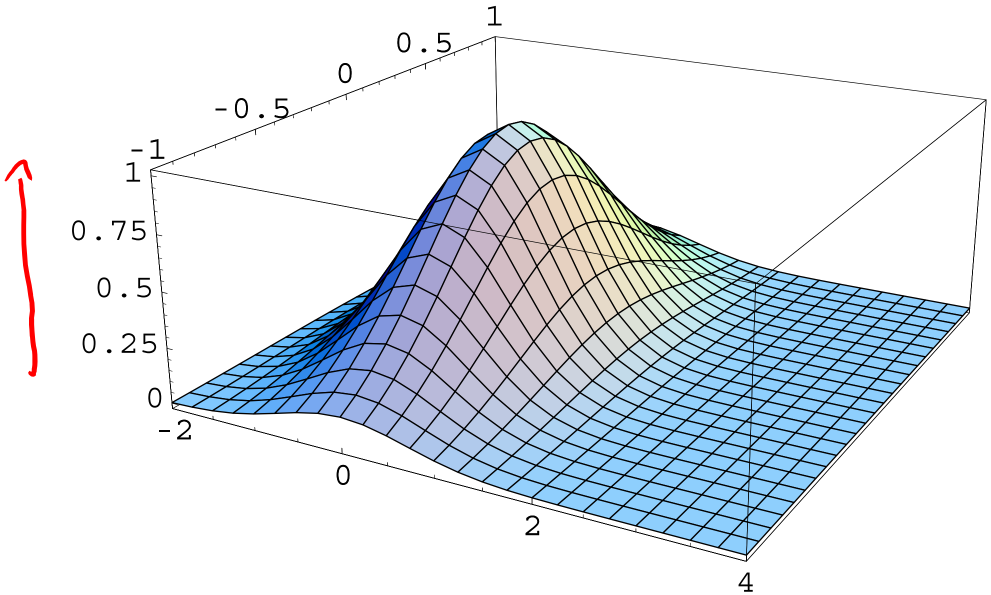
Surface and Flux Integrals

A surface can be defined in 3 ways:

1. <u>Explicitly:</u>

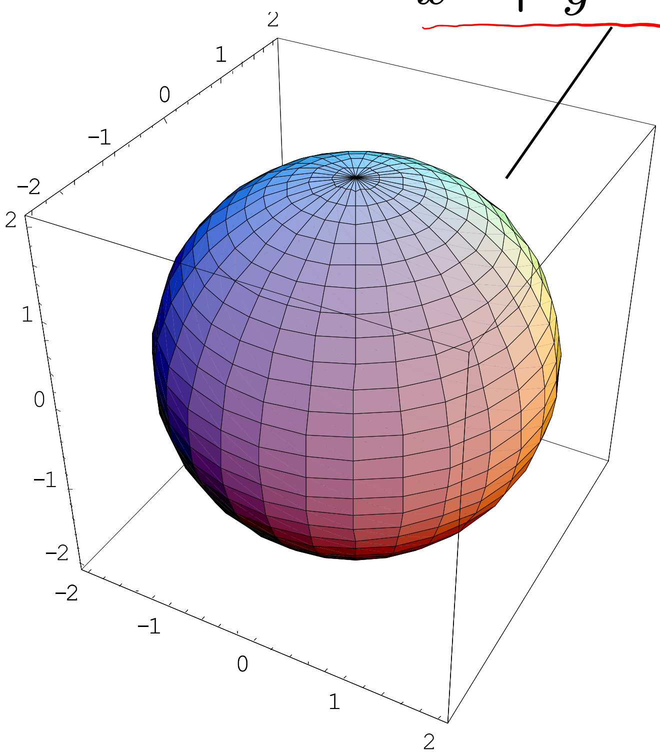
$$z = \underline{g(x, y)}$$



2. Implicitly:

$$\underline{h(x, y, z)} = 0$$

$$\underline{x^2 + y^2 + z^2 - 4} = 0$$



$$z = \pm\sqrt{4 - x^2 - y^2}$$

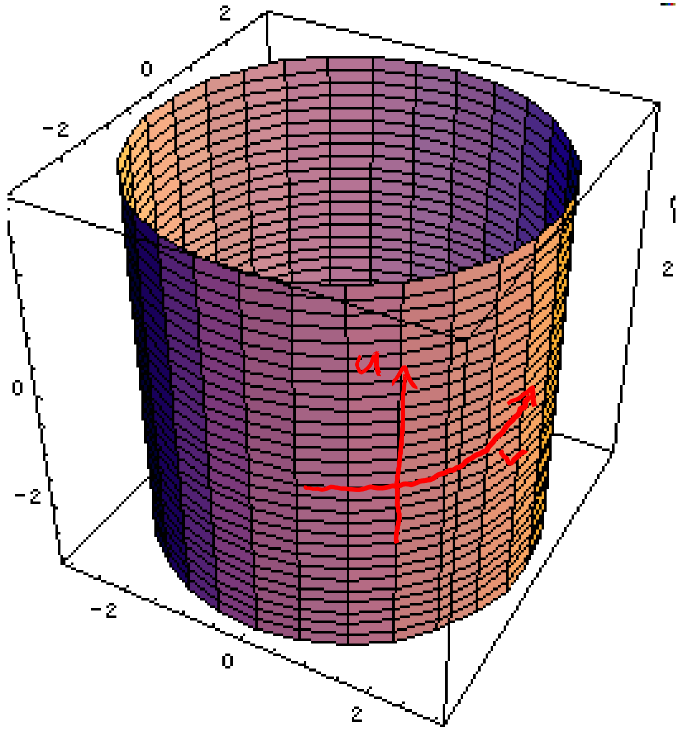
$$z = g(x, y)$$

$$\underbrace{z - g(x, y)} = 0$$

$$h(x, y, z) = 0.$$

3. Parametrically:

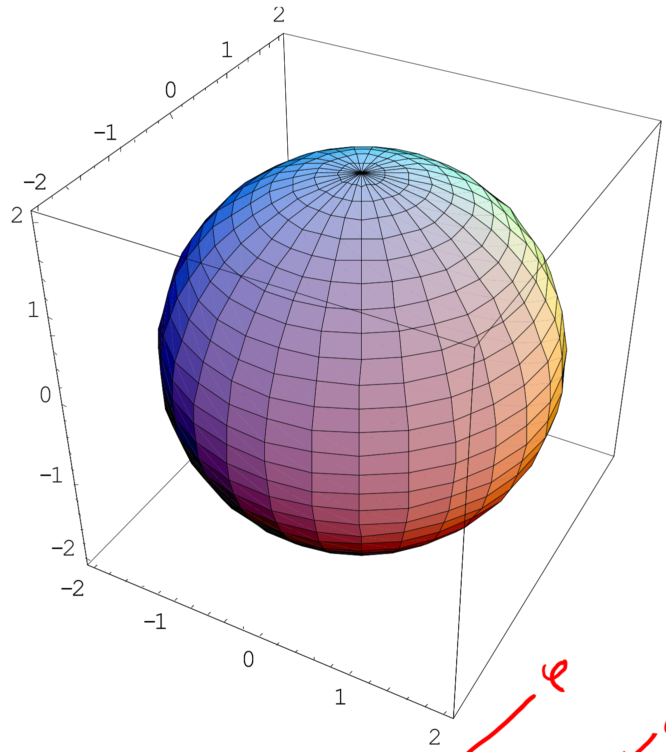
$$\begin{aligned} x &= x(u,v) \\ y &= y(u,v) \\ z &= z(u,v) \end{aligned}$$



$$\begin{aligned} x &= 2\cos v & &= 2\cos\theta \\ y &= 2\sin v & &= 2\sin\theta \\ z &= u & &= z \end{aligned}$$

3. Parametrically:

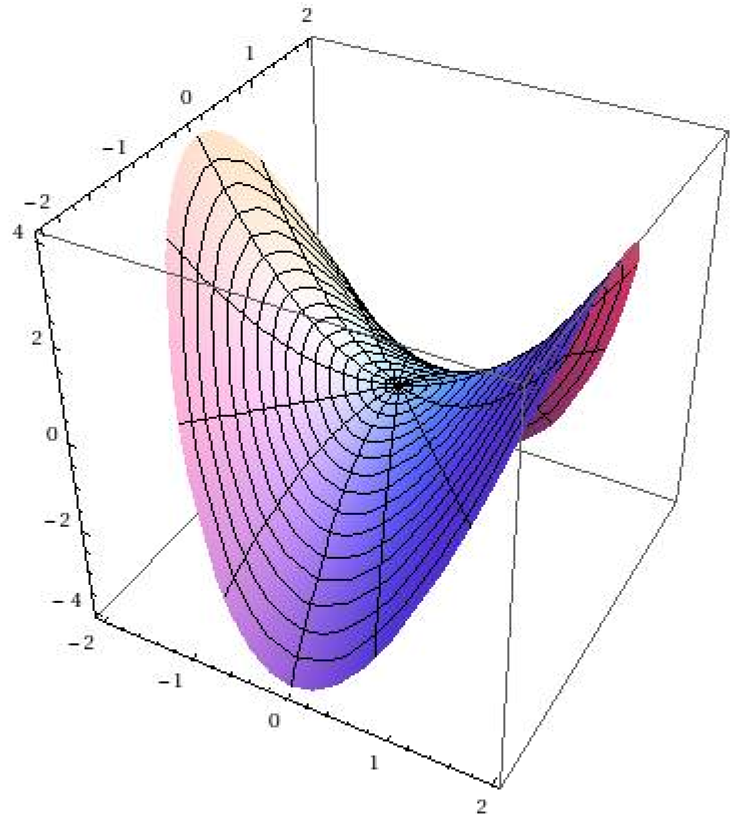
$$
\begin{aligned}
x &= x(u,v) \\
y &= y(u,v) \\
z &= z(u,v)
\end{aligned}
$$



$$
\begin{aligned}
x &= 2\cos \overset{\varphi}{v} \cos \overset{\theta}{u} \\
y &= 2\sin v \cos \overset{\theta}{u} \\
z &= 2\sin \underset{\theta}{u}
\end{aligned}
$$

3. Parametrically:

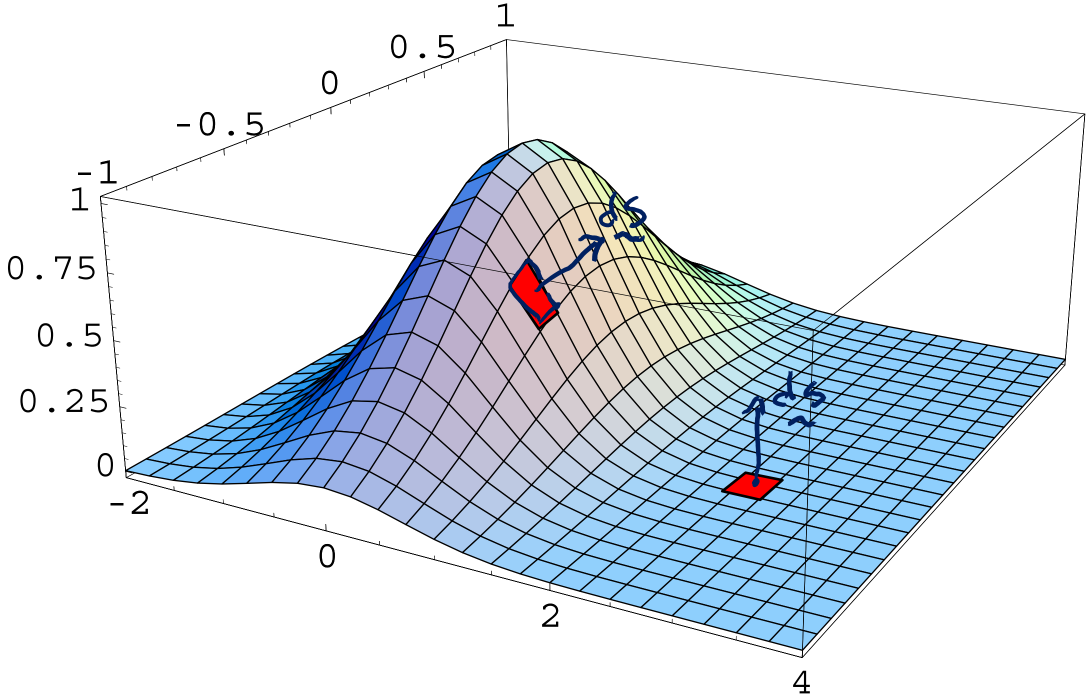
$$
\begin{aligned}
x &= x(u,v) \\
y &= y(u,v) \\
z &= z(u,v)
\end{aligned}
$$



$$
\begin{aligned}
x &= v \cos u \\
y &= v \sin u \\
z &= v^2 \sin(2u)
\end{aligned}
$$

The surface element $d\mathbf{S}$

The infinitessimal surface element dS
Is a vector. Its magnitude is the *area* of the element and its *direction* is normal to the surface



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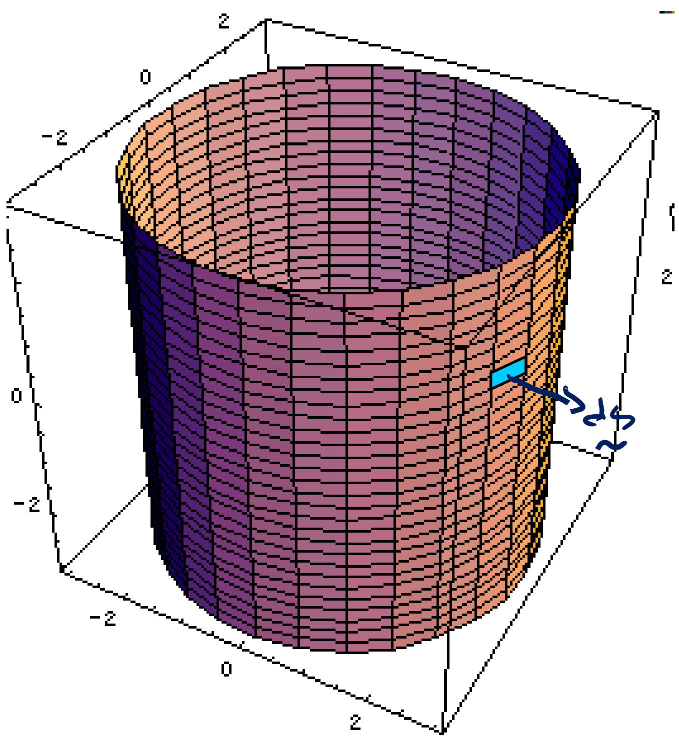

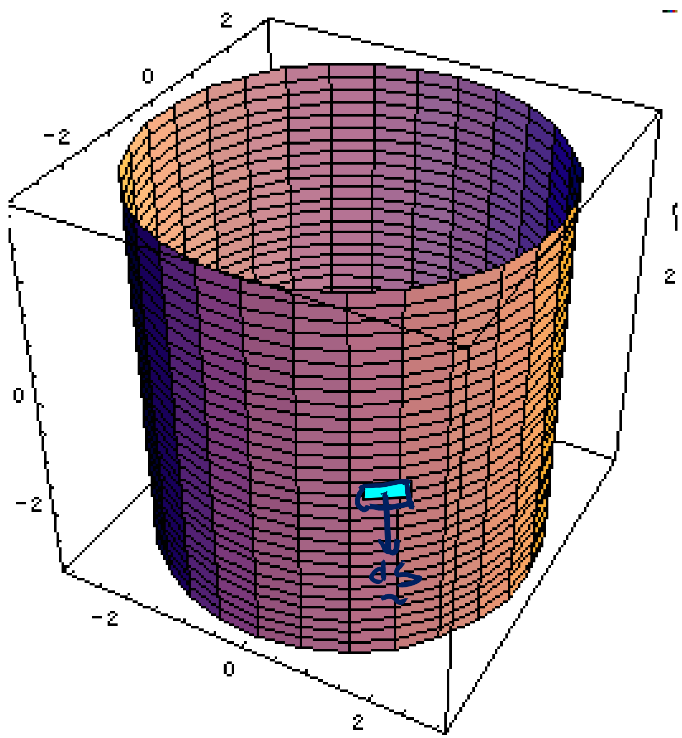

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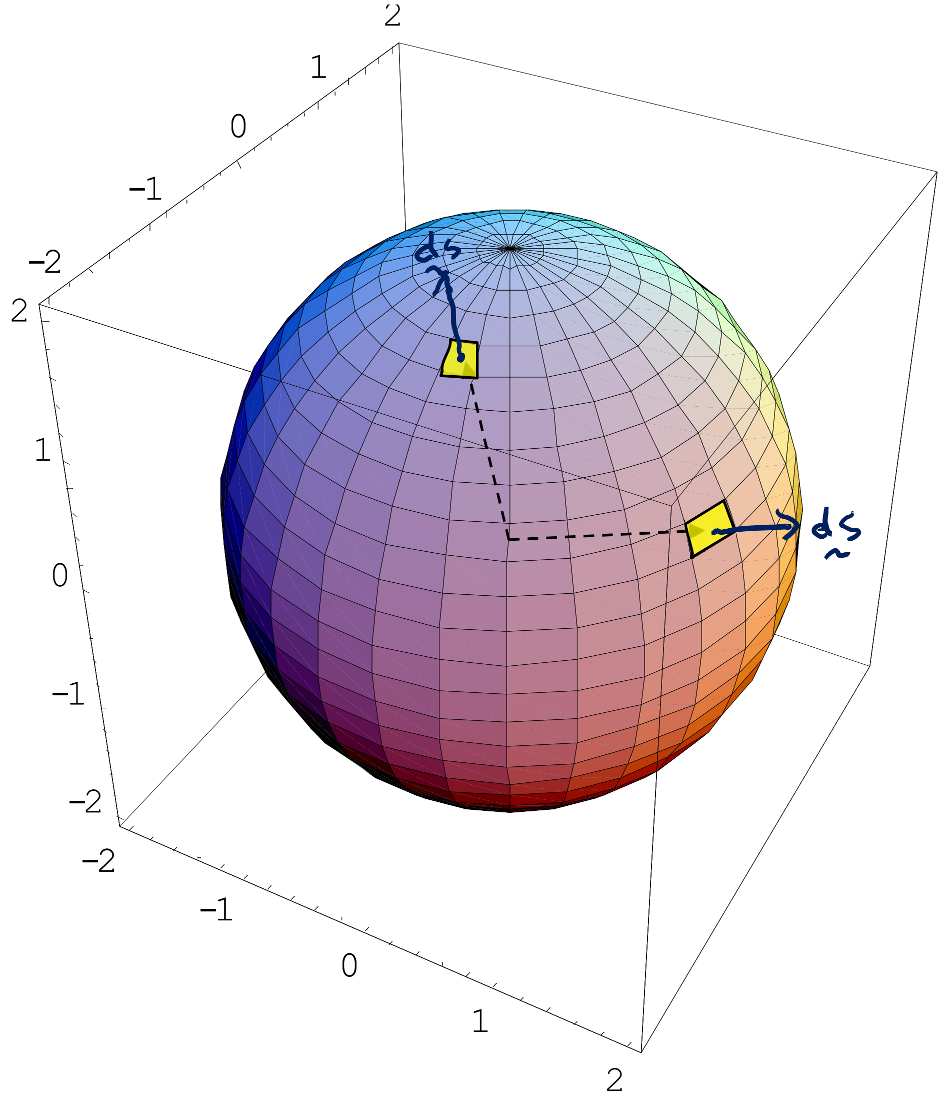
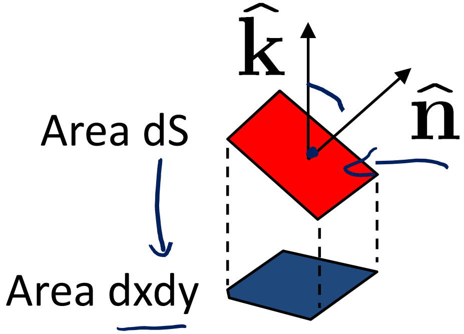
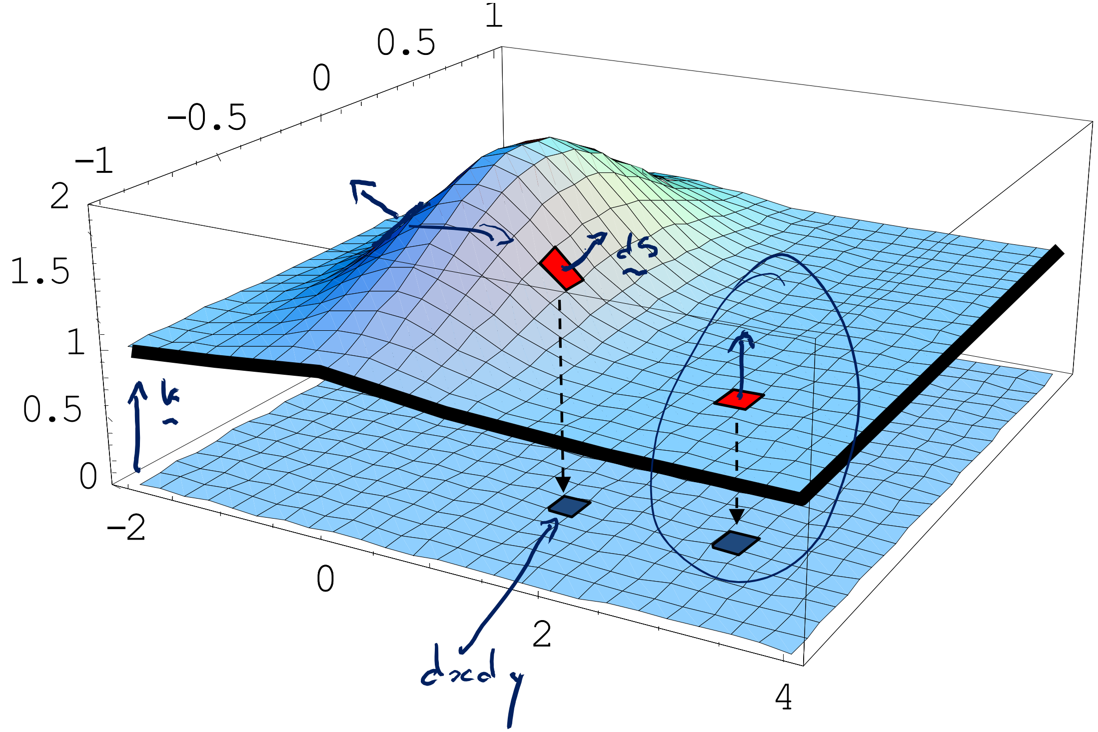


For <u>explicit</u> surfaces

$$z = g(x, y)$$

we can project onto the x-y plane:





$$dx\,dy = |\hat{\mathbf{n}} \cdot \hat{\mathbf{k}}|\, dS$$

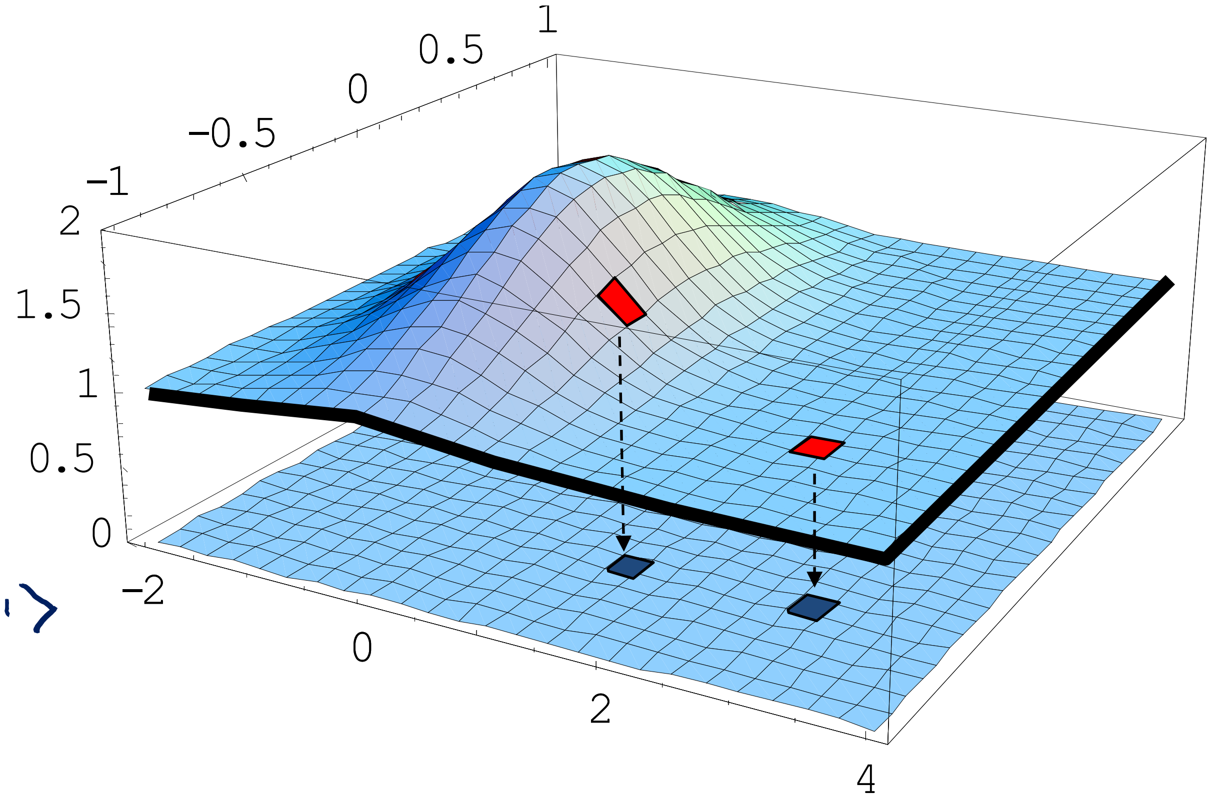
The normal vector is $\quad \mathbf{n} = \nabla\big(\underbrace{z - g(x,y)}_{h(x,y,z)=0}\big) = \nabla h$

The unit normal is $\quad = \left\langle -\dfrac{\partial g}{\partial x}, -\dfrac{\partial g}{\partial y}, 1 \right\rangle$

$$\hat{\mathbf{n}} = \frac{\mathbf{n}}{|\mathbf{n}|} = \frac{\left\langle -\frac{\partial g}{\partial x}, -\frac{\partial g}{\partial y}, 1 \right\rangle}{\sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}}$$

So,

$$|\mathrm{d}\mathbf{S}| = \frac{\mathrm{d}x\mathrm{d}y}{|\hat{\mathbf{n}}\cdot\hat{\mathbf{k}}|}$$



$$= \frac{dxdy}{\left(\frac{\left\langle -\frac{\partial g}{\partial x}, -\frac{\partial g}{\partial y}, 1\right\rangle}{\sqrt{\left(\frac{\partial g}{\partial x}\right)^2+\left(\frac{\partial g}{\partial y}\right)^2+1}}\right)\cdot\langle 0,0,1\rangle}$$

$$= \frac{dxdy}{\frac{1}{\sqrt{\left(\frac{\partial g}{\partial x}\right)^2+\left(\frac{\partial g}{\partial x}\right)^2+1}}} = \underline{\sqrt{1+\left(\frac{\partial g}{\partial x}\right)^2+\left(\frac{\partial g}{\partial y}\right)^2}\;dxdy}$$

The full area element is

$$d\underset{\sim}{S} = \underset{\sim}{\hat{n}}\,|d\underset{\sim}{S}|$$

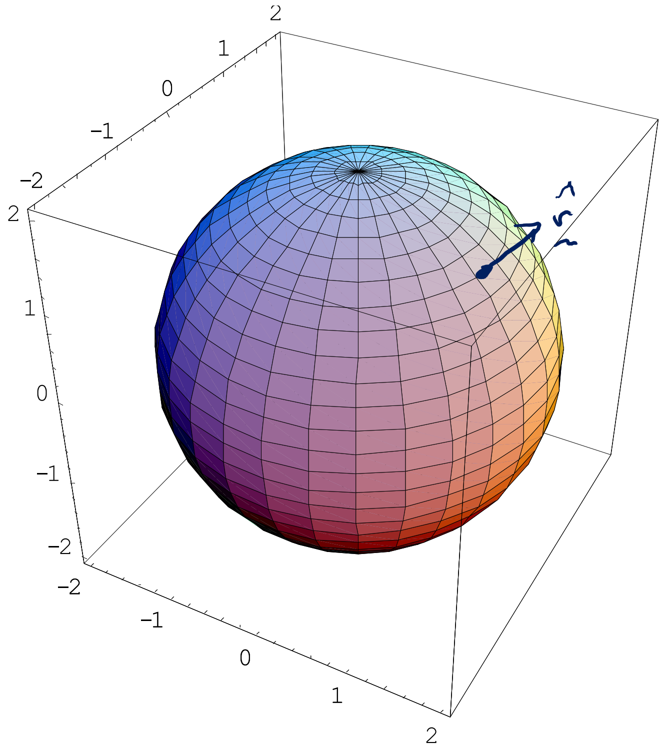
For implicit surfaces

$$h(x, y, z) = 0$$

the unit normal is

$$\hat{\underset{\sim}{n}} = \frac{\nabla h}{|\nabla h|}$$

$$x^2 + y^2 + z^2 - 4 = 0$$



Important:

We can sometimes *guess* the normal. E.g. for the sphere centred at the origin:

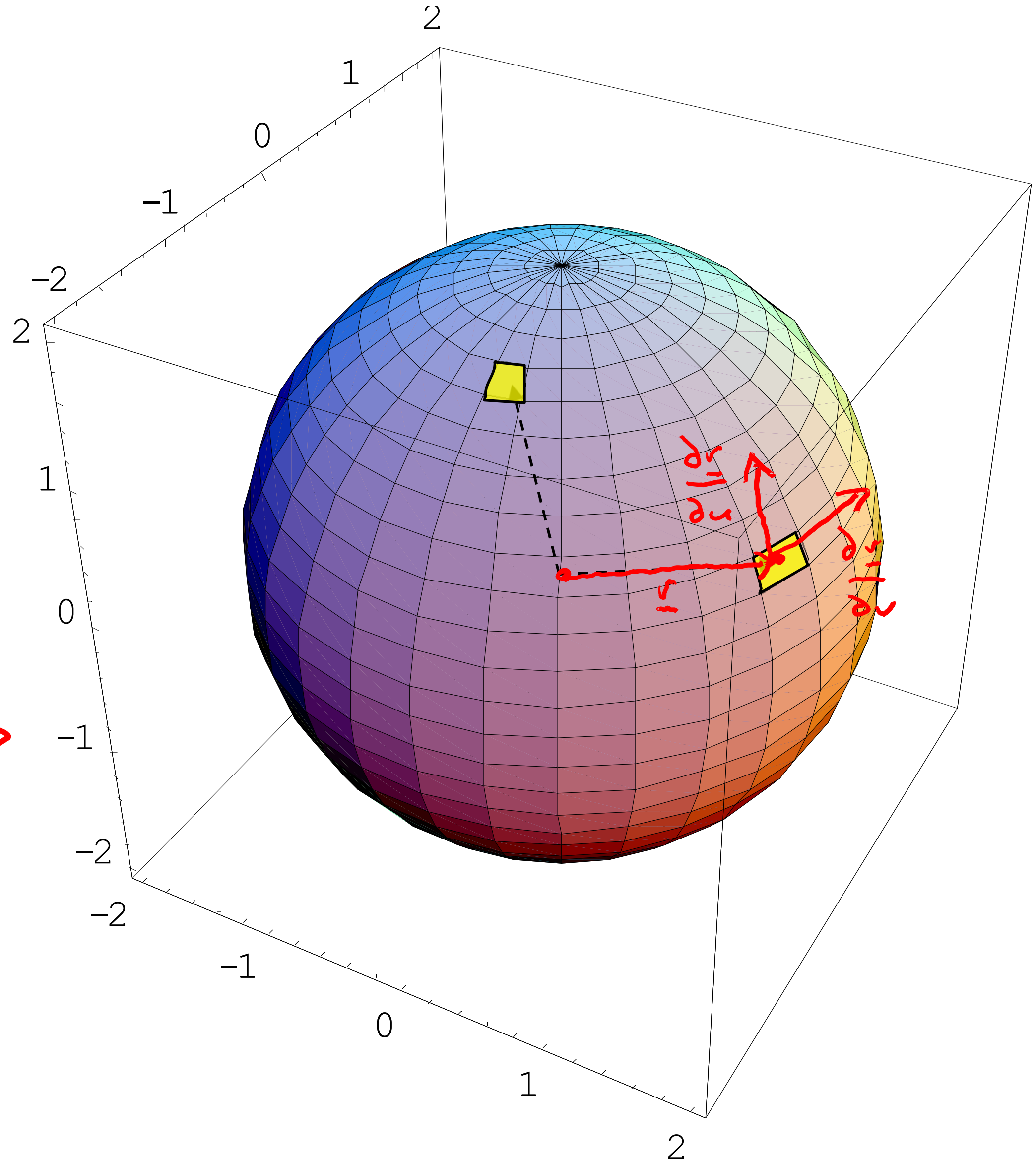
$$\hat{\underset{\sim}{n}} = \hat{\underset{\sim}{r}}$$

For parametric surfaces, we can obtain d**S** from the parameters u and v:

$$
\begin{aligned}
x &= x(u,v) \\
y &= y(u,v) \\
z &= z(u,v)
\end{aligned}
$$

$\underset{\sim}{r} = \langle x, y, z \rangle$

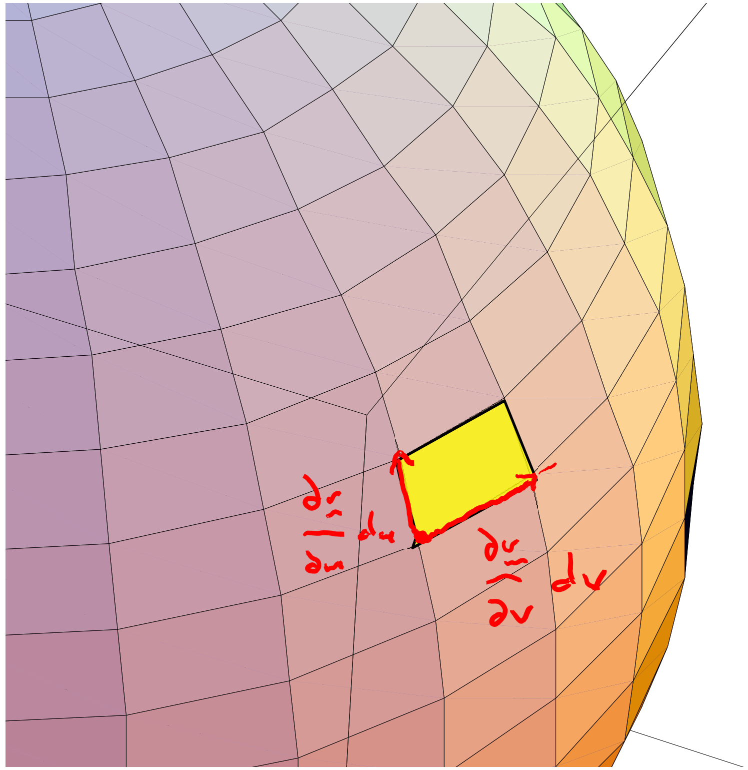
$= \langle \underline{x(u,v)}, y(u,v), z(u,v) \rangle$



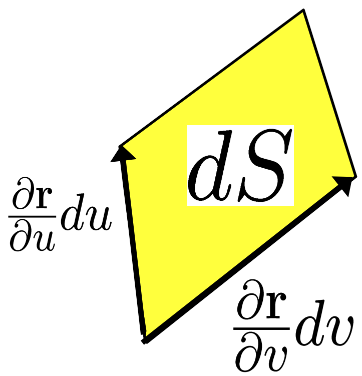
For <u>parametric</u> surfaces, we can obtain dS from the parameters u and v:

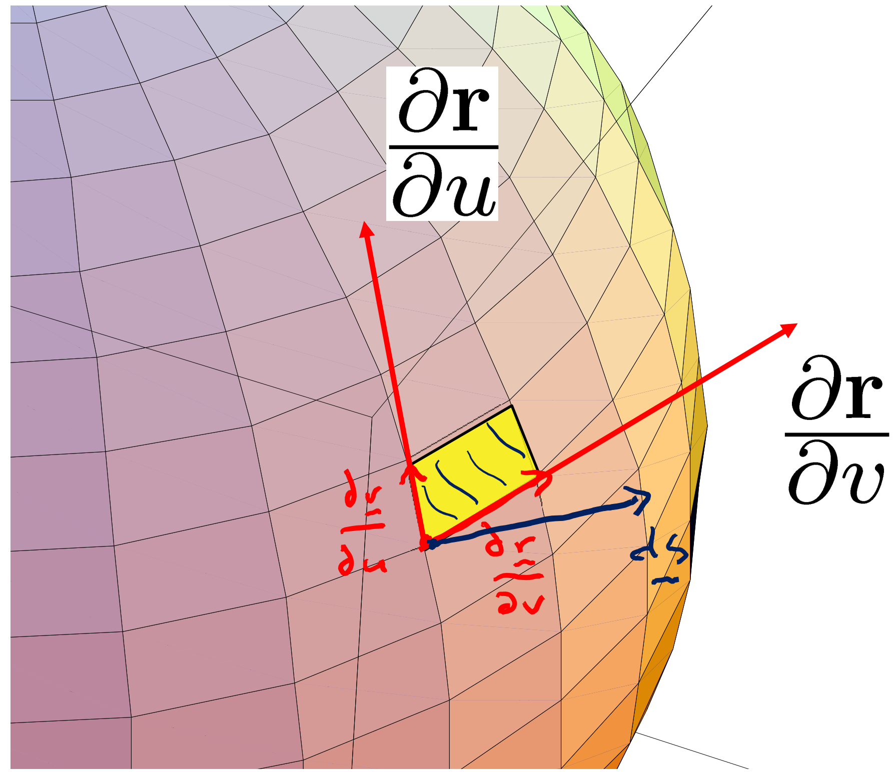
$$\begin{aligned} x &= x(u,v) \\ y &= y(u,v) \\ z &= z(u,v) \end{aligned}$$

$$\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$$



$$\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$$



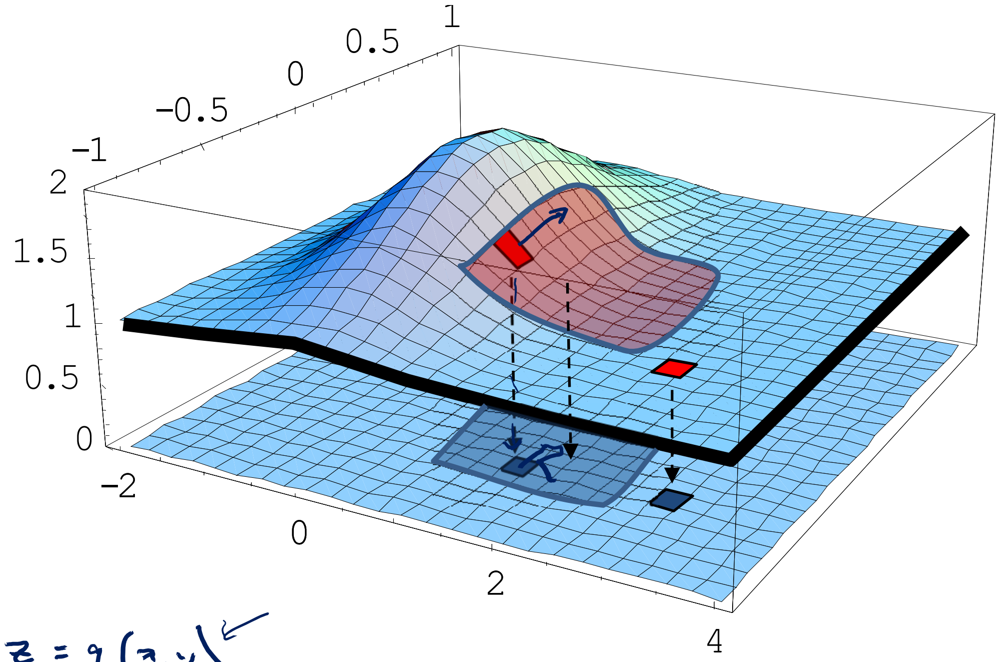


$$\boxed{\mathrm{d}\mathbf{S} = \frac{\partial \mathbf{r}}{\partial v} dv \times \frac{\partial \mathbf{r}}{\partial u} du}$$

Area $dS = \underbrace{\left|\frac{\partial \mathbf{r}}{\partial v} dv \times \frac{\partial \mathbf{r}}{\partial u} du\right|}$ $= \left| \frac{\partial r}{\partial v} \times \frac{\partial r}{\partial u} \right| du\,dv$

The surface integral of a scalar function f(x,y,z) over a surface S is defined to be

$$\iint_S f(x,y,z)\,dS = \lim_{\Delta S \to 0} \sum_{\text{all } \Delta S} f(x,y,z)\,\Delta S$$



Where dS is the magnitude of the infinitessimal area element $d\boldsymbol{S}$

For explicit surfaces $z = g(x,y)$

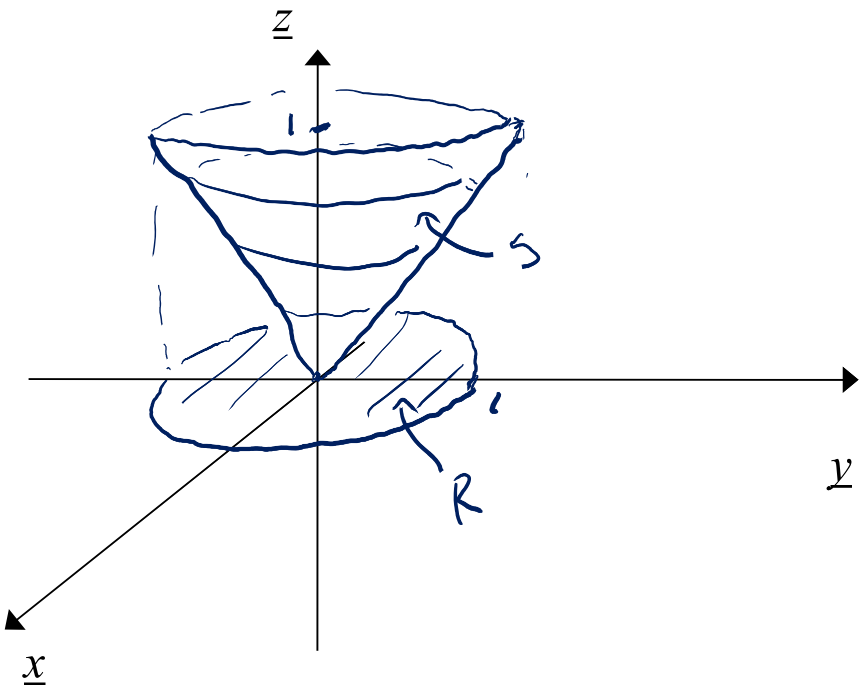
$$\iint_S f\,dS = \iint_R f(x,y,g(x,y))\sqrt{1+\left(\frac{\partial g}{\partial x}\right)^2+\left(\frac{\partial g}{\partial y}\right)^2}\,dx\,dy.$$

Area

where R is the projection of S onto the x-y plane.

Example: Find the surface area of the cone

$$z = \underbrace{\sqrt{x^2 + y^2}}_{g(x,y)} \quad \text{with} \quad 0 \le z \le 1 \; .$$



Surface area is

$$SA = \iint_S 1 \, dS$$

Now

$$dS = \sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2} \, dx\,dy$$

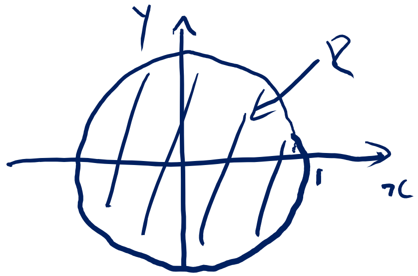
$$g = (x^2+y^2)^{\frac{1}{2}} \implies \frac{\partial g}{\partial x} = \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}} \times 2x = \frac{x}{\sqrt{x^2+y^2}} \qquad \frac{\partial g}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} .$$

So

$$dS = \sqrt{1 + \frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2}} \, dx\,dy = \underline{\sqrt{2} \, dx\,dy}$$

Then

$$\iint_S dS = \iint_R \sqrt{2}\, dx\,dy$$



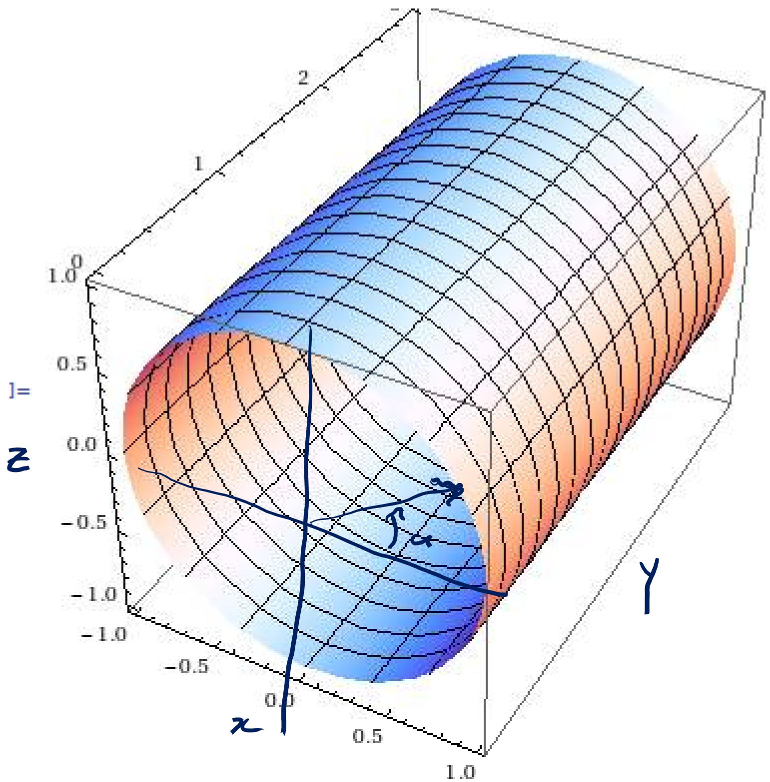
$$= \sqrt{2} \underbrace{\iint_R 1 . dx\,dy}$$

$$= \sqrt{2} \times (\text{Area of } R)$$

$$= \sqrt{2}\pi .$$

Example: integrate $f(x, y, z) = y^2$ over the surface S, defined as

$$\mathbf{r}(u, v) = \langle \cos u, v, \sin u \rangle \qquad 0 \le u \le 2\pi, \quad 0 \le v \le 3$$



Example: integrate $f(x, y, z) = y^2$ over the surface *S*, defined as

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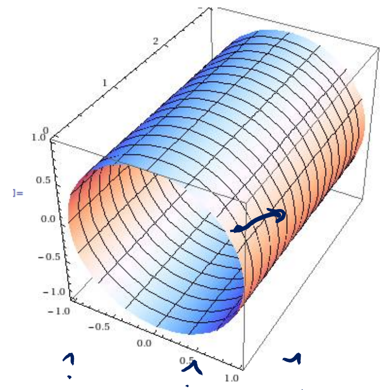


The area element $\underset{\sim}{dS}$ is

$$\underset{\sim}{dS} = \frac{\partial \underset{\sim}{r}}{\partial v} \times \frac{\partial \underset{\sim}{r}}{\partial u} \, du\,dv$$

Now,

$$\frac{\partial \underset{\sim}{r}}{\partial v} = \langle 0, 1, 0 \rangle$$

$$\frac{\partial \underset{\sim}{r}}{\partial u} = \langle -\sin u, 0, \cos u \rangle$$

So

$$\frac{\partial \underset{\sim}{r}}{\partial v} \times \frac{\partial \underset{\sim}{r}}{\partial u} = \begin{vmatrix} \hat{\underset{\sim}{i}} & \hat{\underset{\sim}{j}} & \hat{\underset{\sim}{k}} \\ 0 & 1 & 0 \\ -\sin u & 0 & \cos u \end{vmatrix}$$

$$= \hat{\underset{\sim}{i}} \cos u + \hat{\underset{\sim}{k}} \sin u$$

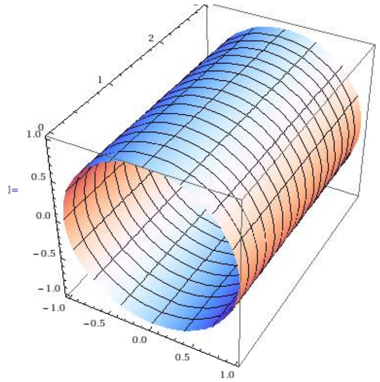
So

$$\underline{dS} = |\underset{\sim}{dS}| = |\hat{\underset{\sim}{i}} \cos u + \hat{j} \sin u| \, du\,dv$$

$$= \sqrt{\cos^2 u + \sin^2 u} \, du\,dv = 1.\,du\,dv = \underline{du\,dv}.$$

Example: integrate $f(x,y,z) = y^2$ over the surface *S*, defined as

$$\mathbf{r}(u,v) = \langle \cos u, v, \sin u \rangle \qquad \underline{0 \le u \le 2\pi}, \quad \underline{0 \le v \le 3}$$



So

$$\iint_S y^2 \, dS = \int_0^3 \int_0^{2\pi} y^2 \, du\, dv$$

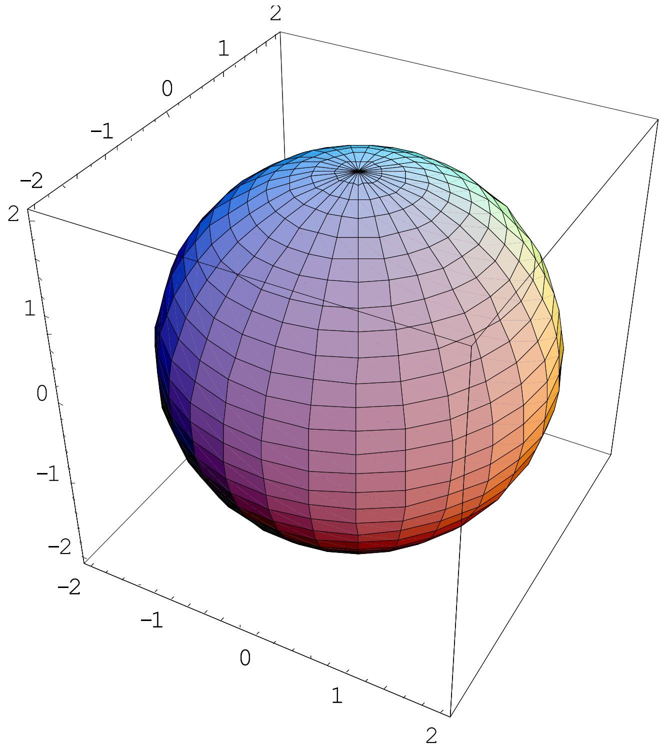
$$= \int_0^3 \int_0^{2\pi} v^2 \, du\, dv$$

$$= \int_0^3 \left[v^2 u \right]_0^{2\pi} dv = \int_0^3 v^2(2\pi - 0)\, dv$$

$$= 2\pi \int_0^3 v^2 \, dv = 2\pi \left[\frac{v^3}{3} \right]_0^3 = 2\pi \times \left(\frac{27}{3} \right)$$

$$= 18\pi \qquad \square.$$

Example: Compute the surface area of a sphere of radius R.



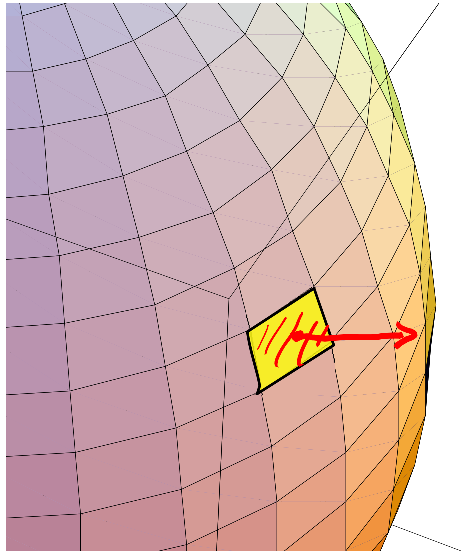
Ans: $\frac{4}{3}\pi R^3$.

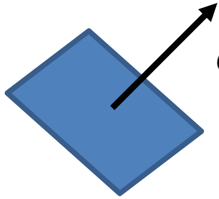
Flux integrals

The integral of a vector field F over a surface integral S is

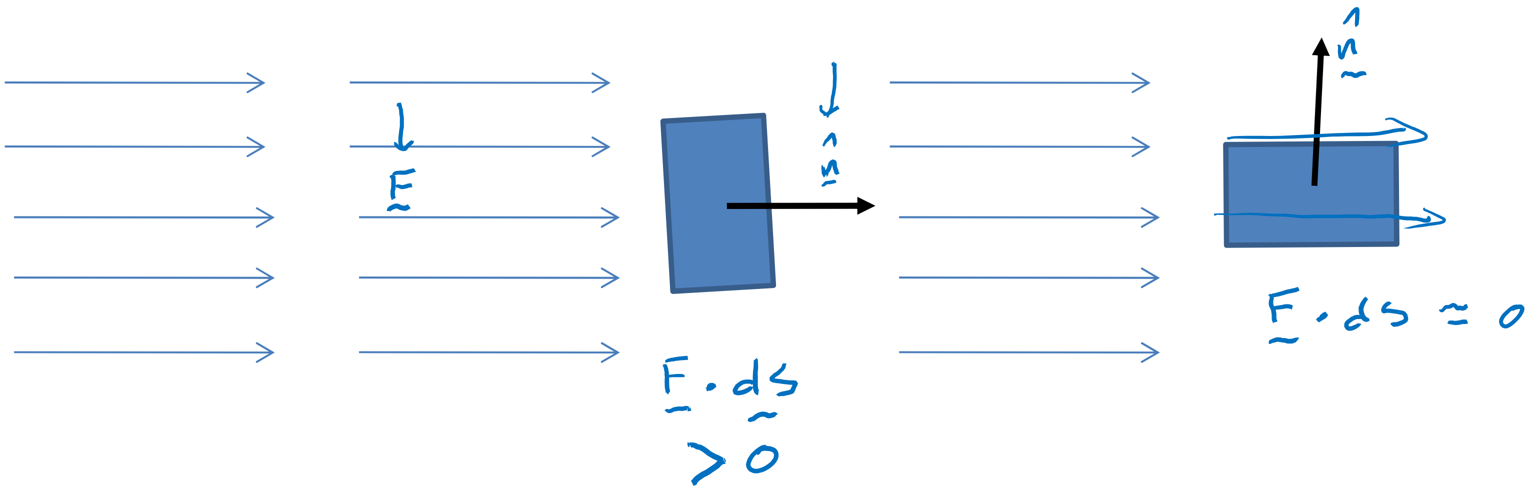
$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS$$

and is known as a *flux* integral.

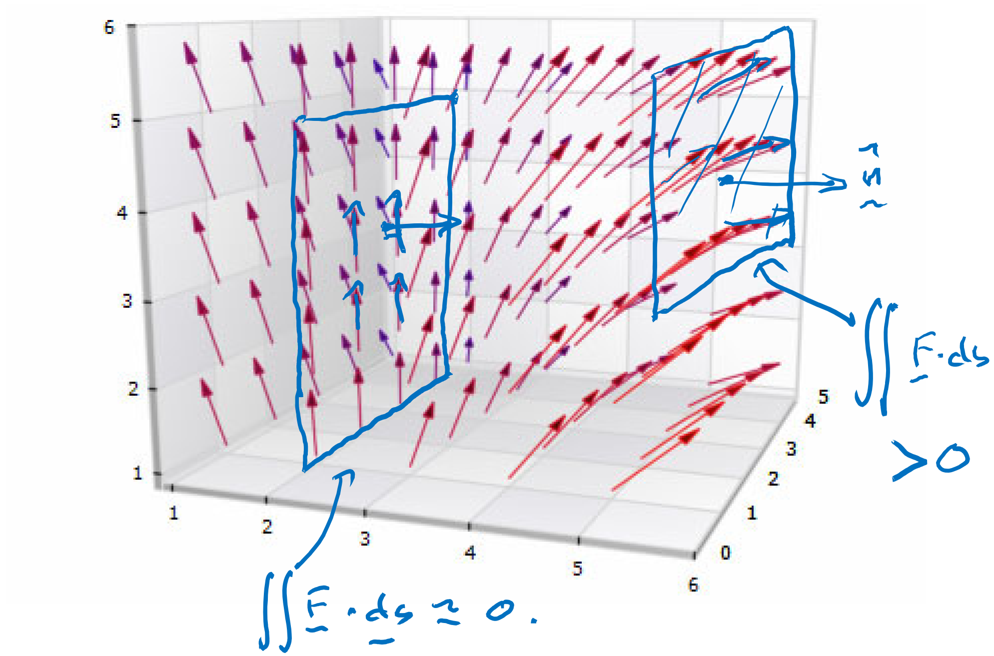


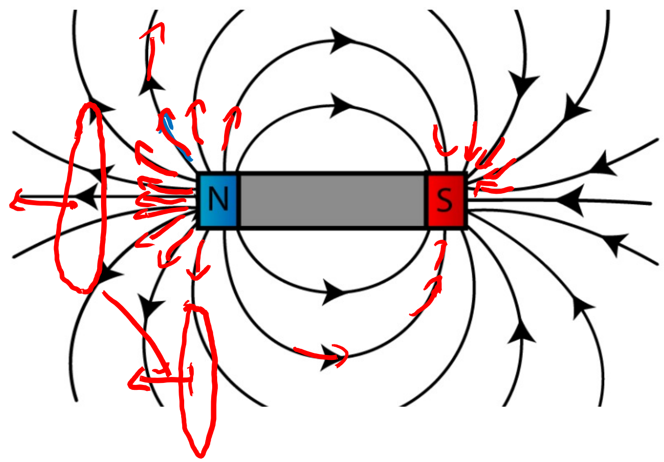


$$d\mathbf{S} = \hat{\mathbf{n}} dS$$



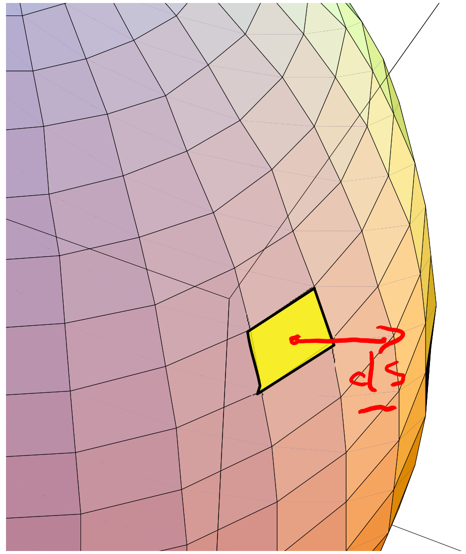
The flux integral tells you *how much the field in 3D "goes through" a surface.*





To compute a flux integral:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \underline{\mathbf{F} \cdot \hat{\mathbf{n}}}\, dS$$



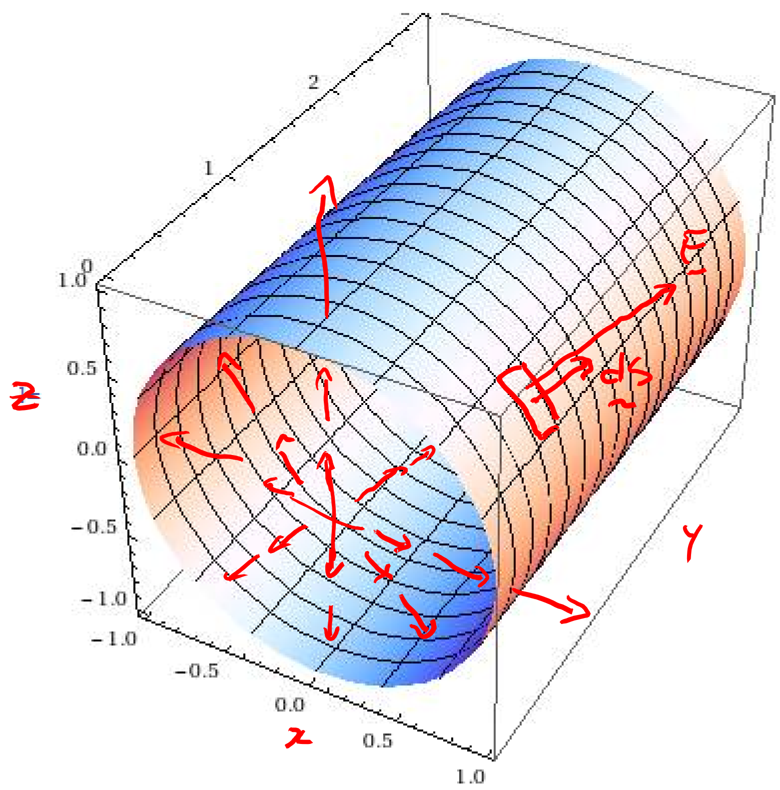
1. Compute the surface element $d\mathbf{S}$

2. Put $\mathbf{F}$ and $d\mathbf{S}$ **in the same 2D coordinate system** and form the dot product between them

3. Integrate over the coordinates.

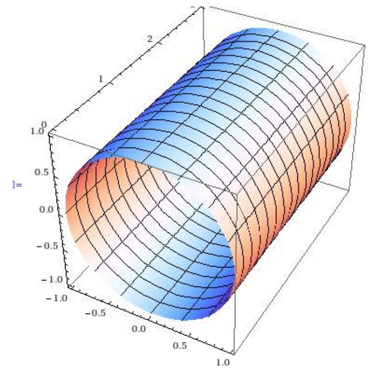
Example: Integrate the vector field $\underbrace{\langle 2x, 1, 2z \rangle}_{\underset{\sim}{F}}$ over the outward-oriented surface

$$\mathbf{r}(u,v) = \langle \cos u, v, \sin u \rangle \qquad 0 \le u \le 2\pi, \quad 0 \le v \le 3$$



Example: Integrate the vector field F = <2x,1,2z>
over the outward-oriented surface

$$\mathbf{r}(u,v) = \langle \cos u, v, \sin u \rangle \qquad 0 \le u \le 2\pi, \quad 0 \le v \le 3$$



$$\iint_S \underset{\sim}{F} \cdot d\underset{\sim}{S} = ?$$

Now $$\boxed{d\underset{\sim}{S} = \frac{\partial \underline{r}}{\partial v} \times \frac{\partial \underline{r}}{\partial u}\, du\, dv}$$

$$= \begin{vmatrix} \hat{\underline{i}} & \hat{\underline{j}} & \hat{\underline{k}} \\ 0 & 1 & 0 \\ -\sin u & 0 & \cos u \end{vmatrix} du\,dv = \hat{\underline{i}} \cos u - \hat{\underline{j}}\, 0 + \hat{\underline{k}} \sin u = (\hat{\underline{i}} \cos u + \hat{\underline{k}} \sin u)\, du\,dv$$

$$= \langle \cos u, 0, \sin u \rangle\, du\,dv$$

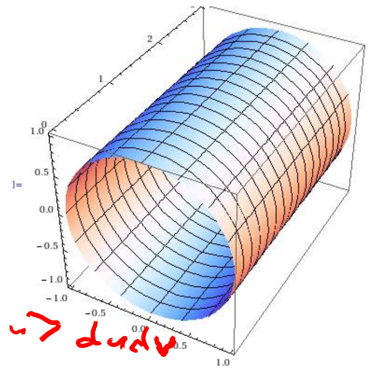
$$\underset{\sim}{F} = \langle 2x, 1, 2z \rangle$$

$$= \langle 2\cos u, 1, 2\sin u \rangle$$

$$\iint_{dS} \underline{F} \cdot d\underset{\sim}{S} = \int_0^3 \int_0^{2\pi} \langle 2\cos u, 1, 2\sin u \rangle \cdot \langle \cos u, 0, \sin u \rangle\, du\,dv$$

Example: Integrate the vector field F = <2x,1,2z> over the outward-oriented surface

$$\mathbf{r}(u,v) = \langle \cos u, v, \sin u \rangle \qquad 0 \le u \le 2\pi, \quad 0 \le v \le 3$$



So

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \int_0^3 \int_0^{2\pi} \langle 2\cos u, 1, 2\sin u \rangle \cdot \langle \cos u, 0, \sin u \rangle \, du\, dv$$

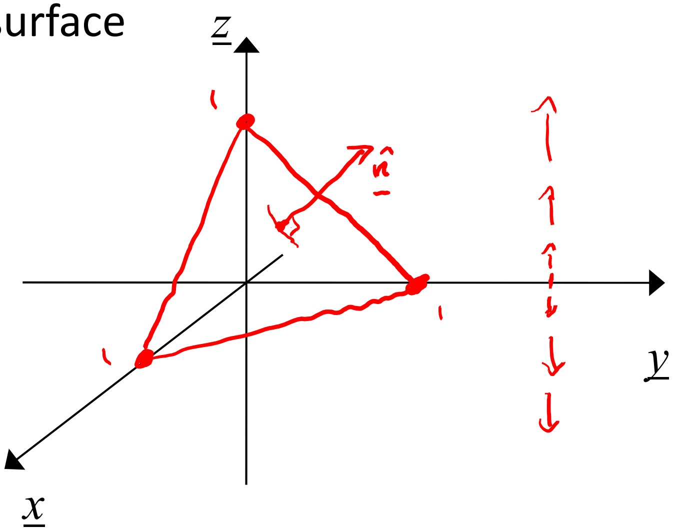
$$= \int_0^3 \int_0^{2\pi} \left(2\cos^2 u + 0 + 2\sin^2 u \right) du\, dv$$

$$= \int_0^3 \int_0^{2\pi} 2 \, du\, dv = \int_0^3 \left[2u \right]_0^{2\pi} dv$$

$$= \int_0^3 4\pi \, dv = 4\pi \int_0^3 dv$$

$$= 12\pi$$

Example: Find the integral of $\boldsymbol{F} = < 0,0,z >$ over the surface $x + y + z = 1,\ x > 0, y > 0, z > 0$.



(choose $u = x$, $v = z$

then $y = 1 - x - z$

then $\underline{r} = < u, 1-u-v, v >$)

Here the surface is

$$z = g(x,y) = 1 - x - y.$$

The normal is

$$\underline{n} = \nabla(z - g(x,y)) = < -\frac{\partial g}{\partial x}, -\frac{\partial g}{\partial y}, 1 >$$

$$= < 1, 1, 1 >$$

The unit normal is

$$\hat{\underline{n}} = \frac{\underline{n}}{|\underline{n}|} = \frac{1}{\sqrt{3}} < 1, 1, 1 >$$

$$\hat{\underline{n}} \cdot \hat{\underline{k}} = \frac{1}{\sqrt{3}}$$

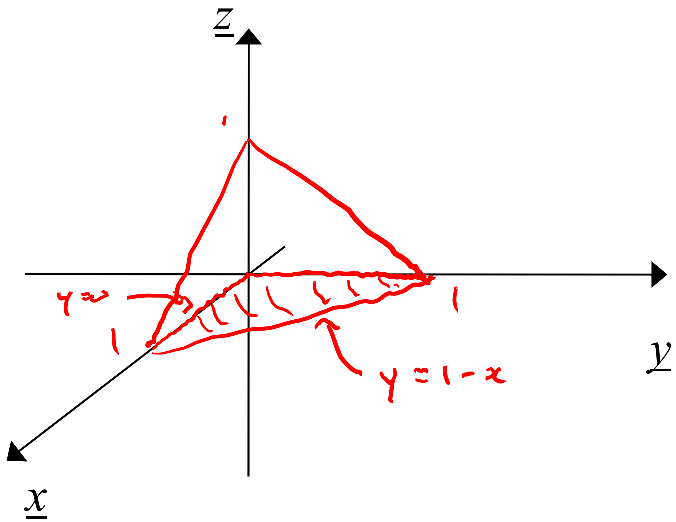
Now

$$dxdy = |\hat{\underline{n}} \cdot \hat{\underline{k}}| \, ds \Rightarrow ds = \frac{dxdy}{|\hat{\underline{n}} \cdot \hat{\underline{k}}|} = \frac{dxdy}{\frac{1}{\sqrt{3}}} = \sqrt{3}\,dxdy!$$

$$dS = \sqrt{3}\, dx\, dy$$

and

$$\hat{n} = \frac{1}{\sqrt{3}} \langle 1, 1, 1 \rangle$$



So

$$d\underset{\sim}{S} = \hat{n}\, dS$$

$$= \frac{1}{\cancel{\sqrt{3}}} \langle 1, 1, 1 \rangle \cancel{\sqrt{3}}\, dx\, dy$$

$$= \langle 1, 1, 1 \rangle\, dx\, dy .$$

$$z = g(x,y) = 1 - x - y$$

So

$$\iint_S \underset{\sim}{F} \cdot d\underset{\sim}{S} = \iint \langle 0, 0, z \rangle \cdot \langle 1, 1, 1 \rangle\, dx\, dy$$

$$= \iint z\, dx\, dy = \int_0^1 \int_0^{1-x} \underline{(1 - x - y)}\, dy\, dx .$$

$$= \int_0^1 \left[-\frac{1}{2} (1 - x - y)^2 \right]_0^{1-x} dx = \int_0^1 \left(-\frac{1}{2} \left(\cancel{1 - x} - (\cancel{1 - x}) \right)^2 + \frac{1}{2} (1 - x - 0)^2 \right) dx$$

$$= \frac{1}{2} \int_0^1 (1 - x)^2\, dx = \frac{1}{2} \left[-\frac{1}{3} (1 - x)^3 \right]_0^1 = \frac{1}{2} \left(-\cancel{\frac{1}{3}(0)} + \frac{1}{3} \right) = \frac{1}{6} .$$

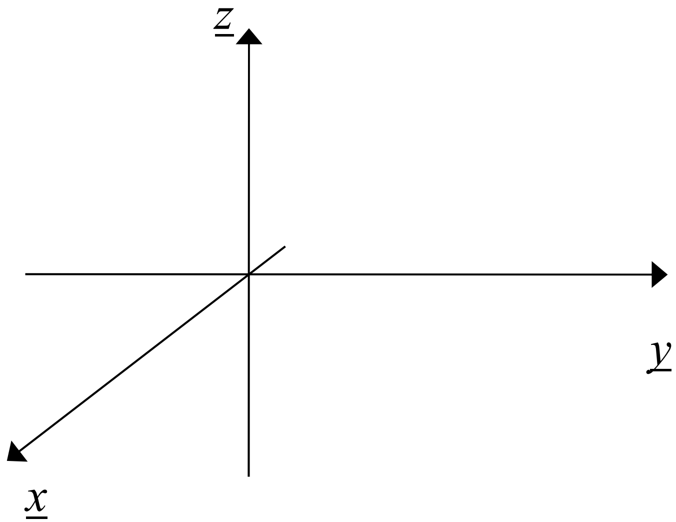
Example: integrate

$$\mathbf{F}(x, y, z) = \left\langle x^2 + y^2, x^2 + y^2, 0 \right\rangle$$

over the half-cylinder

$$\mathbf{r}(u, v) = \langle 3\cos u, 3\sin u, v \rangle$$

with $0 \leq u \leq \pi$, and $-1 \leq v \leq 1$.



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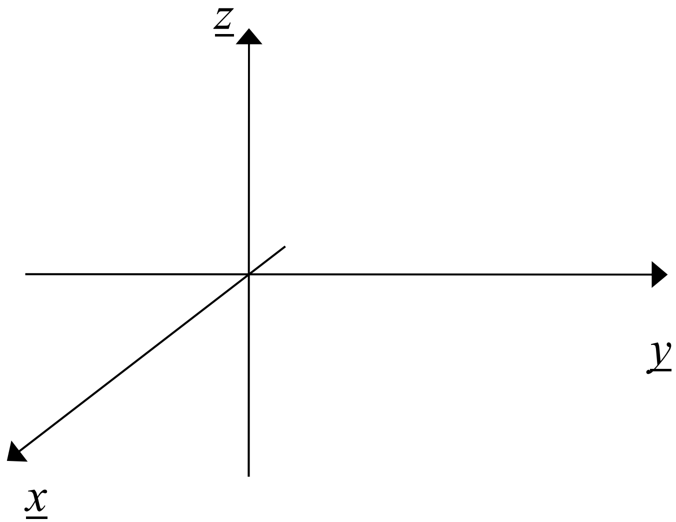


Compute the flux integral

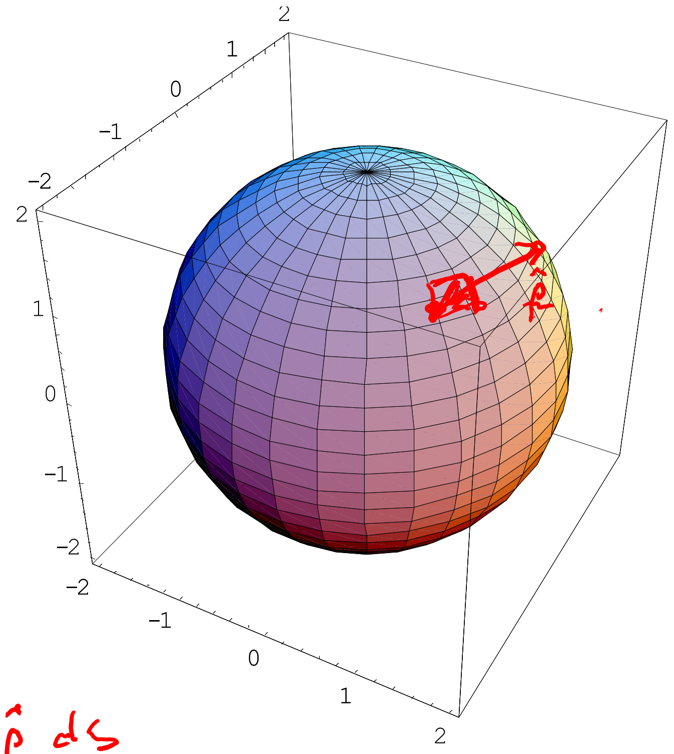
$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\; dS$$

$= \hat{\rho}\, dS$

Where

$$\mathbf{F} = \rho\sin\theta\hat{\boldsymbol{\rho}} + \rho\cos\theta\hat{\boldsymbol{\phi}}$$

And S is the surface of a sphere of radius R.



$$\iint_S \underset{\sim}{F}\cdot d\underset{\sim}{S} = \iint_S (\rho\sin\theta\,\hat{\rho} + \rho\cos\theta\,\hat{\phi})\cdot\hat{\rho}\; dS$$

$$= \iint_S \rho\sin\theta\; dS$$

Now

$$dS = \rho^2\sin\theta\, d\theta\, d\phi$$

So

$$\iint_S \underline{F}\cdot d\underline{S} = \int_0^{2\pi}\int_0^{\pi} \rho\sin\theta\;\rho^2\sin\theta\; d\theta\, d\phi = \int_0^{2\pi}\int_0^{\pi} R^3\sin^2\theta\; d\theta\, d\phi.$$

$$= \int_0^{2\pi} \int_0^{\pi} R^3 \sin^2\theta \, d\theta \, d\varphi$$

$$= R^3 \int_0^{2\pi} \int_0^{\pi} \frac{1}{2}\left(1 - \cos 2\theta\right) d\theta \, d\varphi$$

$$= R^3 \int_0^{2\pi} \left[\frac{\theta}{2} - \frac{1}{4} \sin 2\theta \right]_0^{\pi} d\varphi$$

$$= R^3 \int_0^{2\pi} \frac{\pi}{2} \, d\varphi = R^3 \frac{\pi}{2} \int_0^{2\pi} d\varphi$$

$$= R^3 \frac{\pi}{2} \times 2\pi = R^3 \pi^2$$