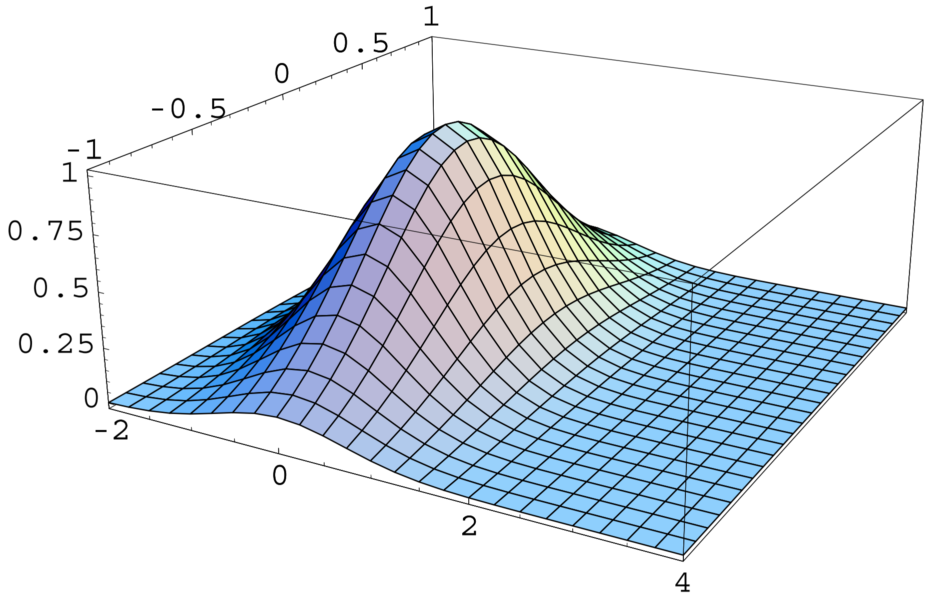
Surface and Flux Integrals

A surface can be defined in 3 ways:

1. Explicitly:

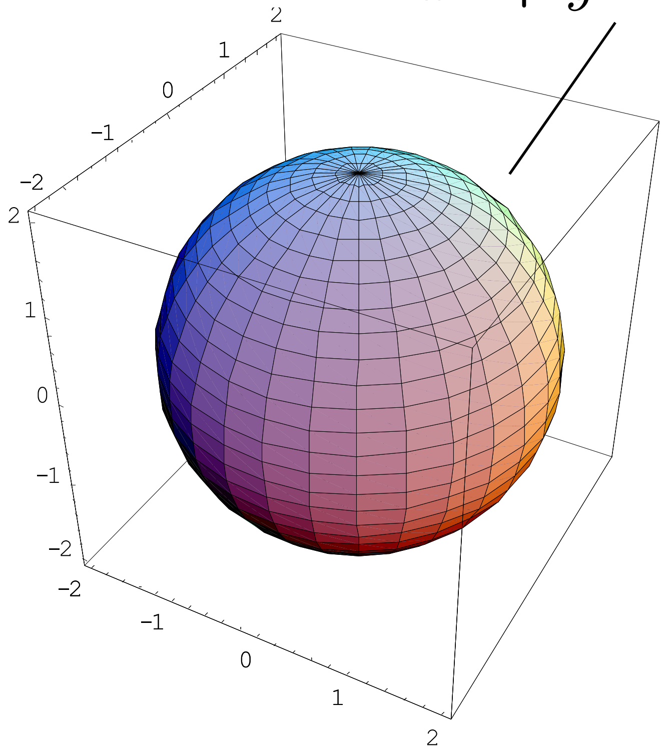
$$z = g(x, y)$$



2. Implicitly:

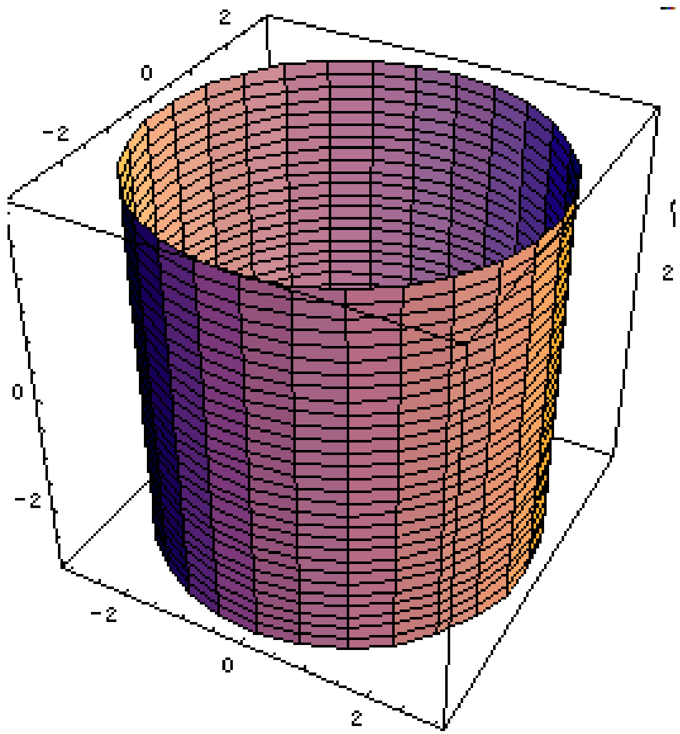
$$h(x, y, z) = 0$$

$$x^2 + y^2 + z^2 - 4 = 0$$



3. Parametrically:

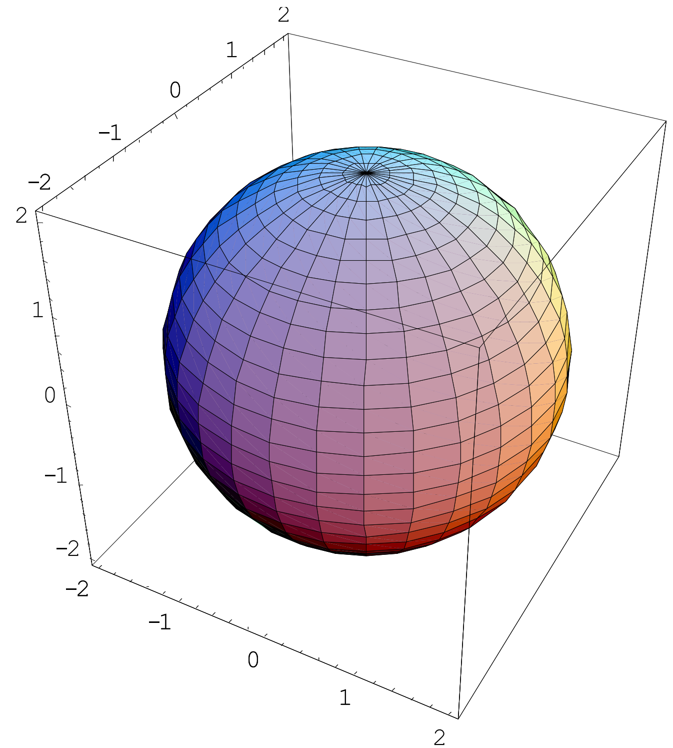
$$
\begin{aligned}
x &= x(u,v) \\
y &= y(u,v) \\
z &= z(u,v)
\end{aligned}
$$



$$
\begin{aligned}
x &= 2\cos v \\
y &= 2\sin v \\
z &= u
\end{aligned}
$$

3. Parametrically:

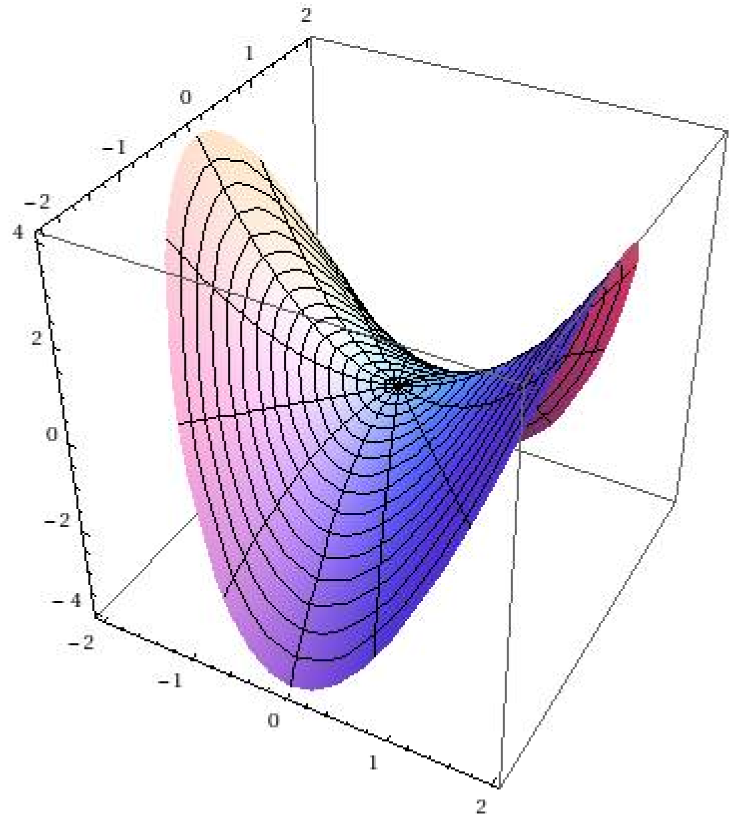
$$\begin{aligned} x &= x(u,v) \\ y &= y(u,v) \\ z &= z(u,v) \end{aligned}$$



$$\begin{aligned} x &= 2\cos v \cos u \\ y &= 2\sin v \cos u \\ z &= 2\sin u \end{aligned}$$

3. Parametrically:

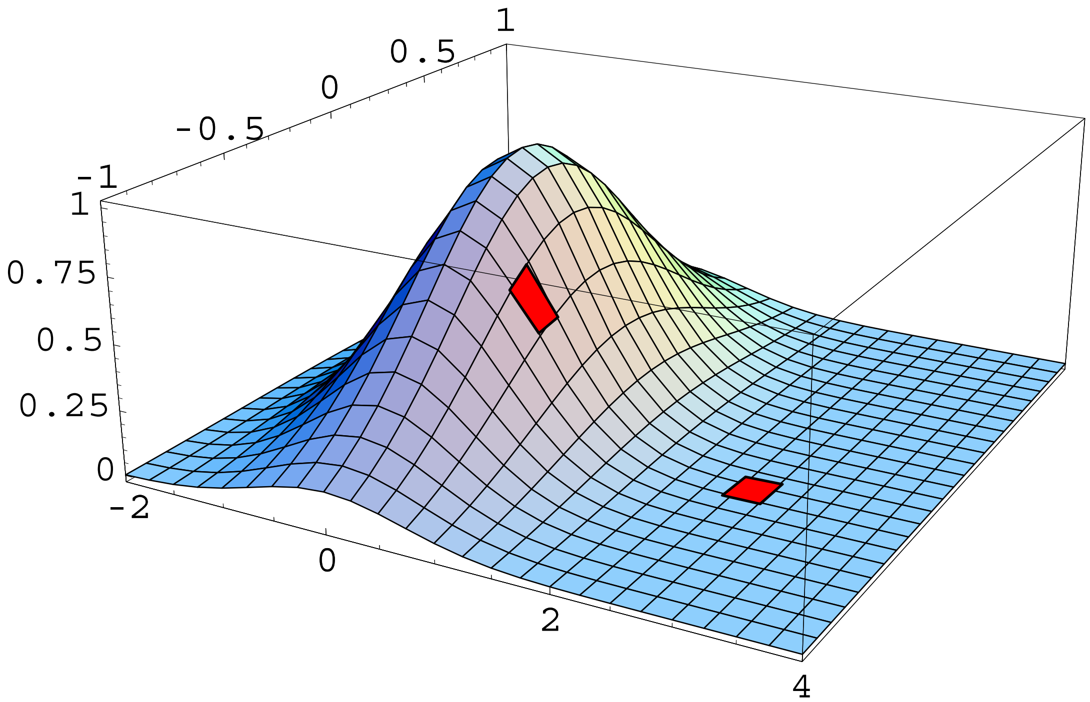
$$\begin{aligned} x &= x(u,v) \\ y &= y(u,v) \\ z &= z(u,v) \end{aligned}$$



$$\begin{aligned} x &= v\cos u \\ y &= v\sin u \\ z &= v^2\sin(2u) \end{aligned}$$

The surface element **dS**

The infinitessimal surface element dS Is a vector. Its magnitude is the *area* of the element and its *direction* is normal to the surface



The surface element $d\mathbf{S}$

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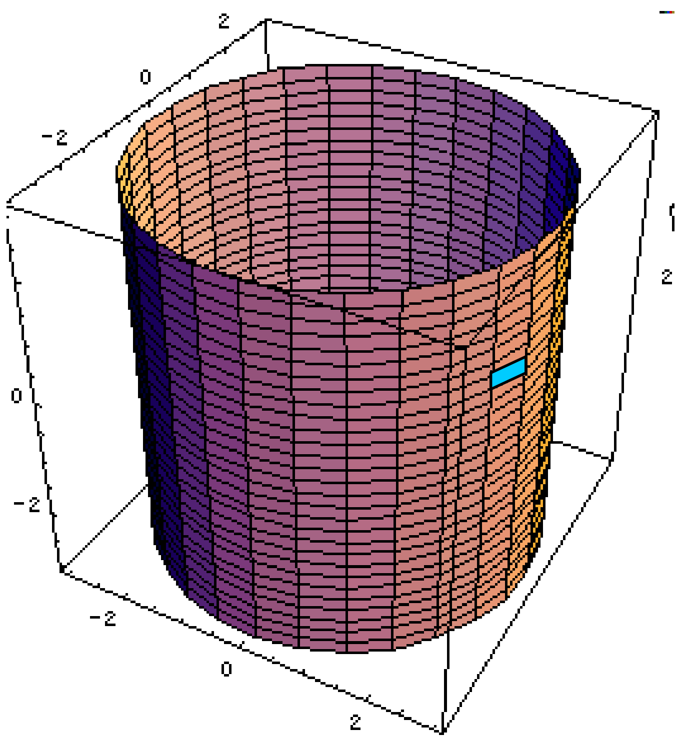

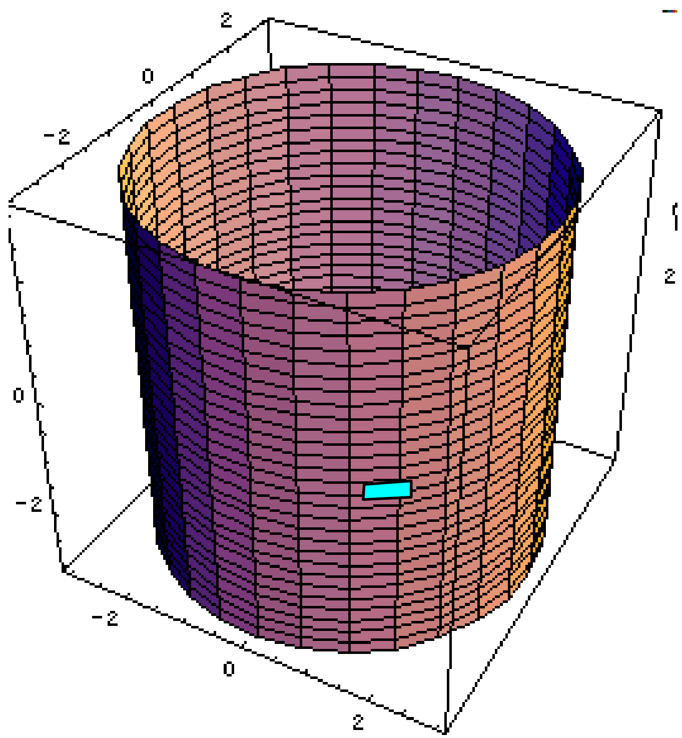

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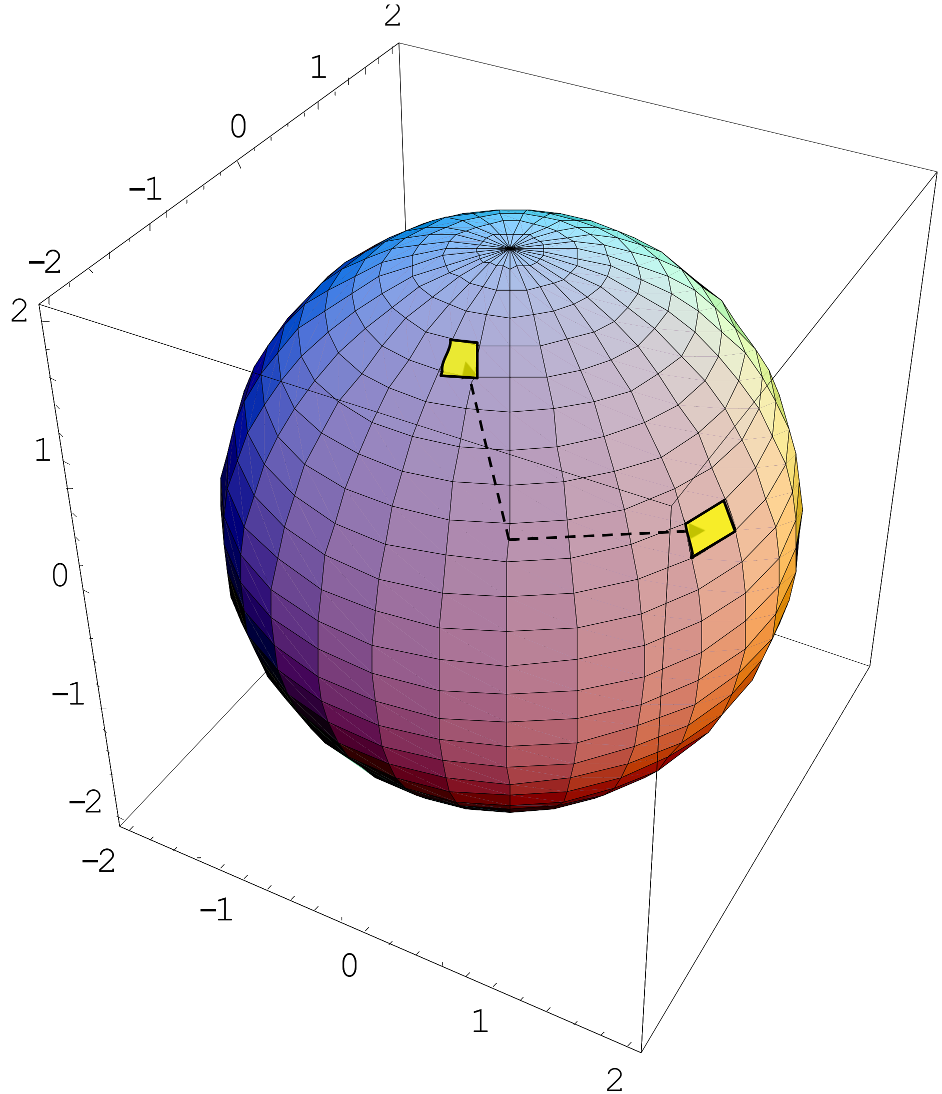
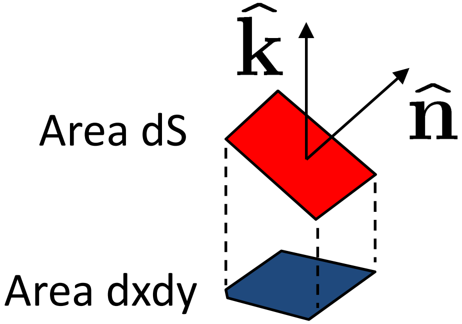
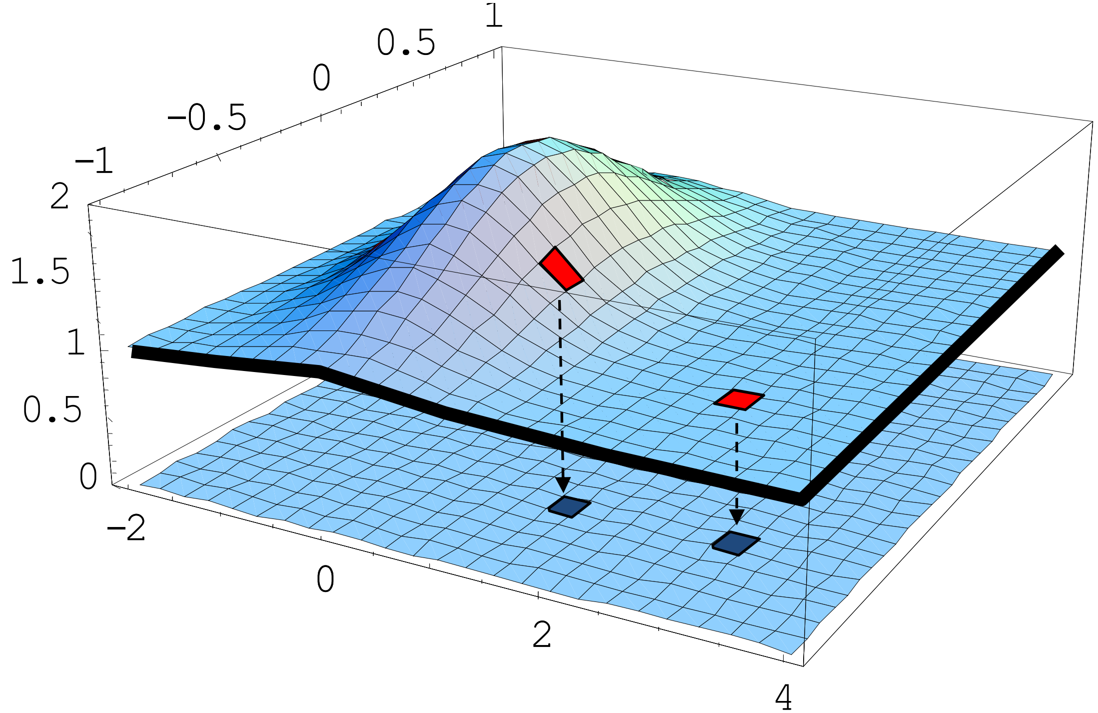


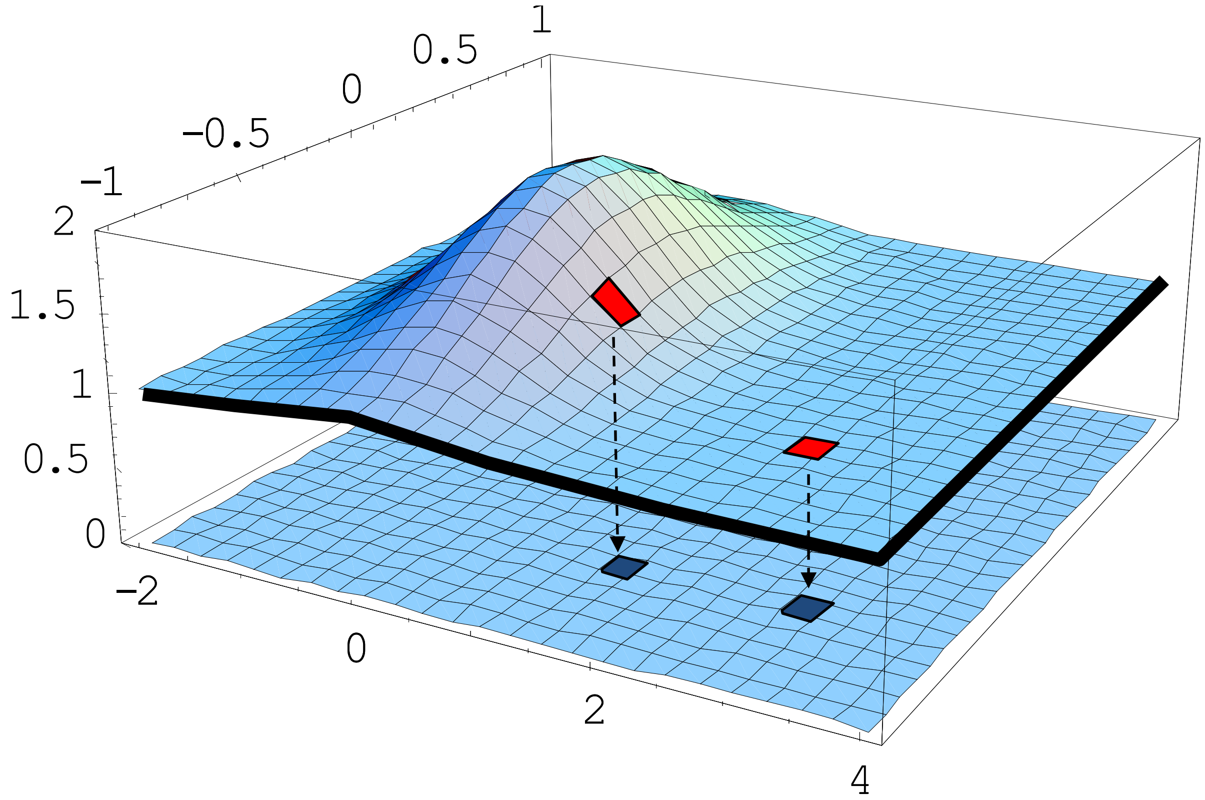
For <u>explicit</u> surfaces

$$z = g(x, y)$$

we can project onto the x-y plane:







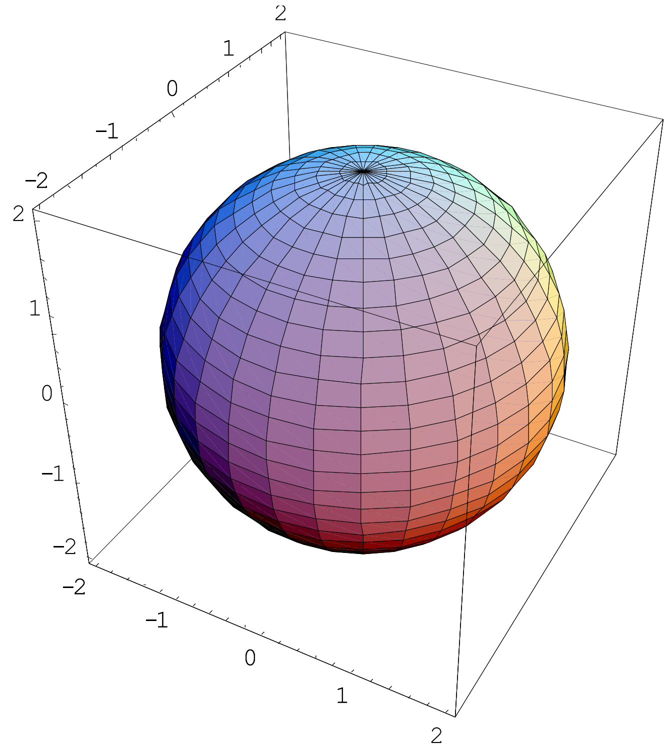
1
0.5
0
−0.5
−1
2
1.5
1
0.5
0
−2
0
2
4

For <u>implicit surfaces</u>

$$h(x, y, z) = 0$$

the unit normal is

$$x^2 + y^2 + z^2 - 4 = 0$$

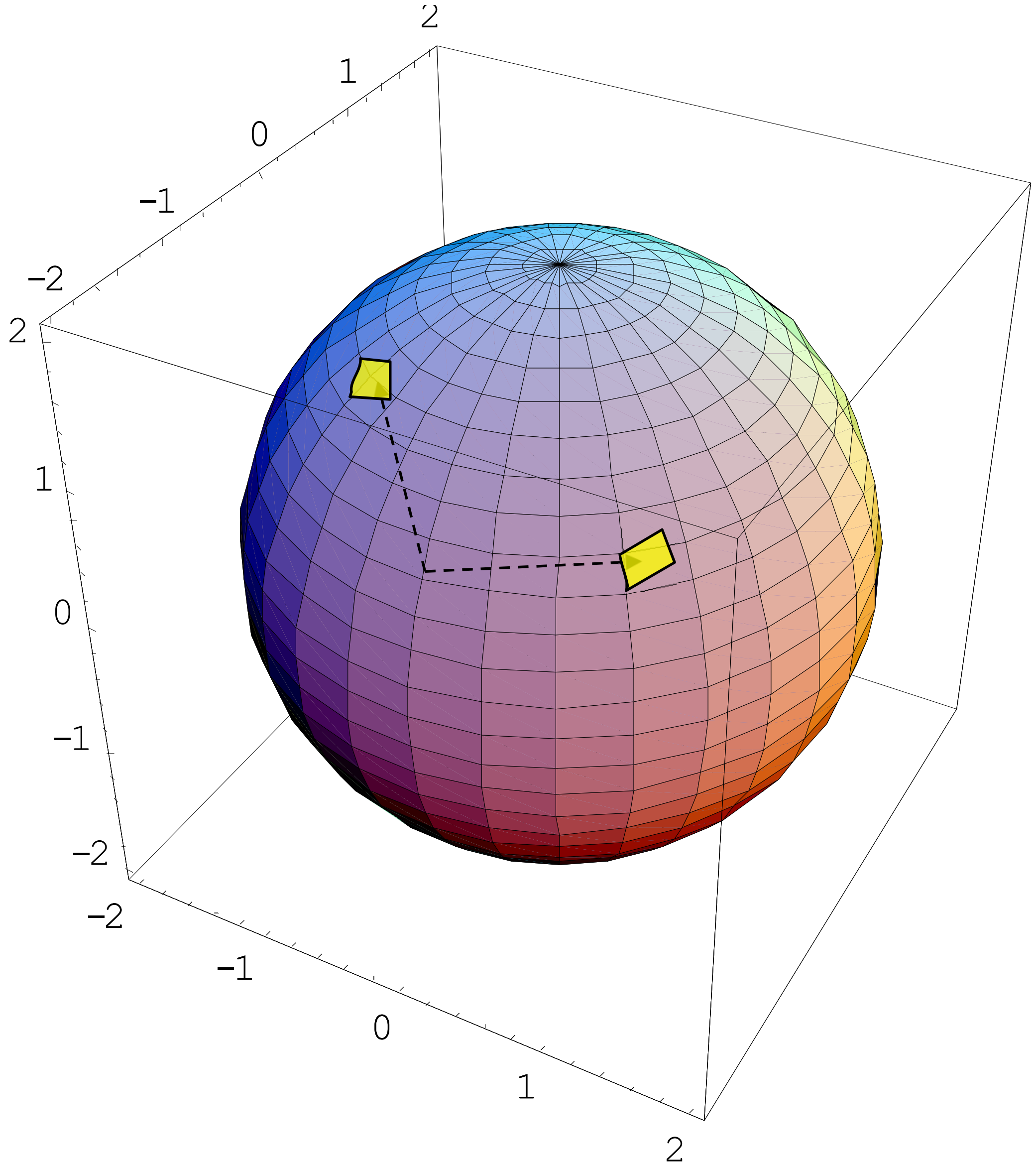


Important:

We can sometimes *guess* the normal. E.g. for the sphere above.

For parametric surfaces, we can obtain dS from the parameters u u and v:

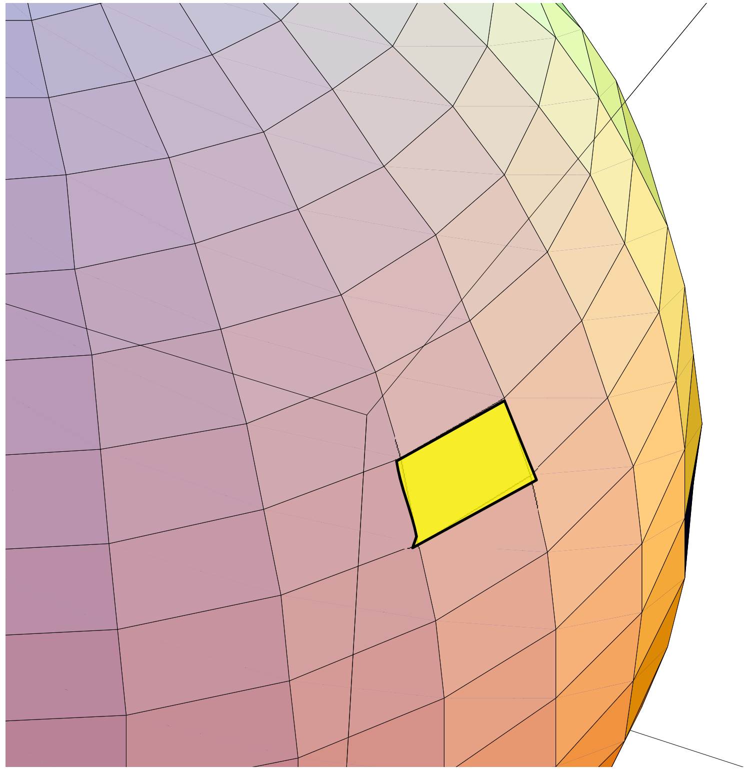
$$\begin{aligned} x &= x(u,v) \\ y &= y(u,v) \\ z &= z(u,v) \end{aligned}$$



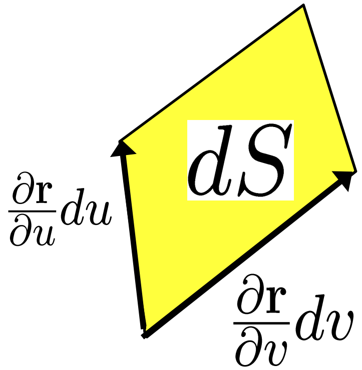
For parametric surfaces, we can obtain dS from the parameters u and v:

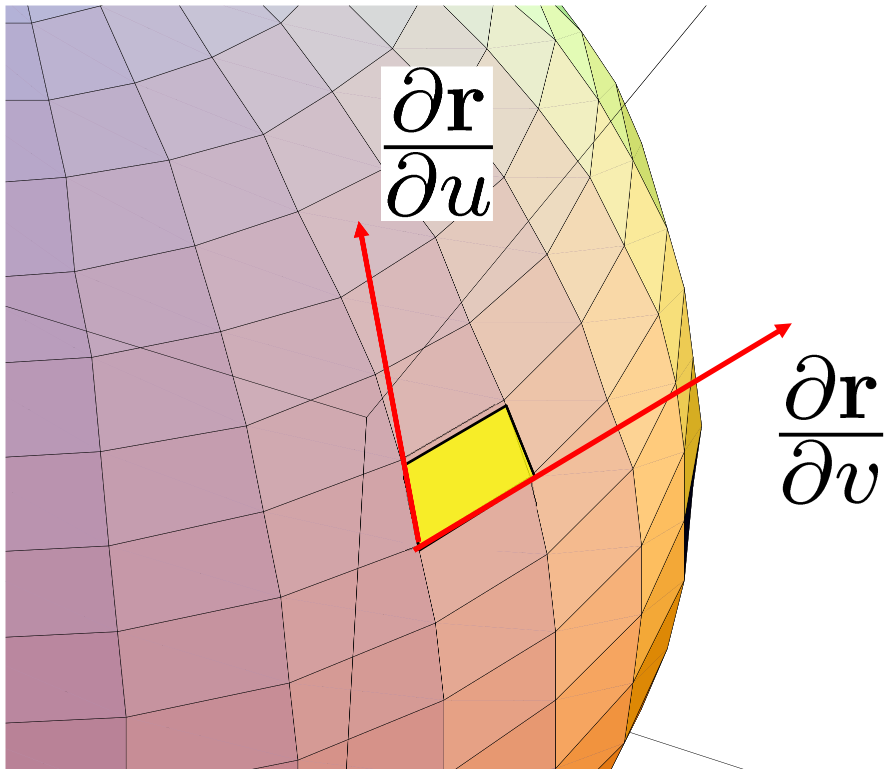
$$\begin{aligned} x &= x(u,v) \\ y &= y(u,v) \\ z &= z(u,v) \end{aligned}$$

$$\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$$



$$\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$$





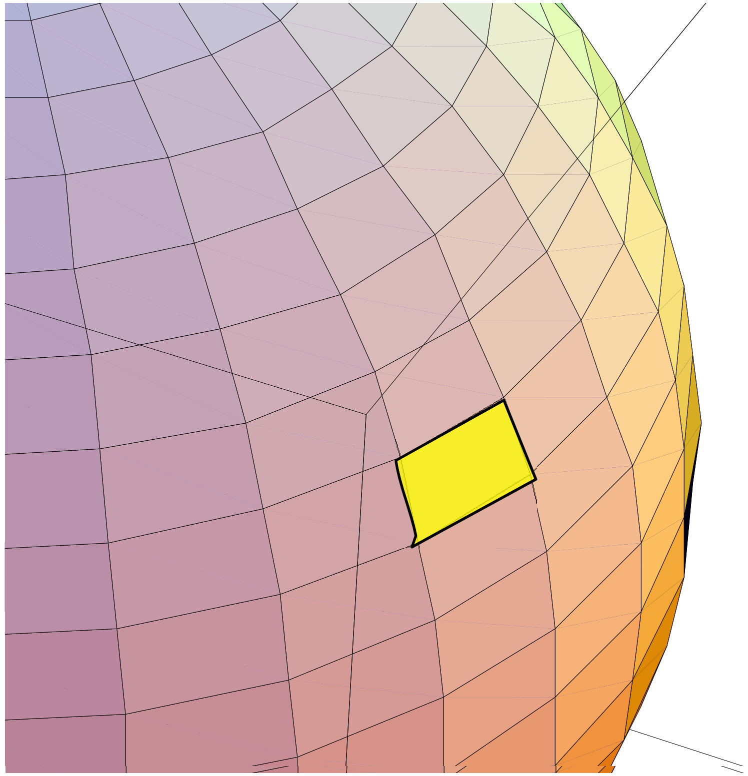
$$\mathrm{d}\mathbf{S} = \frac{\partial \mathbf{r}}{\partial v} dv \times \frac{\partial \mathbf{r}}{\partial u} du$$

Area $dS = \left|\frac{\partial \mathbf{r}}{\partial v} dv \times \frac{\partial \mathbf{r}}{\partial u} du\right|$

The surface integral of a scalar function f(x,y,z) over a surface S is defined to be

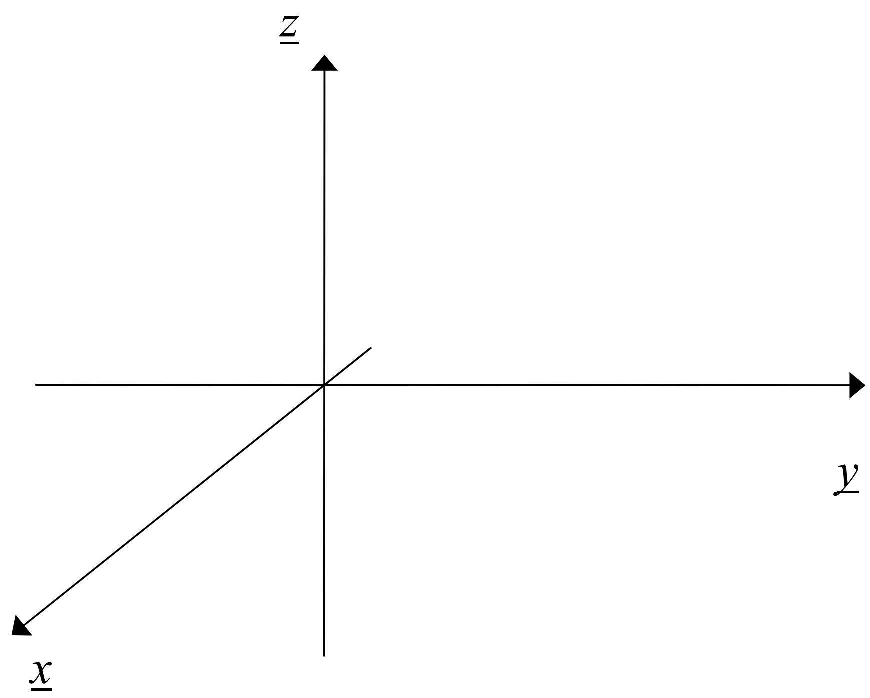
$$\iint_S f(x,y,z)dS = \lim_{\Delta S \to 0} \sum_{\text{all } \Delta S} f(x,y,z)\Delta S$$

Where dS is the magnitude of the infinitessimal area element $d\boldsymbol{S}$



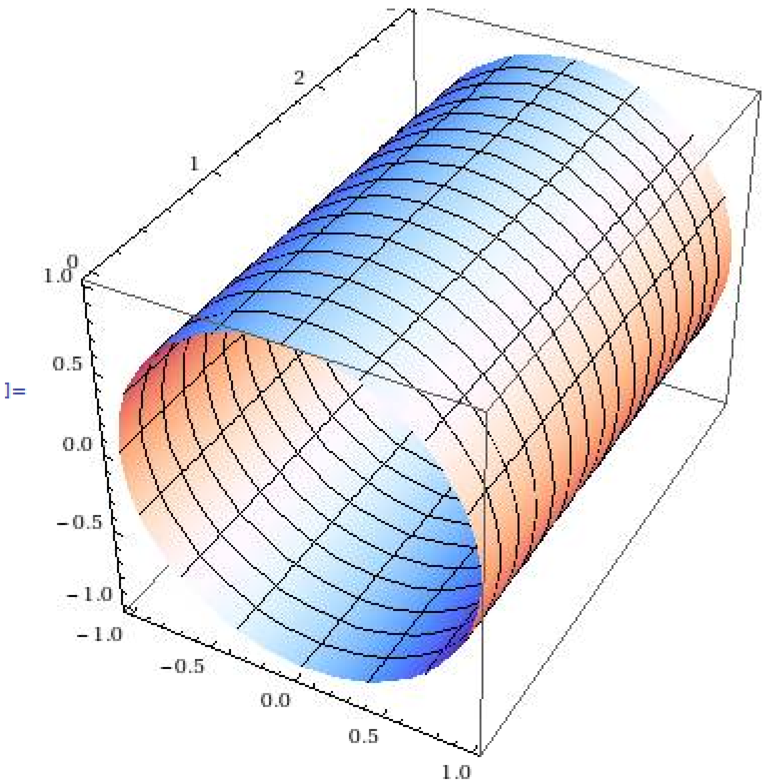
Example: Find the surface area of the cone

$$z = \sqrt{x^2 + y^2} \quad \text{with} \quad 0 \leq z \leq 1 \ .$$



Example: integrate $f(x, y, z) = y^2$ over the surface *S*, defined as

$$\mathbf{r}(u, v) = \langle \cos u, v, \sin u \rangle \qquad 0 \le u \le 2\pi, \quad 0 \le v \le 3$$



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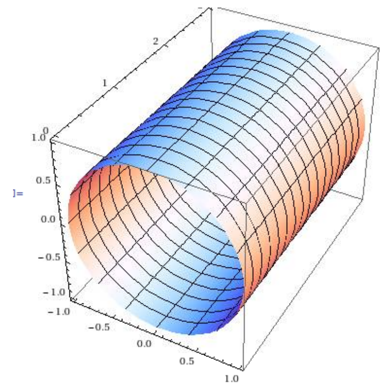

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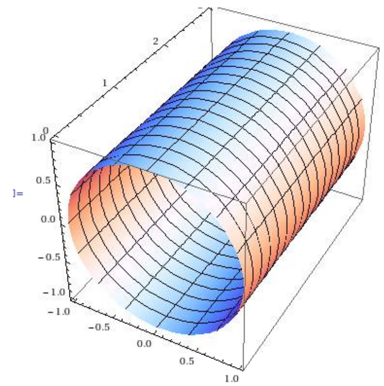
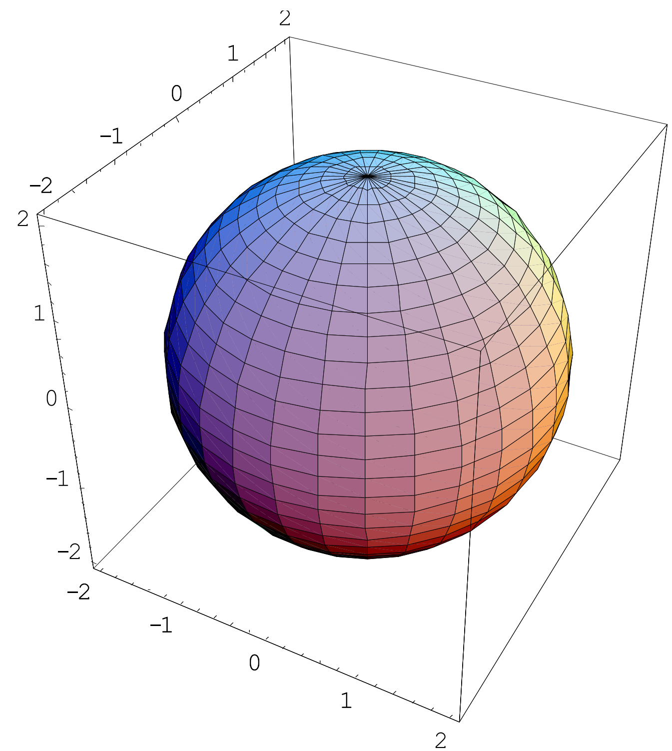


Example: Compute the surface area of a sphere of radius R.

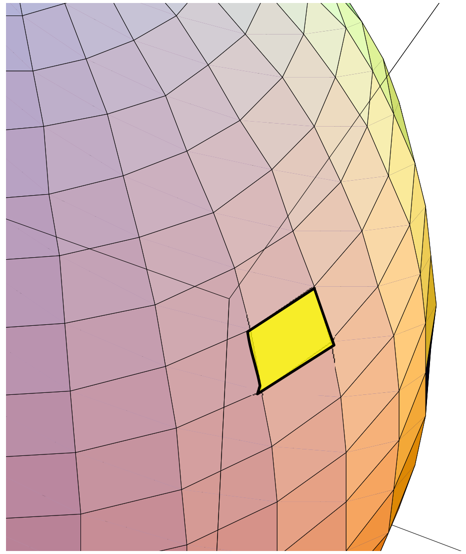


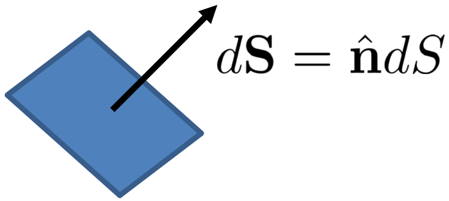
Flux integrals

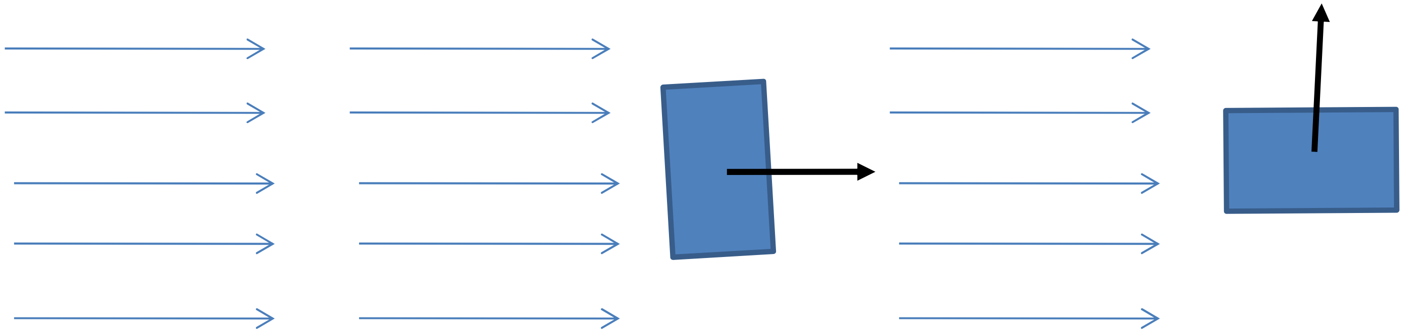
The integral of a vector field F over a surface integral S is

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\; dS$$

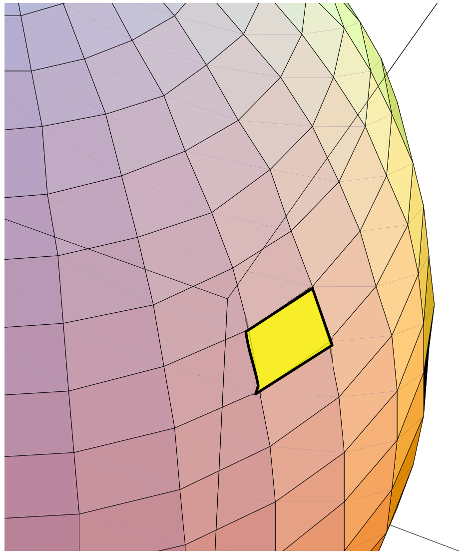
and is known as a *flux* integral.







To compute a flux integral:



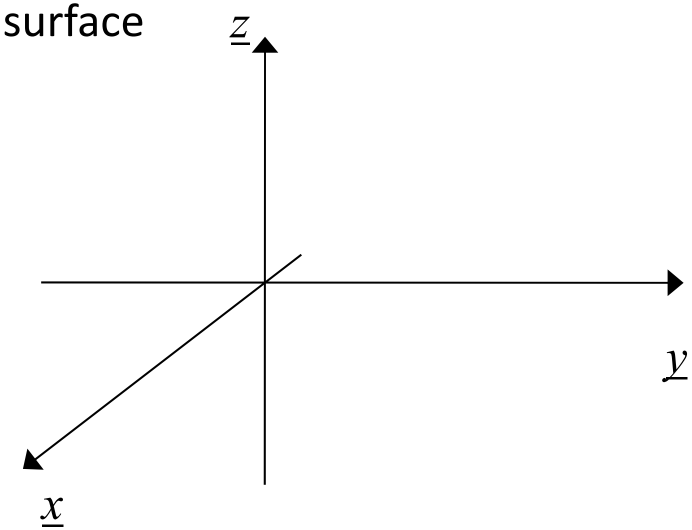
$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\ dS$$

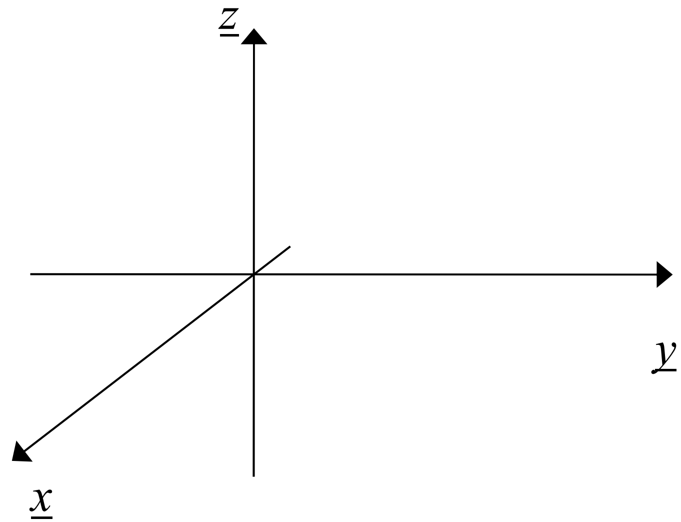
1. Compute the surface element $d\mathbf{S}$

2. Put $\mathbf{F}$ and $d\mathbf{S}$ **in the same 2D coordinate system** and form the dot product between them

3. Integrate over the coordinates.

Example: Find the integral of $\boldsymbol{F} = < 0,0,z >$ over the surface $x + y + z = 1, \; x > 0, y > 0, z > 0$.





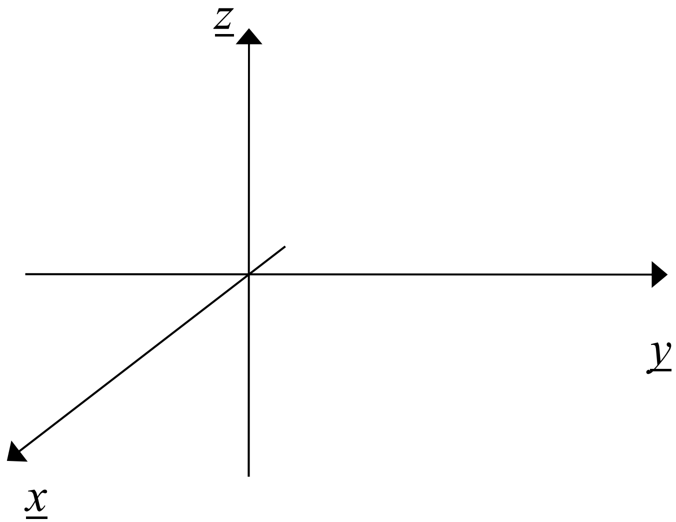
z
y
x

Example: integrate

$$\mathbf{F}(x, y, z) = \langle x^2 + y^2, x^2 + y^2, 0 \rangle$$

over the cylinder

$$\mathbf{r}(u, v) = \langle 3\cos u, 3\sin u, v \rangle$$



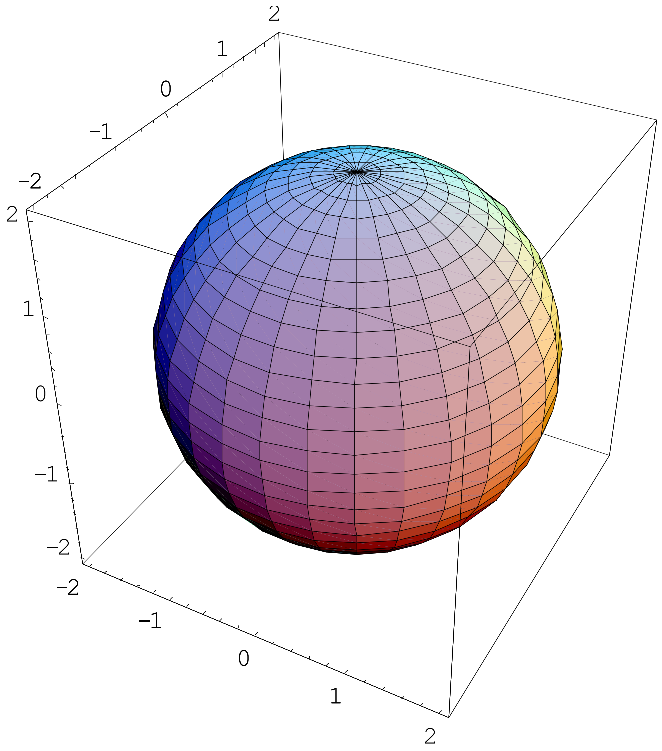
Compute the flux integral

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\ dS$$

Where

$$\mathbf{F} = \rho \sin^2\theta \hat{\boldsymbol{\rho}} + \rho \cos^2\theta \hat{\boldsymbol{\phi}}$$

And S is the surface of a sphere of radius R.



Compute

$$\iint_S \frac{x^2}{x^2+y^2}\hat{\mathbf{r}} \cdot d\mathbf{S}$$

Over the cylindrical surface S defined by $x^2 + y^2 = 4$.