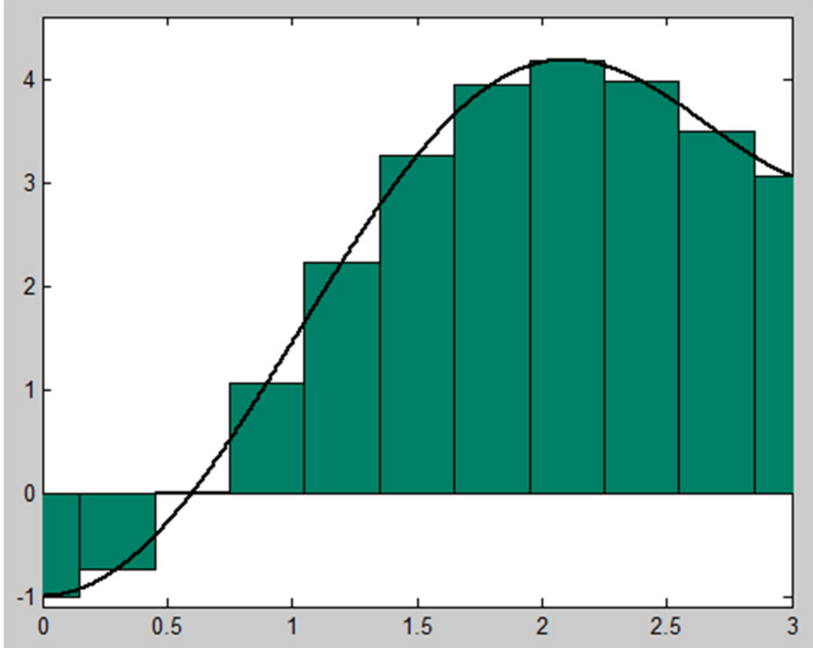
Mass of a volume:
$$M = \int\int\int_V \rho(x,y,z)\mathrm{d}x\ \mathrm{d}y\ \mathrm{d}z$$

Other examples: centre of mass, moment of inertia, total charge, etc.

To compute all these quantities for real applications
we have to be able to integrate in multiple dimensions.

1D definite integrals

We think of a one-dimensional definite integral as the sum of an infinite number of rectangles:



$$\int_a^b f(x)\mathrm{d}x = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$$

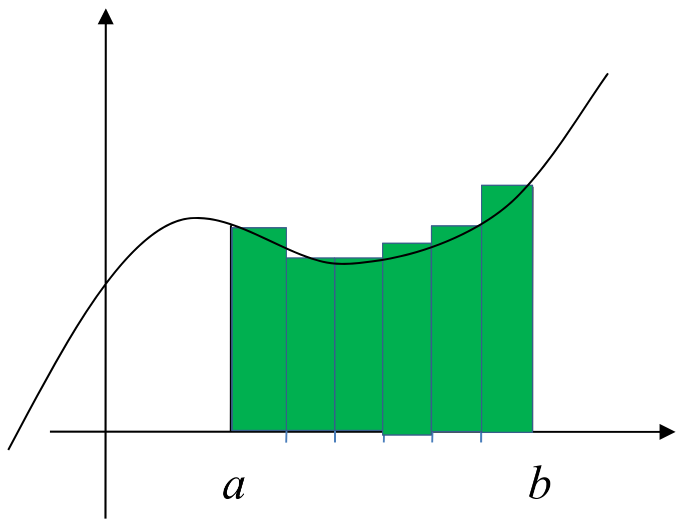
where

$$\Delta x = \frac{b-a}{n} \quad , \quad x_i = a + i\Delta x$$

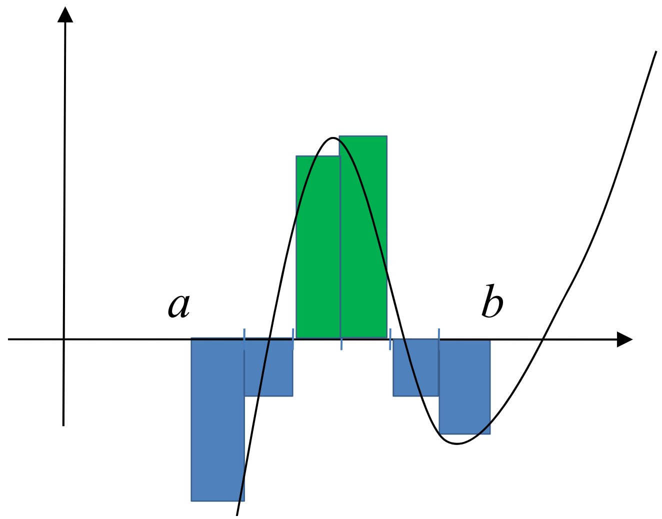
This is known as the *Riemann sum* of the integral.

As the number of rectangles increases, a better and better approximation for the area under the curve is obtained.

NB: The integral is often thought of as the *area* under a graph.



However, integrals can also be *negative* or *zero* (unlike areas).



Double integrals

We can extend this definition to integrals of 2D functions over *rectangular domains* .

$$\int\int_R f(x,y)\mathrm{d}x\mathrm{d}y = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij},y_{ij})\Delta A$$

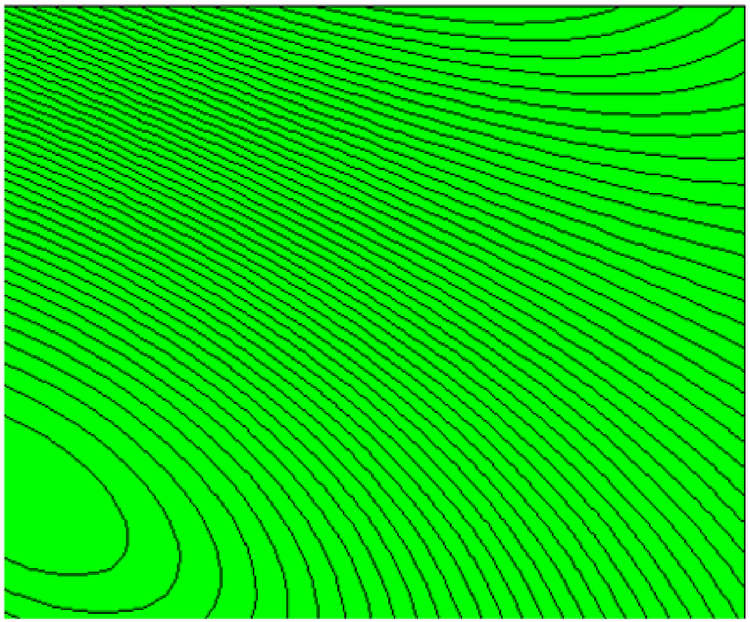
$$\Delta y = \frac{c-d}{n} \quad ,$$

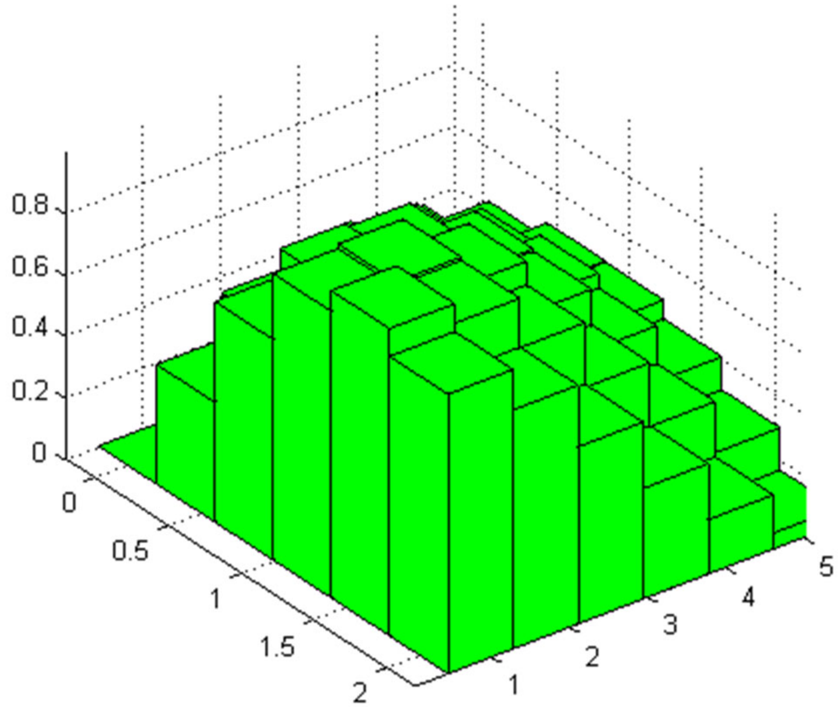
$$\Delta x = \frac{b-a}{n} \quad ,$$

$$x_{ij} = a + i\Delta x$$

$$y_{ij} = c + j\Delta y$$

This time, the integral represents a *signed volume* under the 2D surface.



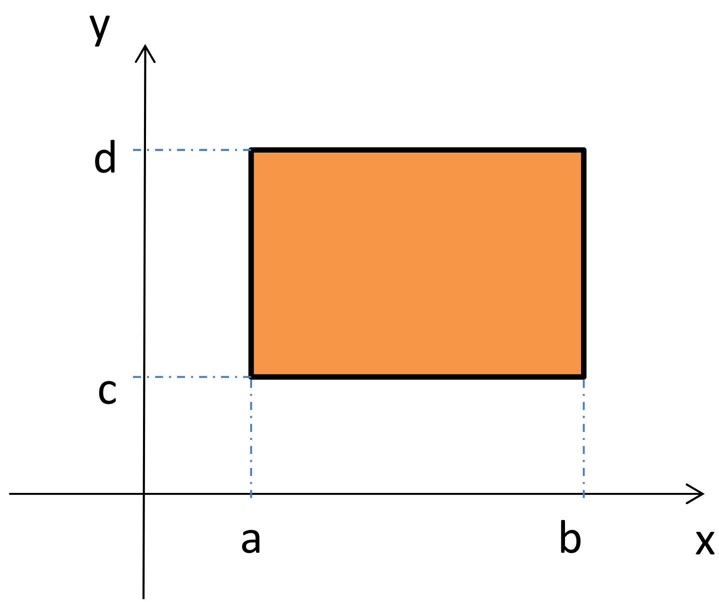


To perform an integral in 2D, we use nested (or *iterated)* integration:

For a rectangular domain, this means that we pick one variable to integrate over first and evaluate this while keeping the other variable constant.

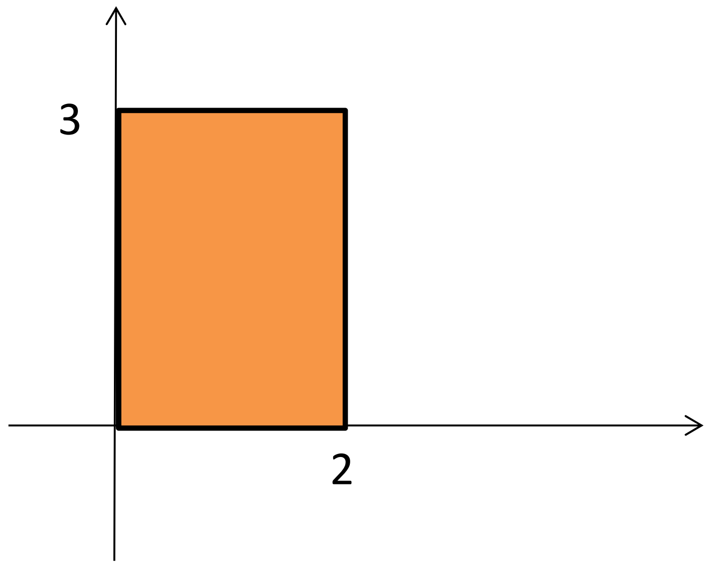
$$\int\int_D f(x,y)\mathrm{d}x\ \mathrm{d}y = \int_a^b \int_c^d f(x,y)\ \mathrm{d}y\ \mathrm{d}x$$

$$= \int_a^b \left[\int_c^d f(x,y)\ \mathrm{d}y \right]\ \mathrm{d}x$$



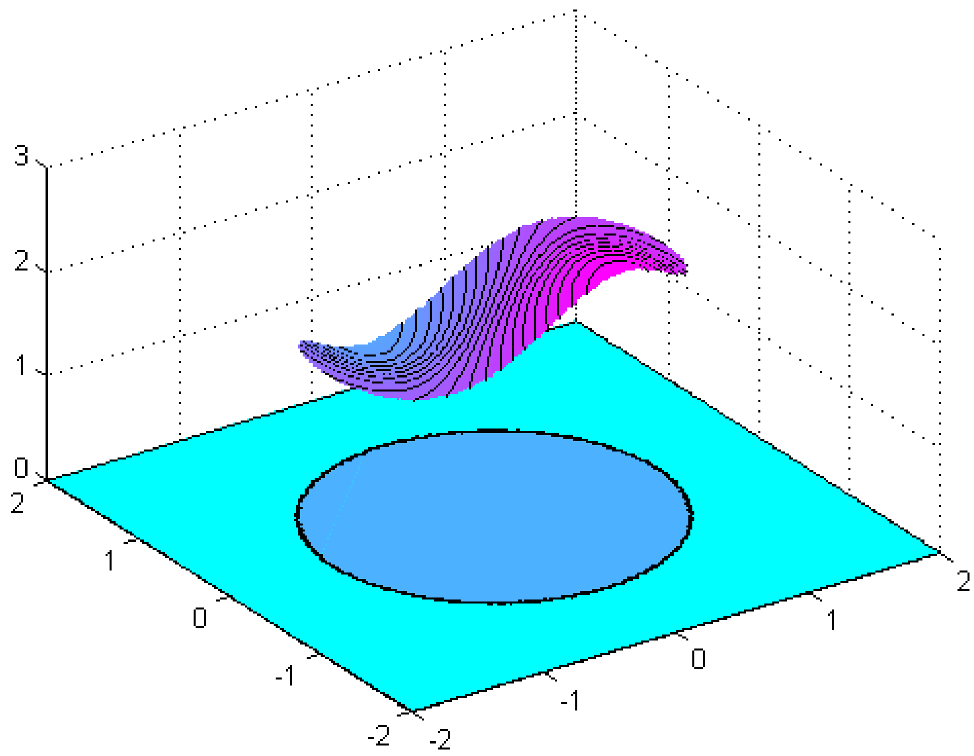
Example: evaluate

$$\int_0^3 \int_0^2 x^2 y \mathrm{d}x \ \mathrm{d}y$$



Ans: 12

Unlike in 1D, the *domain of integration* in 2D can be complicated.



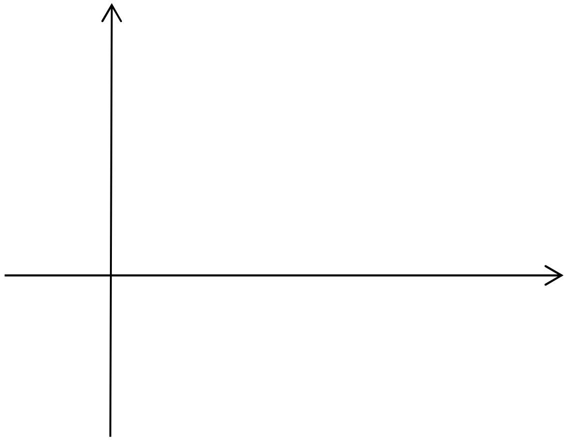
To integrate in 2D, we first have to describe the domain of integration.

The general form is:

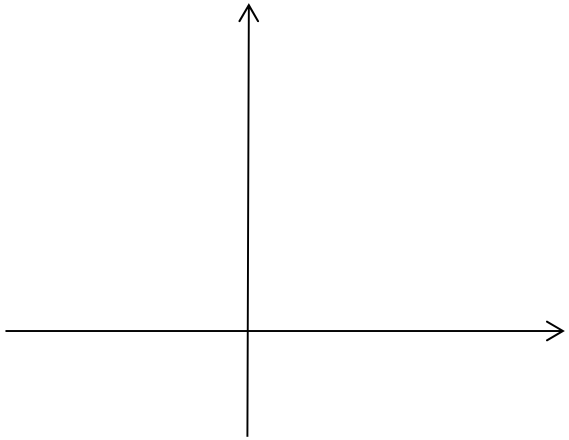
$$D = \{(x, y) | \text{some inequalities involving } x \text{ and } y\}$$

Examples:

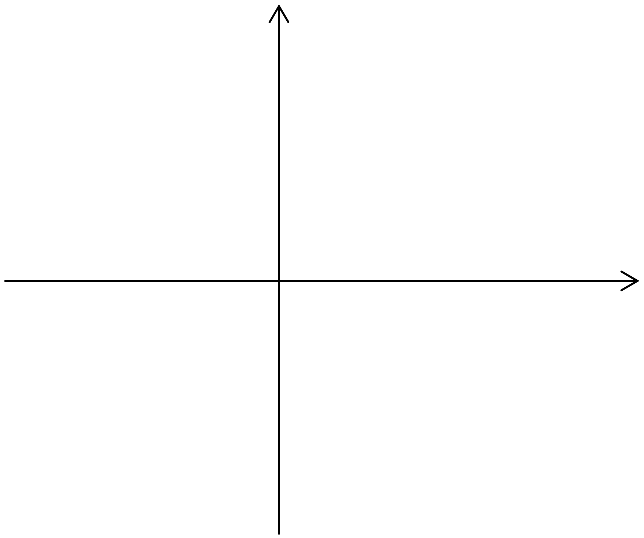
$$D = \{(x, y) | x > 1\ ,\ y \geq -1\}$$



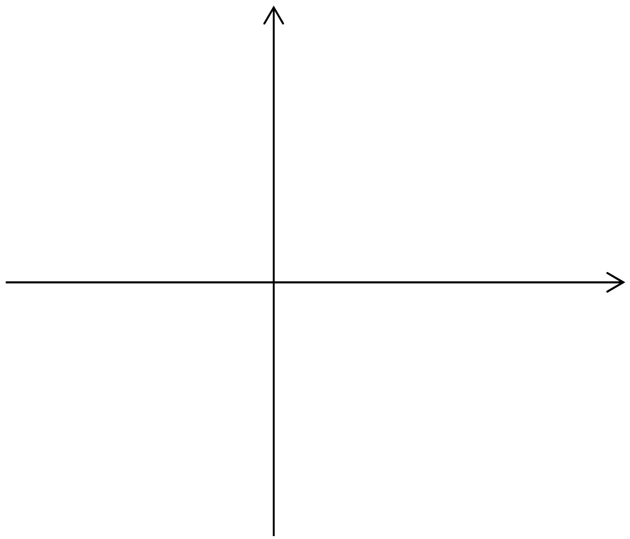
$$D = \{(x, y) | -1 \leq x \leq 2\ ,\ y \leq x^2\}$$



$$D = \{(x, y) | -1 \leq y \leq 1\ ,\ x > y^3\}$$



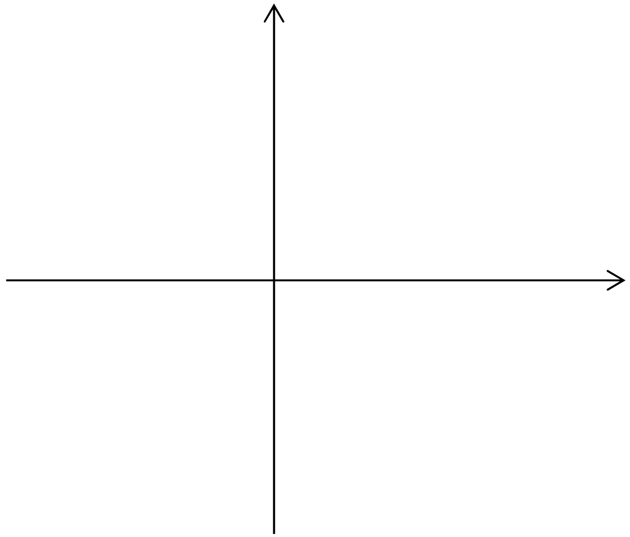
$$D = \{(x, y) | x^2 + y^2 < 4\}$$



Integrating over more complicated domains:

First, write down and draw the domain in 2D, e.g.

$$D = \{(x, y) | -1 \leq x \leq 2 \ , \ 0 \leq y \leq x + 1\}$$



Pick the inequalities and use them
as the limits for you integral:

$$\int\int_D f(x, y) \mathrm{d}x \ \mathrm{d}y = \int_{-1}^{2}\int_{0}^{x+1} f(x, y) \mathrm{d}y \ \mathrm{d}x$$

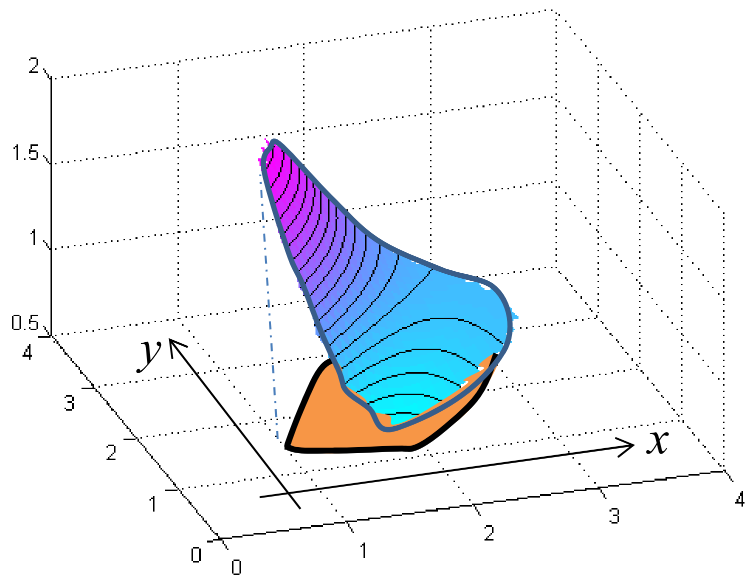
(Important: make sure that the outer limits do not depend on x or y. If this happens, swap the order of integration).

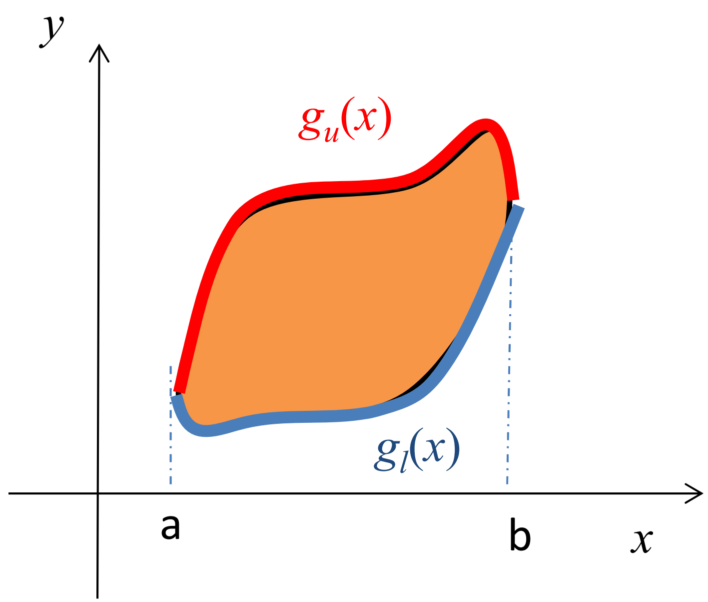
Then integrate, starting with the inner integral.

So far in 2D integration:

Given a function f(x,y) on a domain D,

$$D = \{(x,y) | a \le x \le b \quad , \quad g_\ell(x) \le y \le g_u(x)\}$$



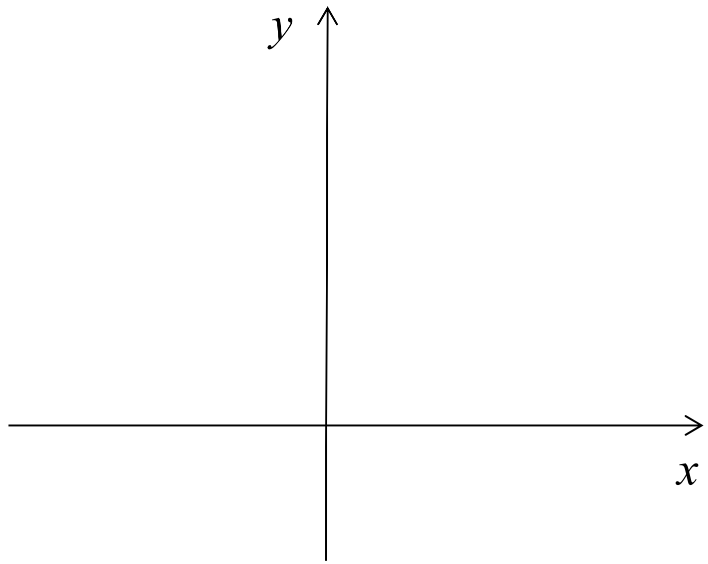


the integral of a function f(x,y) over this domain is

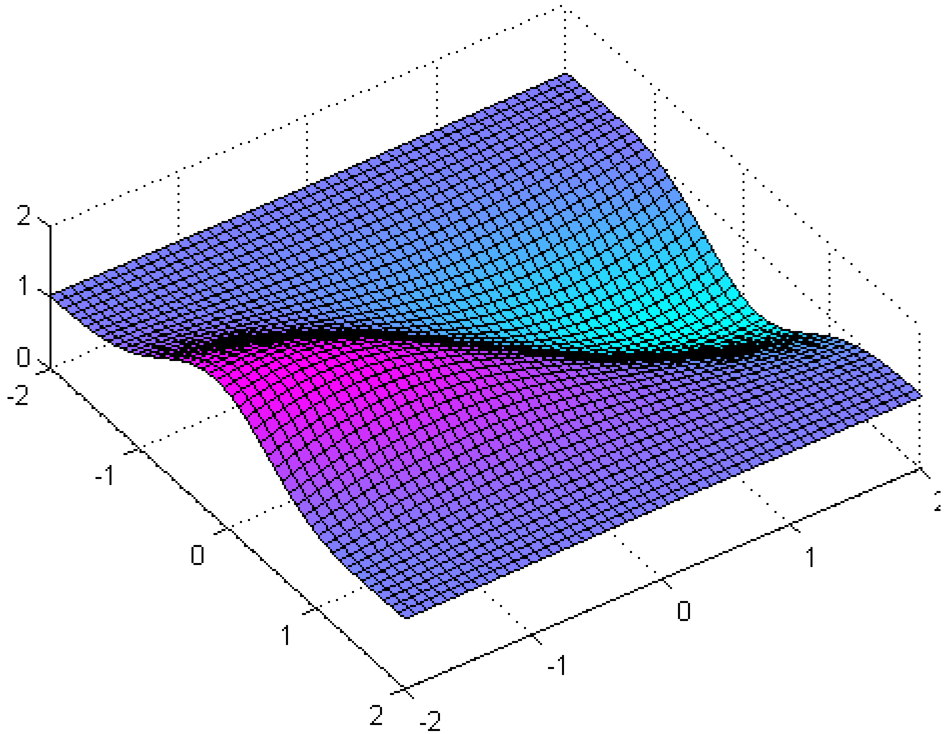
$$\iint_D f \; \mathrm{d}A = \int\int_D f(x,y) \mathrm{d}x \; \mathrm{d}y = \int_a^b \int_{g_\ell(x)}^{g_u(x)} f(x,y) \; \mathrm{d}y \; \mathrm{d}x$$

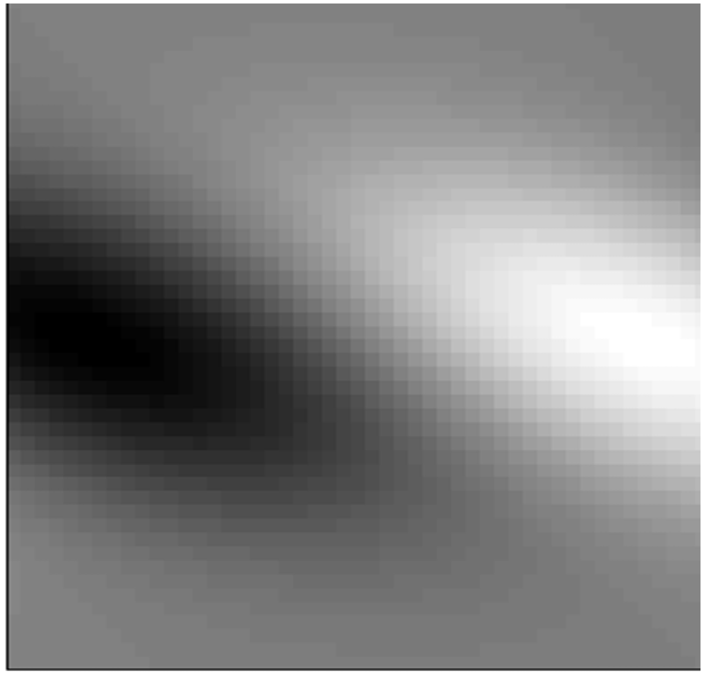
Example:

Find the area of the domain between the curves $y = x^2$ and $y = x^3$.



The integral can be thought of as a *weighted sum* over the domain, where the function f(x,y) gives the weighting.

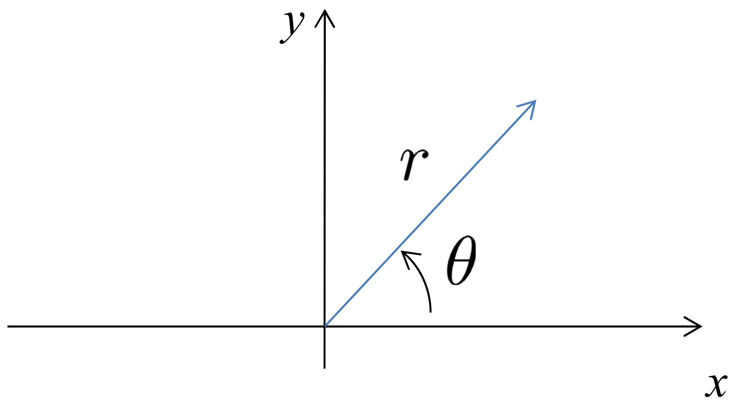




Integrating the function f = 1 over the domain D gives the area of the domain.

Polar coordinates

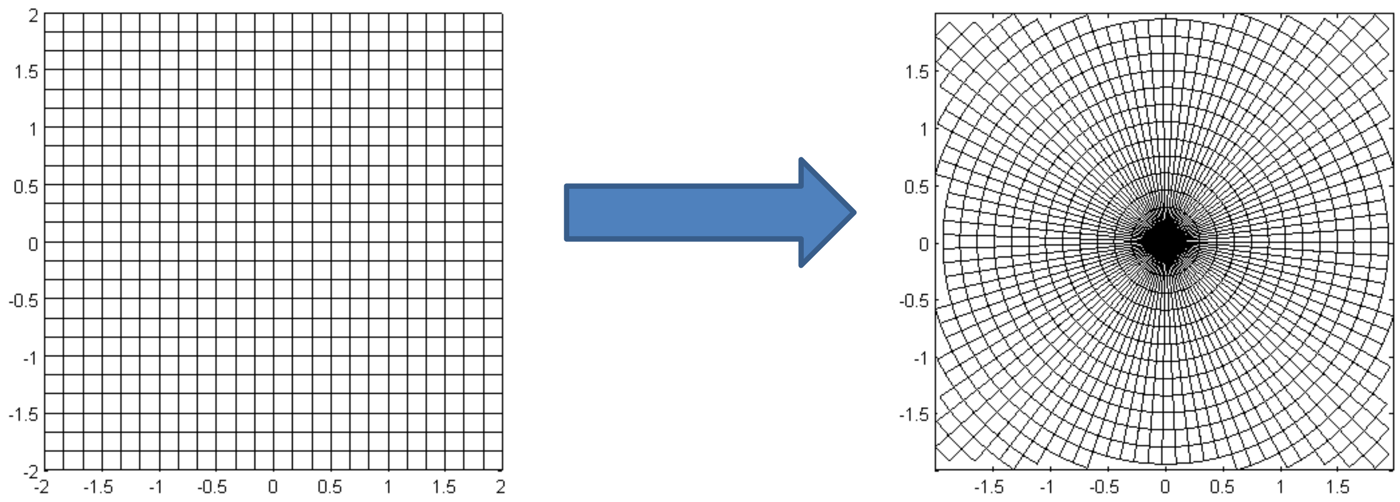
We often want to integrate circular domains, or regions with round elements. For this we need polar coordinates.



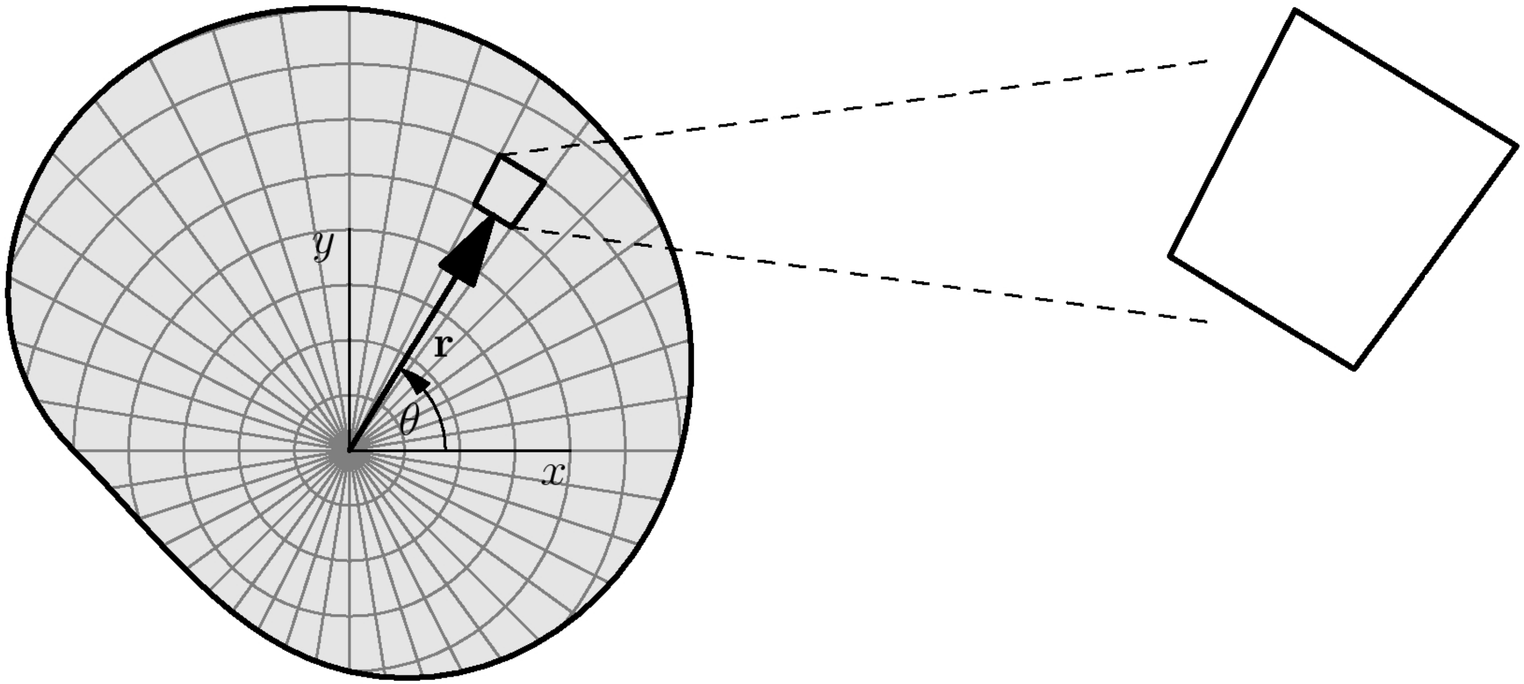
The transformation from (x,y) to (r,„) is

$$x = r\cos\theta$$
$$y = r\sin\theta$$

To integrate, we also need to change from dx dy to differentials involving r and θ:



To integrate, we divide the domain into a large number of small sections, each with area dA.



Note: In polar coordinates, dA *decreases* as we approach the origin.

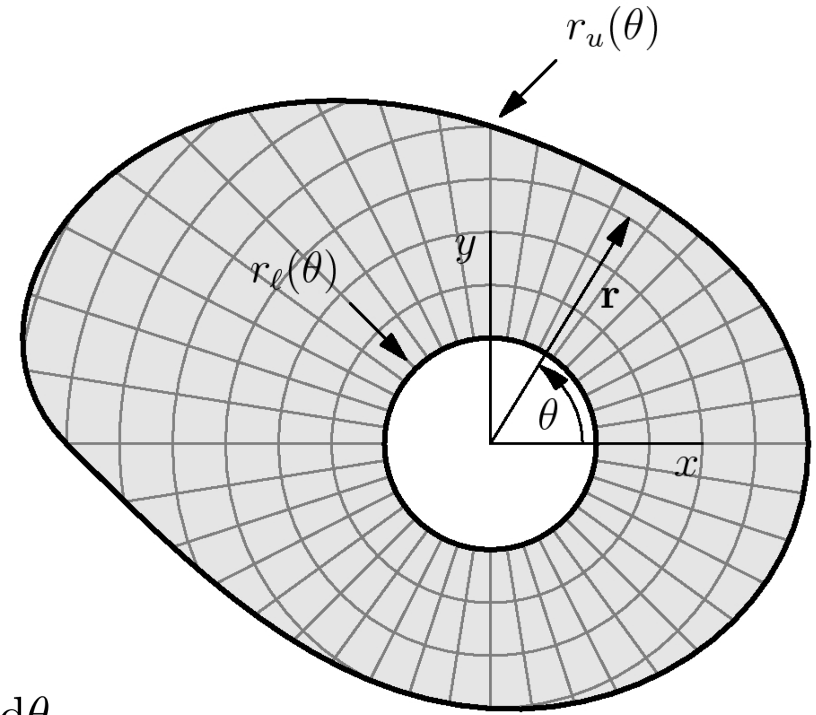
Length of small element ~

Width of small element ~

The area of a small element ~

To go from (x,y) to (r,„), we make the transformation

$$\mathrm{d}x\mathrm{d}y \Rightarrow r\mathrm{d}r\mathrm{d}\theta$$



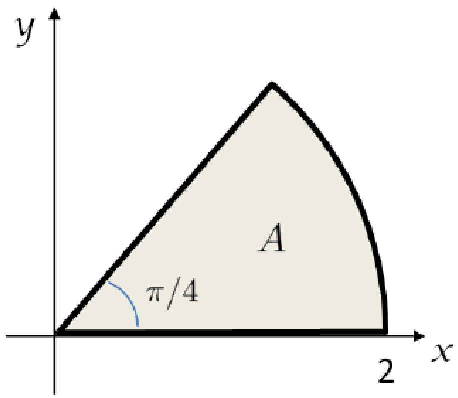
And then integrate, picking one coordinate to integrate over first:

$$\int\int_D f(x,y)\mathrm{d}x\ \mathrm{d}y = \int_{\theta_1}^{\theta_2}\int_{r_\ell(\theta)}^{r_u(\theta)} f(r,\theta)\ r\mathrm{d}r\ \mathrm{d}\theta$$

Example: Integrate

$$\int\int_A x\mathrm{d}x\ \mathrm{d}y$$

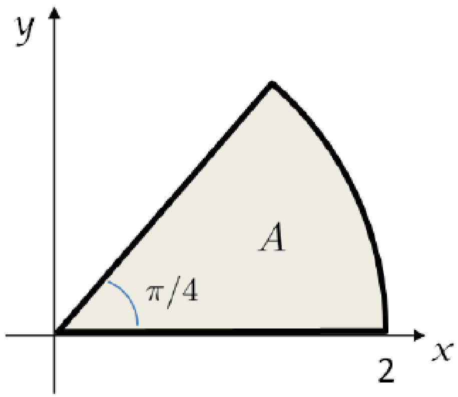
where A is the area shown:



Example: Integrate

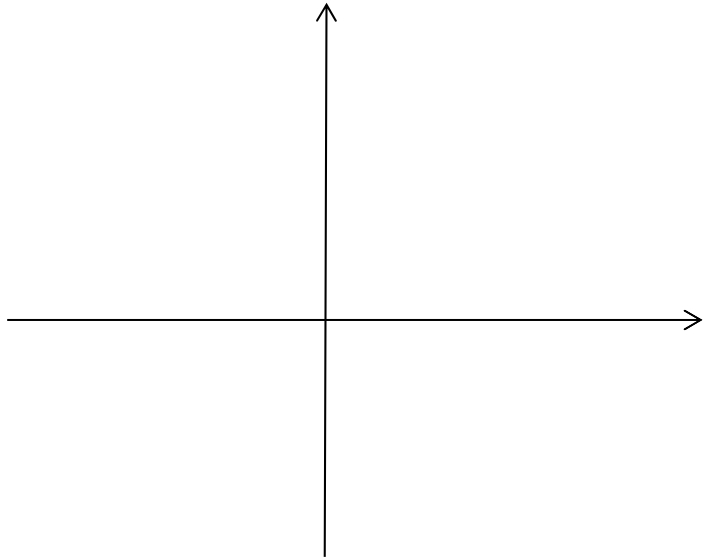
$$\int\int_A x \mathrm{d}x \ \mathrm{d}y$$

where A is the area shown:



Example: evaluate

$$\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}(x^2+y^2)\,dx\,dy$$

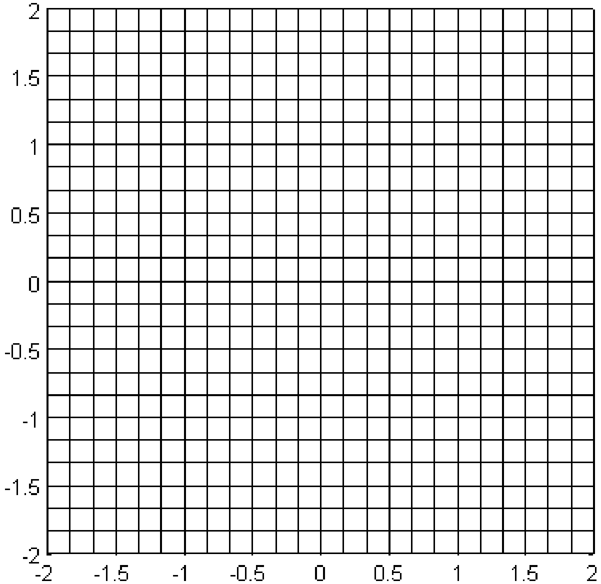


Ans: $\pi/2$

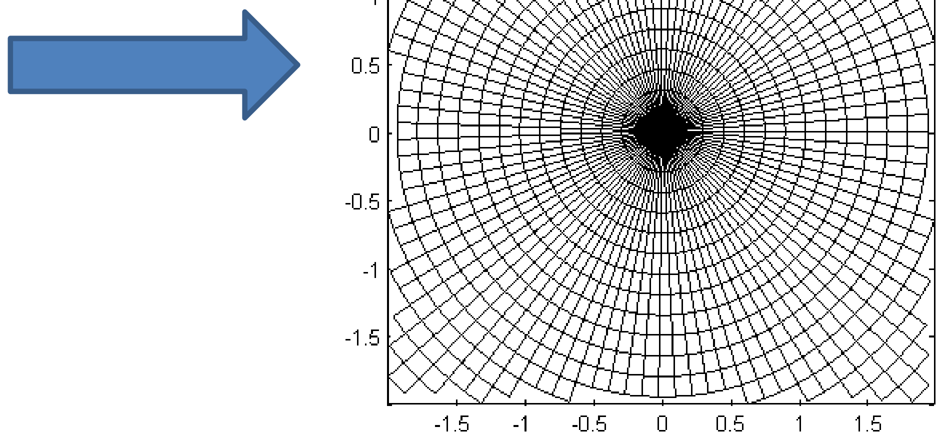
<u>General change of coordinates:</u>

We can write a new coordinate system in terms of the old as

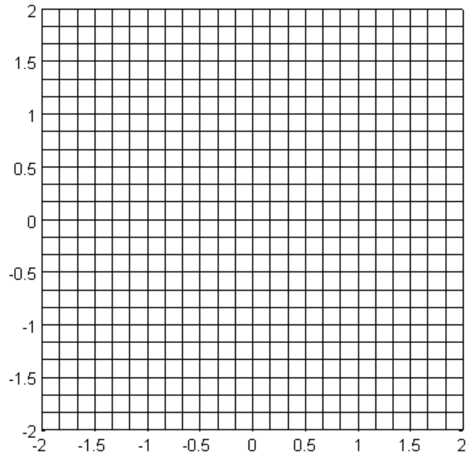
$$x = x(s,t)$$
$$y = y(s,t)$$



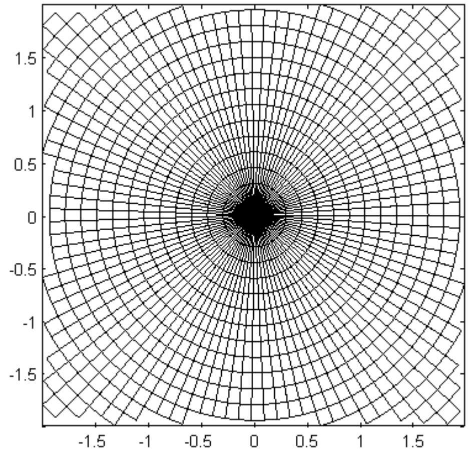
$$x = x(s,t)$$
$$y = y(s,t)$$



We would like a way of writing the area element dA = dx dy in terms of the new coordinates.



$$x = x(s,t)$$
$$y = y(s,t)$$



To change the area element from (x,y) to (s,t) we use the Jacobian:

$$\mathrm{d}x\ \mathrm{d}y = \left| \begin{array}{cc} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{array} \right| \mathrm{d}s\ \mathrm{d}t\ .$$

This is often written

$$dxdy = \left|\frac{\partial(x,y)}{\partial(s,t)}\right| dsdt$$

with

$$\frac{\partial(x,y)}{\partial(s,t)} = \left[\begin{array}{cc} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{array} \right]$$

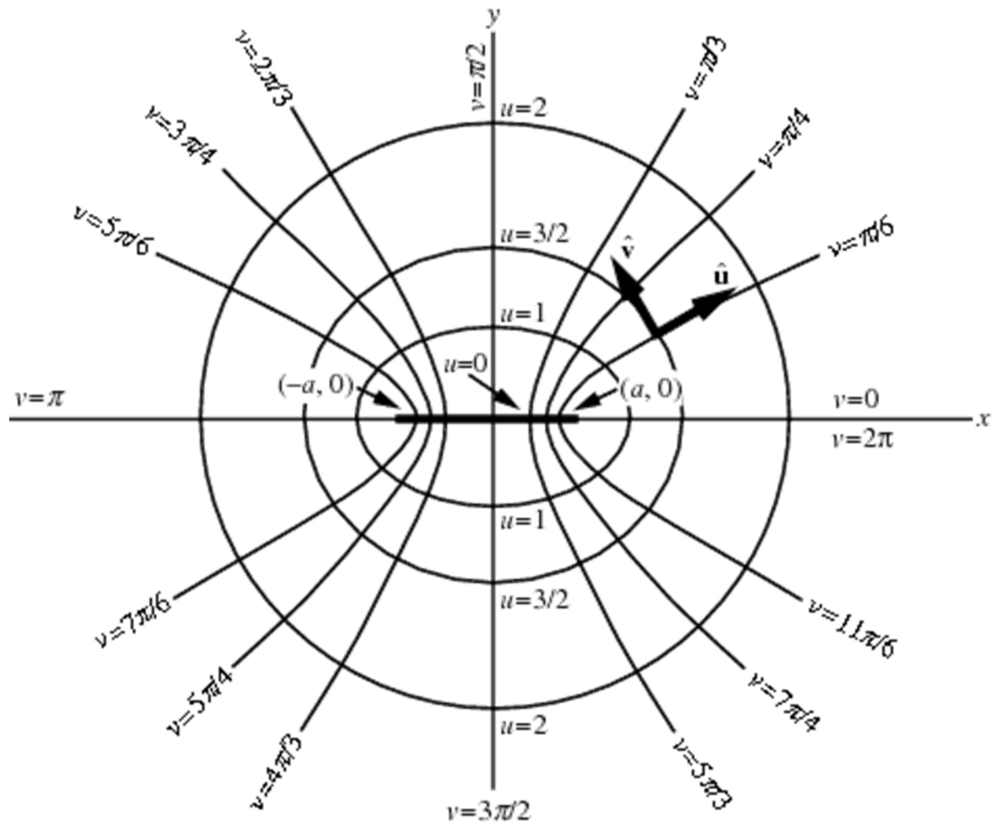
Example: Cartesian to polar coordinates:

$$x = r\cos\theta$$
$$y = r\sin\theta$$

Example: Cartesian to Elliptic coordinates

$$x = a\cosh u \cos v$$

$$y = a\sinh u \sin v$$

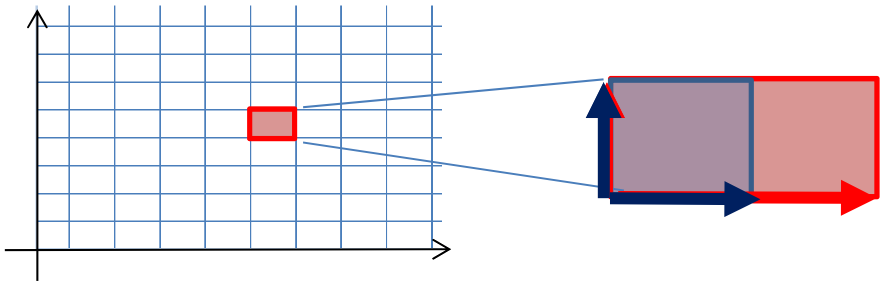


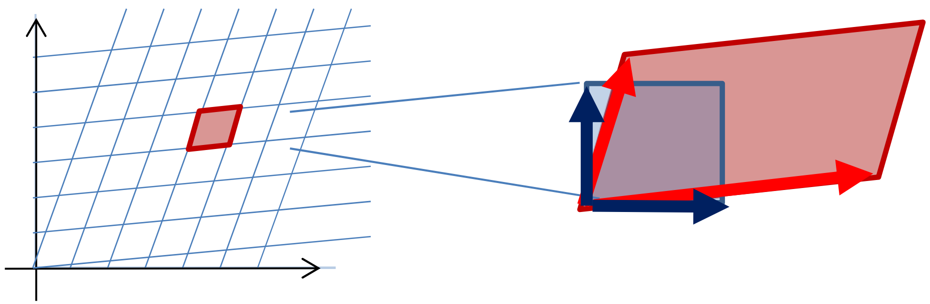
$$dxdy = a^2(\sinh^2 \mu + \sin^2 \nu)d\mu d\nu$$

The <u>reciprocal theorem</u> states that

$$\left|\frac{\partial(u,v)}{\partial(x,y)}\right| = \frac{1}{\left|\frac{\partial(x,y)}{\partial(u,v)}\right|}$$

Why? The <u>determinant</u> gives the scaling factor of the area for a coordinate transformation

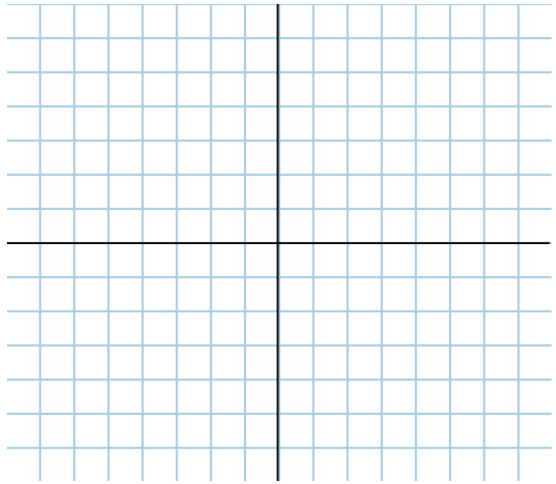




Example: Let R be the parallelogram with vertices (-1,3), (1,-3), (3,-1), (1,5). Evaluate

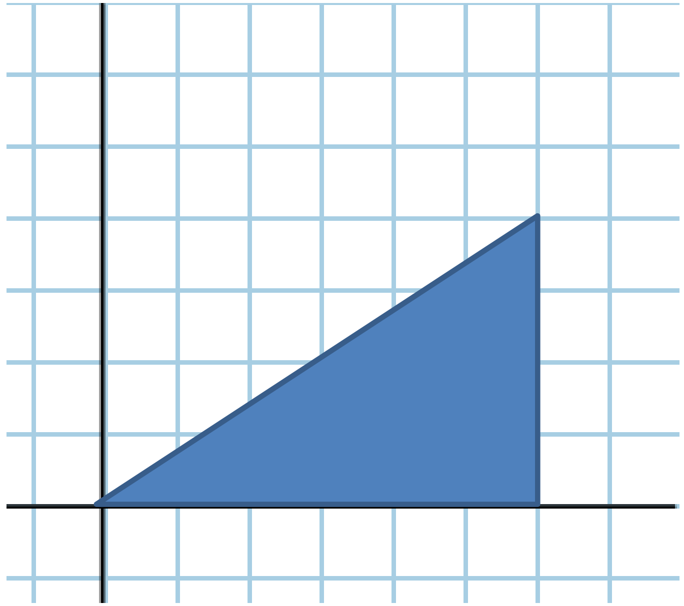
$$\int_R (x-y)^2 dxdy$$

Using the substitution $u = x - y$, $v = 3x + y$.



Areas and averages

The *area* of a region *R* is

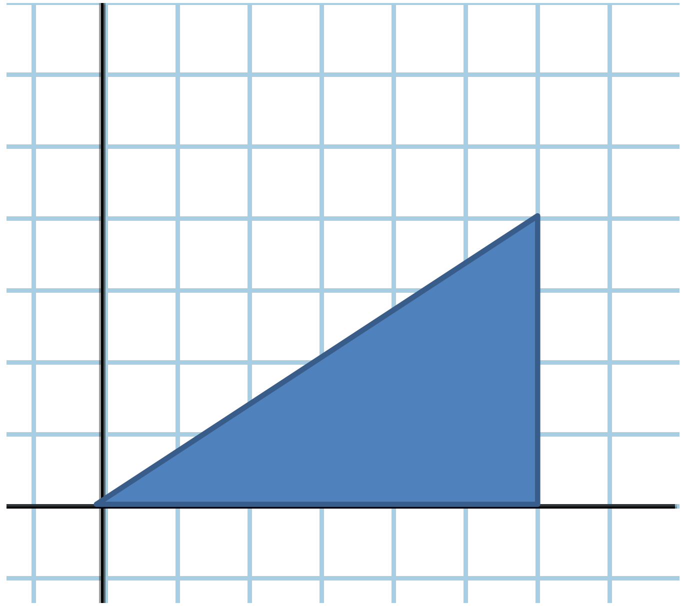


The average of a function *f(x,y)* over R is

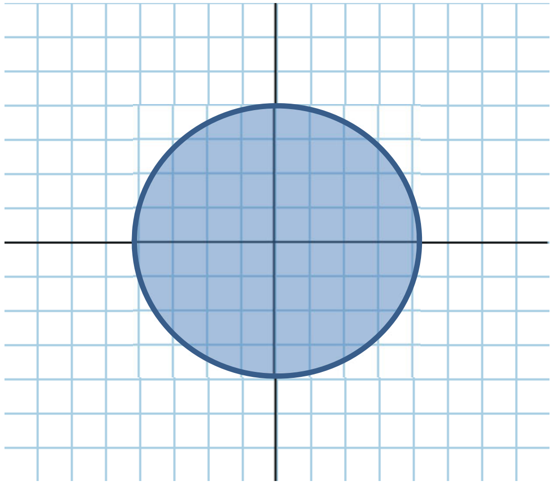
The *average x-position* (often called the *1st moment of Area)* is

The *centroid* is the *average position vector*

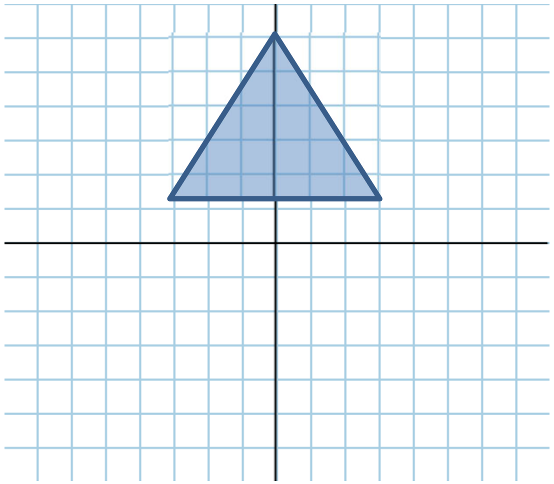
$$< \bar{x}, \bar{y} > = \frac{1}{A} \int\int_R < x, y > dx\ dy$$



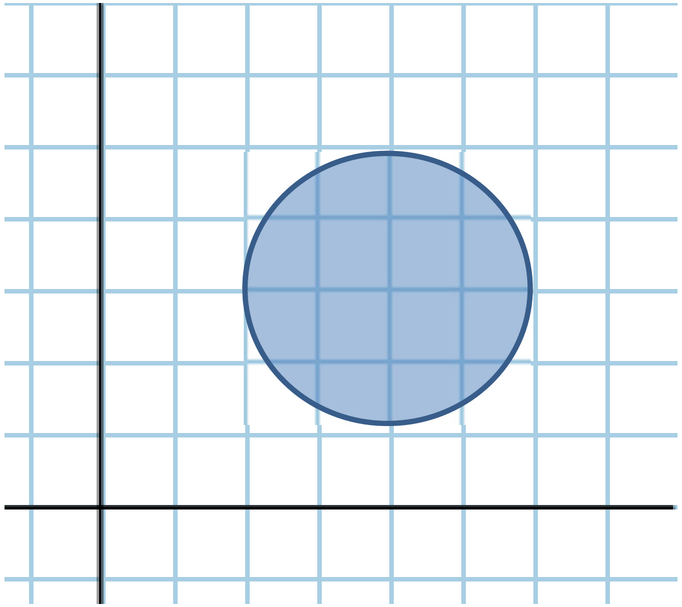
Often the one can "guess" the values of first moment integrals by using the symmetry of the region. E.g:



$$\int\int_R y dx dy =$$

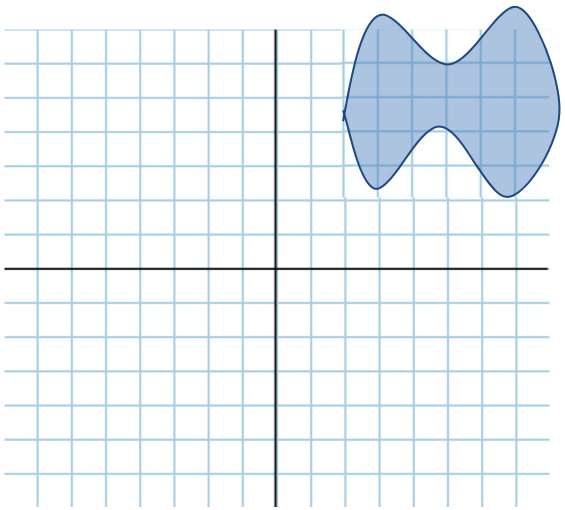


$$\int\int_R x dx dy =$$



$$\int\int_R y\,dx\,dy =$$

$$\int\int_R x\,dx\,dy =$$



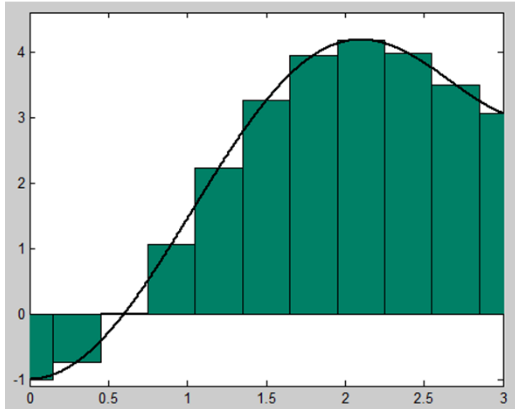
$$\int\int_R y\,dx\,dy =$$

Triple integrals

Often we would like to integrate a function over a *volume, i.e.* over a region in three dimensions.

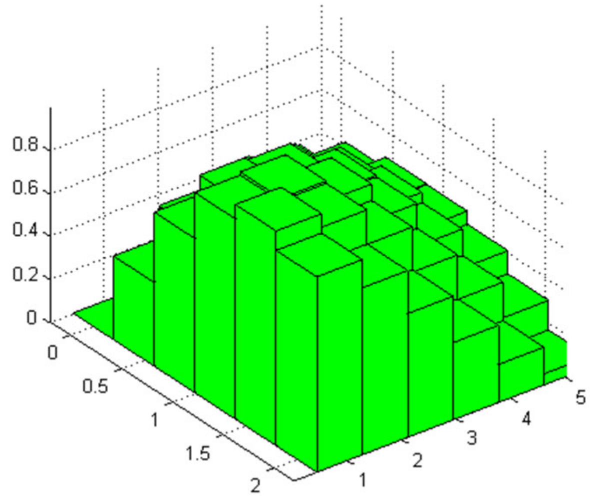
Recall:

Integrals in 1D are defined as the limit of a sum over *intervals of length* Δx*:*



$$\int_a^b f(x)\mathrm{d}x = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$$

Integrals in 2D are defined as the limit of a sum over *squares of area* ΔA*:*



$$\int\int_R f(x,y)\mathrm{d}x\mathrm{d}y = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij},y_{ij})\Delta A$$

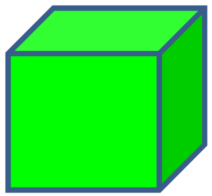
We will define an integral in 3D as the limit of a sum over *boxes of volume* ΔV

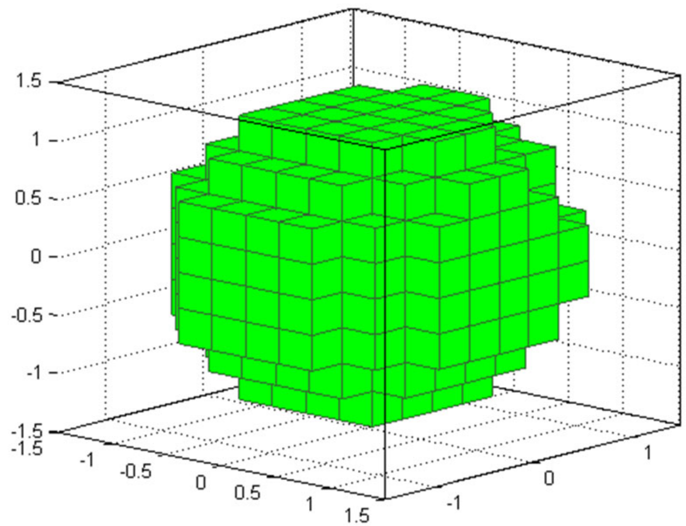
$$\iiint_V f(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z = \lim_{\ell,m,n\to\infty}\sum_{i=1}^{\ell}\sum_{j=1}^{m}\sum_{j=1}^{n} f(x_{ijk},y_{ijk},z_{ijk})\Delta V$$

Here,

$$\Delta V = \Delta x \Delta y \Delta z$$

is the volume of a small box.





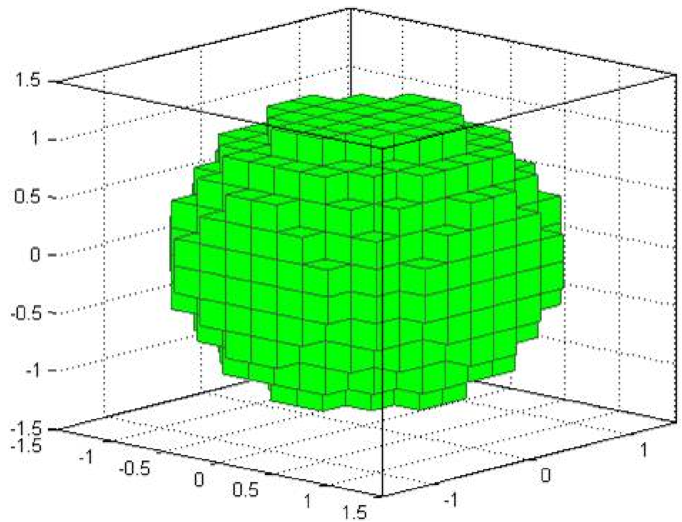
As the number of boxes increases,

$$\Delta V \to dV = dx\ dy\ dz$$

Another way of writing this is:

$$\iiint_V f(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z = \lim_{\Delta V \to 0} \sum_{\text{all boxes}} f(x,y,z)\Delta V$$

That is, the integral is the *weighted sum* of all volume elements in V.



Note that for f = 1,

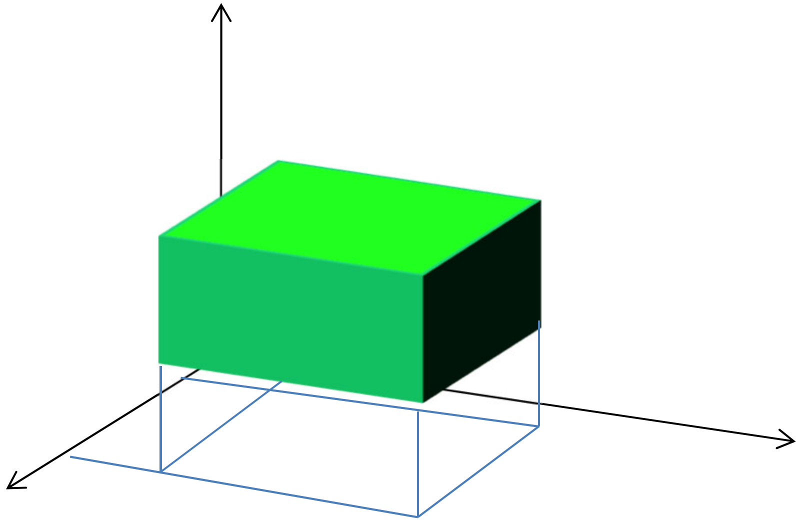
$$\iiint_V 1.\mathrm{d}x\mathrm{d}y\mathrm{d}z = \lim_{\Delta V \to 0} \sum_{\text{all boxes}} \Delta V$$

= the *volume* of the region V

As in 2D, the integral can be computed iteratively.

A rectangular box in 3D is described by

$$V = \{(x, y, z) \quad | \quad a \leq x \leq b \;, \\ c \leq y \leq d \;, \\ p \leq z \leq q \;\}$$



The integral is then

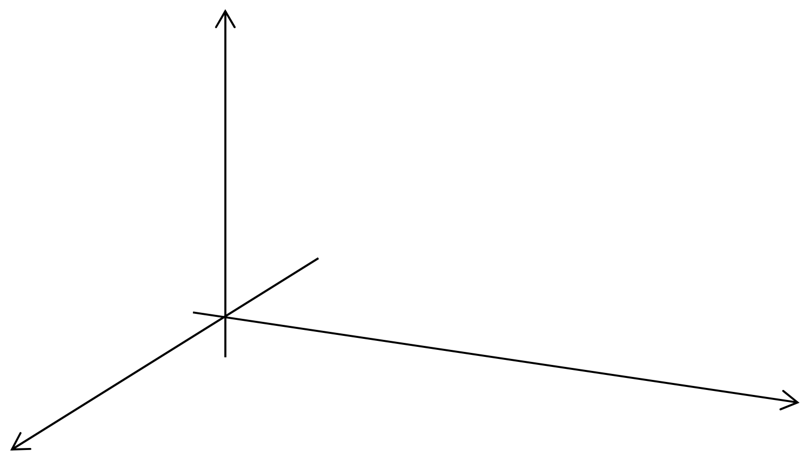
$$\iiint_V f(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z = \int_a^b \left[\int_c^d \left[\int_p^q f(x,y,z)\mathrm{d}z \right] \mathrm{d}y \right] \mathrm{d}x$$

(Fubini’s theorem says that the order of integration is not important.)

Example: Integrate

$$\iiint_B xy^2 z \,\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

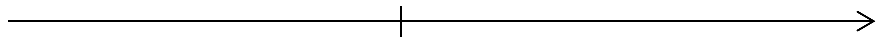
where B is the rectangular box with 1 ≤ x ≤ 2, 1 ≤ y ≤ 3 , 0 ≤ z ≤ 1.



<u>Describing more general regions in 3D</u>

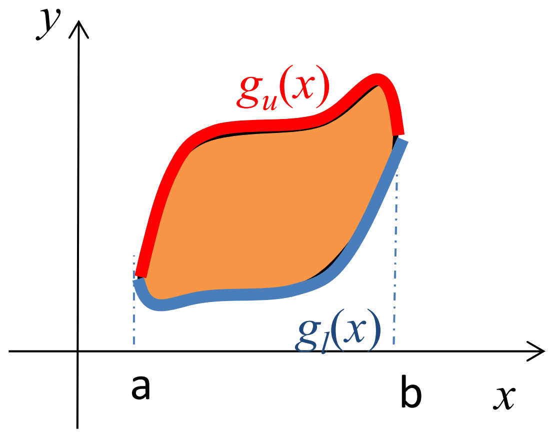
In 1D a region is described by “hard” inequalities

$$D = \{x \mid -1 \leq x \leq 3 \}$$



In 2D we describe a region by a pair of “outer” and “inner” inequalities:

$$D = \{(x, y) | a \leq x \leq b \quad , \quad g_\ell(x) \leq y \leq g_u(x)\}$$

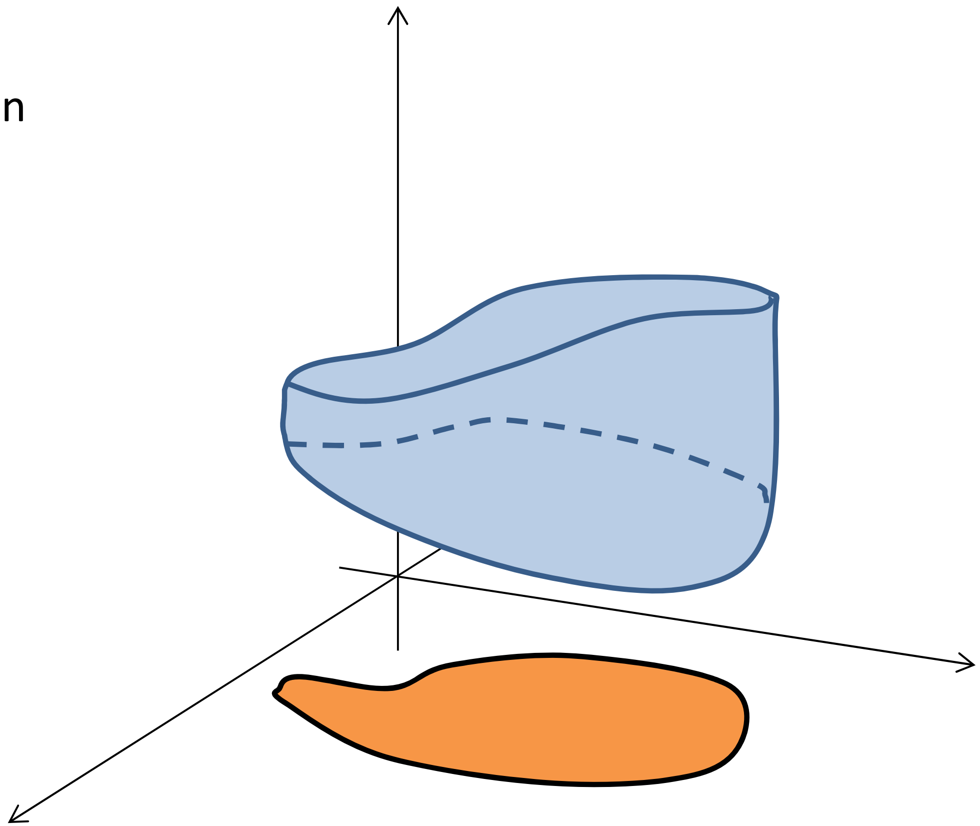


In 3D we need an extra pair of inner inequalities to describe the domain:

$$D = \{(x, y, z) | a \leq x \leq b \quad , \quad g_\ell(x) \leq y \leq g_u(x) \quad , u_1(x, y) \leq z \leq u_2(x, y) \ \}$$

A simple domain in 3D can be written

$$D = \{(x, y, z) | a \leq x \leq b \quad ,$$
$$g_\ell(x) \leq y \leq g_u(x) \quad ,$$
$$u_1(x, y) \leq z \leq u_2(x, y) \ \}$$



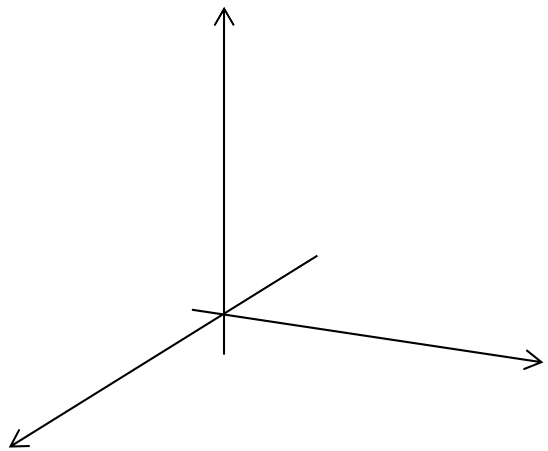
The integral in 3D over this domain is then

$$\iiint_V f(x, y, z) \mathrm{d}x\mathrm{d}y\mathrm{d}z = \int_a^b \left[\int_{g_\ell(x)}^{g_u(x)} \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z) \mathrm{d}z \right] \mathrm{d}y \right] \mathrm{d}x$$

Example: Evaluate

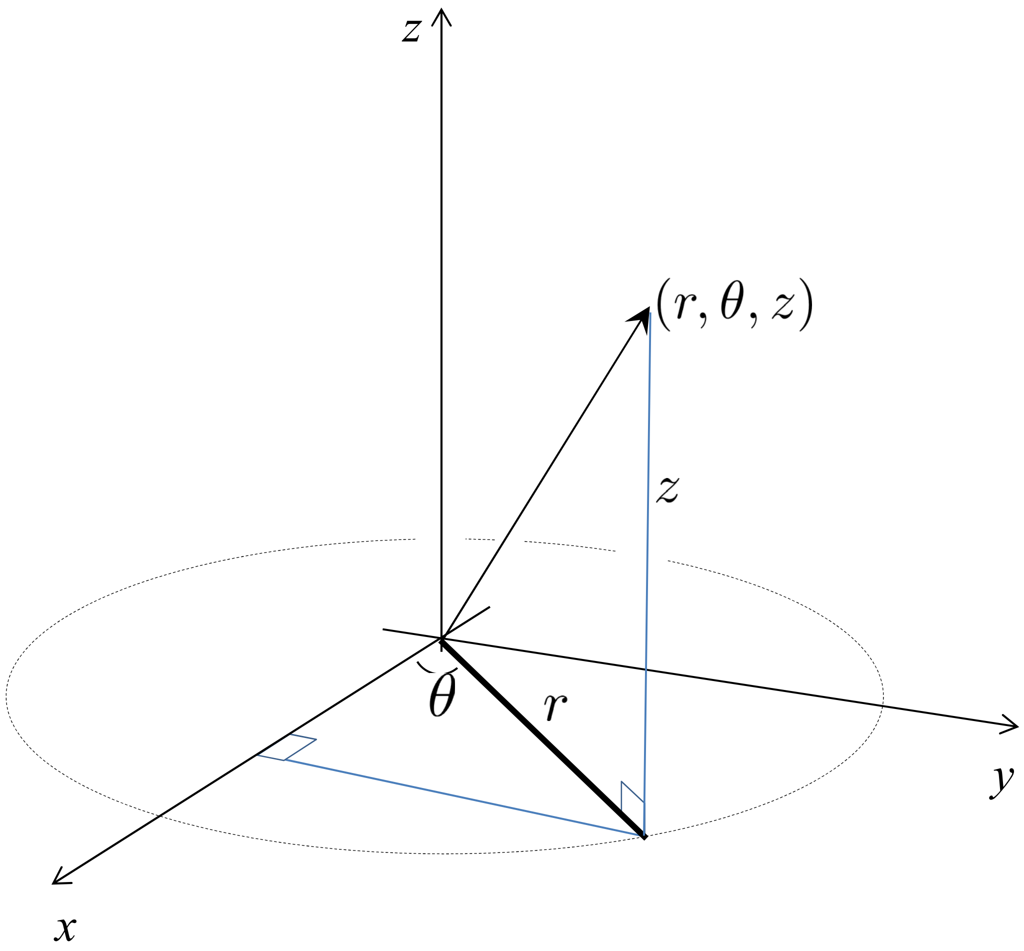
$$\iiint_E z \; \mathrm{d}x\mathrm{d}y\mathrm{d}z$$

where E is the tetrahedron bound by the planes x = 0, y = 0, z = 0 and the plane x + y + z = 1.



Cylindrical coordinates

are a useful hybrid of Cartesian and polar coordinates: the x-y plane is treated in polar coordinates but the z-axis is the same as in the (x,y,z) system.



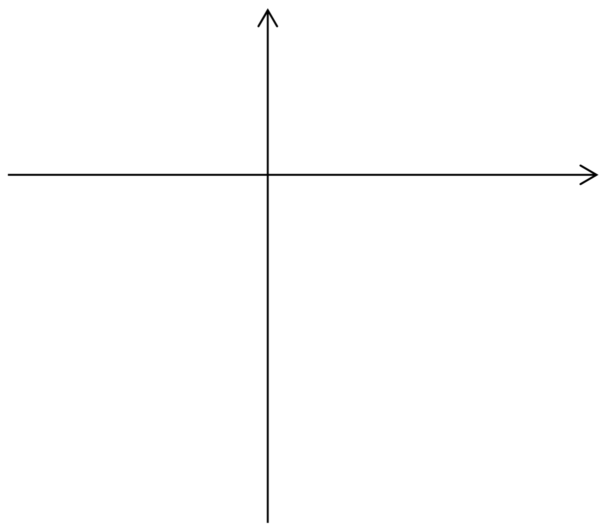
Cylindrical coordinates:

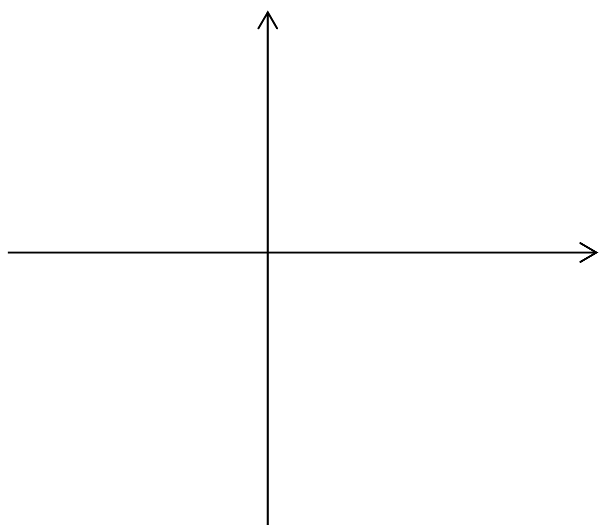
$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}$$

Cylindrical volume element:

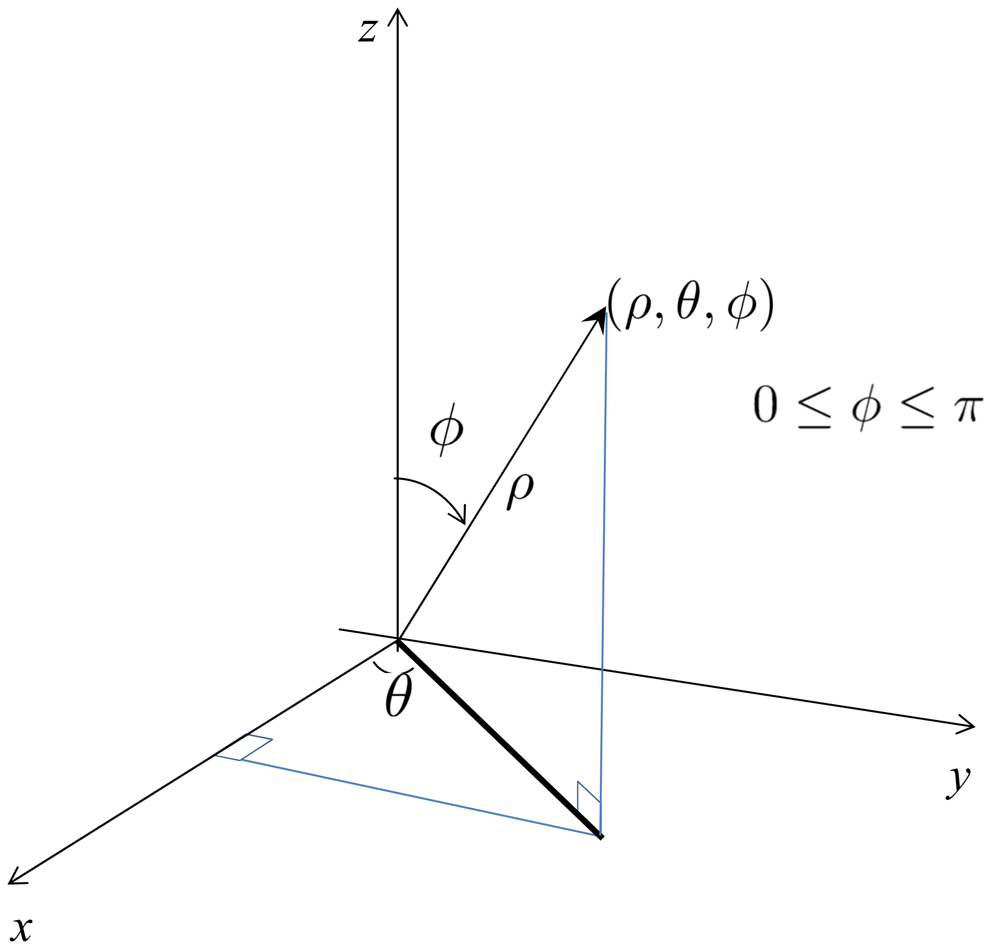
$$\mathrm{d}x\mathrm{d}y\mathrm{d}z = r\mathrm{d}r\mathrm{d}\theta\mathrm{d}z$$

Example: A fuel tank has a circular cross section and consists of two sections: a cone of height 4 metres and an inverted parabola of height 2 metres.
The maximum radius of the tank is 4 metres. Compute the volume of the tank.





Spherical coordinates are essential for computing anything with 3D spherical symmetry



$$0 \leq \phi \leq \pi$$

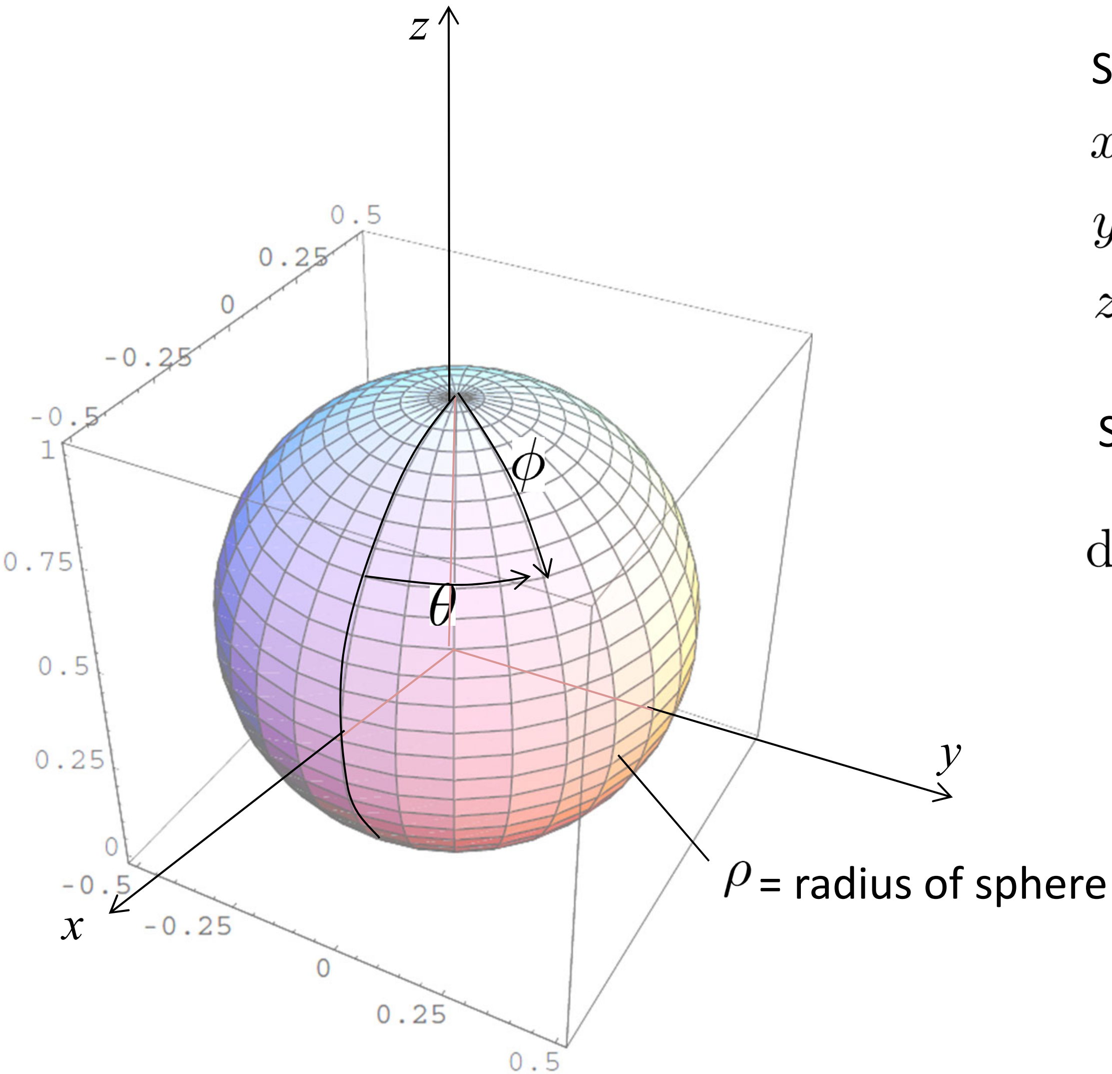
Spherical coordinates:

$$\begin{aligned} x &= \rho \cos\theta \sin\phi \\ y &= \rho \sin\theta \sin\phi \\ z &= \rho \cos\phi \end{aligned}$$

Spherical volume element:

$$\mathrm{d}x\mathrm{d}y\mathrm{d}z = \rho^2 \sin\phi \mathrm{d}\rho\mathrm{d}\theta\mathrm{d}\phi$$

Important note:
The range of Á is



Spherical coordinates:

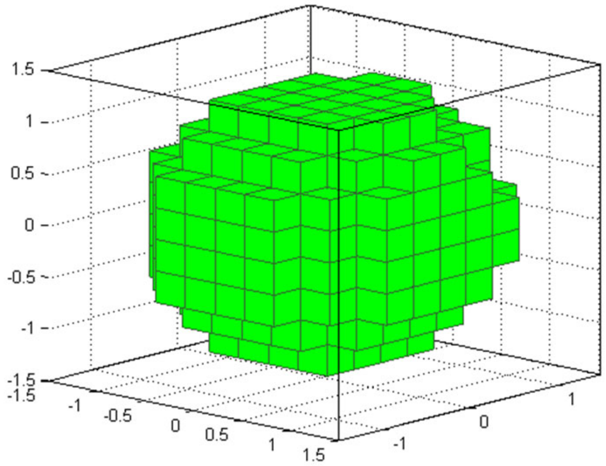
$$x = \rho \cos\theta \sin\phi$$
$$y = \rho \sin\theta \sin\phi$$
$$z = \rho \cos\phi$$

Spherical volume element:

$$\mathrm{d}x\mathrm{d}y\mathrm{d}z = \rho^2 \sin\phi \mathrm{d}\rho\mathrm{d}\theta\mathrm{d}\phi$$

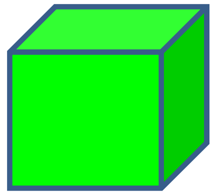
Applications: volume, mass and density

An integral is a *weighted sum* over a volume.
The **meaning of the integral is determined by the weighting.**



E.g. the volume of a solid is given by

$$V = \iiint_E 1.\mathrm{d}x\mathrm{d}y\mathrm{d}z = \lim_{\Delta V \to 0} \sum_{\text{all boxes}} \Delta V$$



Suppose that the function ´(x,y,z) describes the *mass per unit volume* of the solid. Then the total mass is

$$M = \iiint_E \rho(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

Important note:
Do not get mixed up between the density ´ and the spherical coordinate ´#!

Use spherical coordinates to find the volume of a solid that lies above the cone $z = \sqrt{x^2 + y^2}$ and within the sphere $x^2 + x^2 + z^2 \leq 1$.