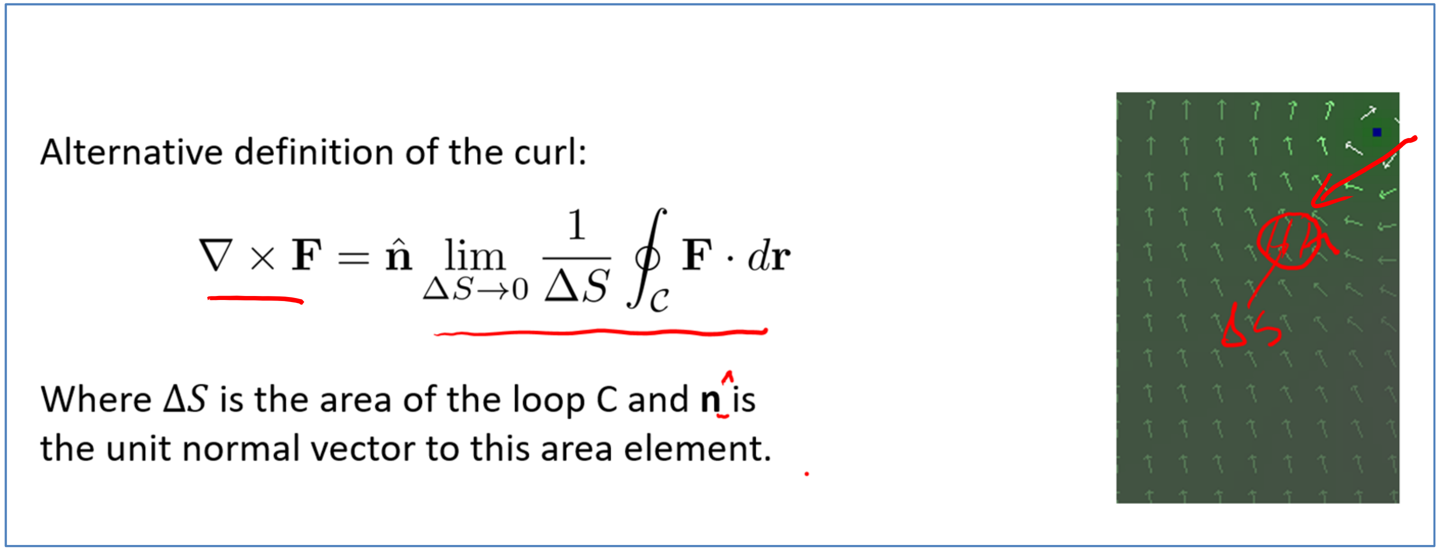
Integral theorems

Recall: We could define the *curl* of a vector field as a line integral around a loop:



Alternative definition of the curl:

$$\underline{\nabla \times \mathbf{F}} = \hat{\mathbf{n}} \underline{\lim_{\Delta S \to 0} \frac{1}{\Delta S} \oint_C \mathbf{F} \cdot d\mathbf{r}}$$

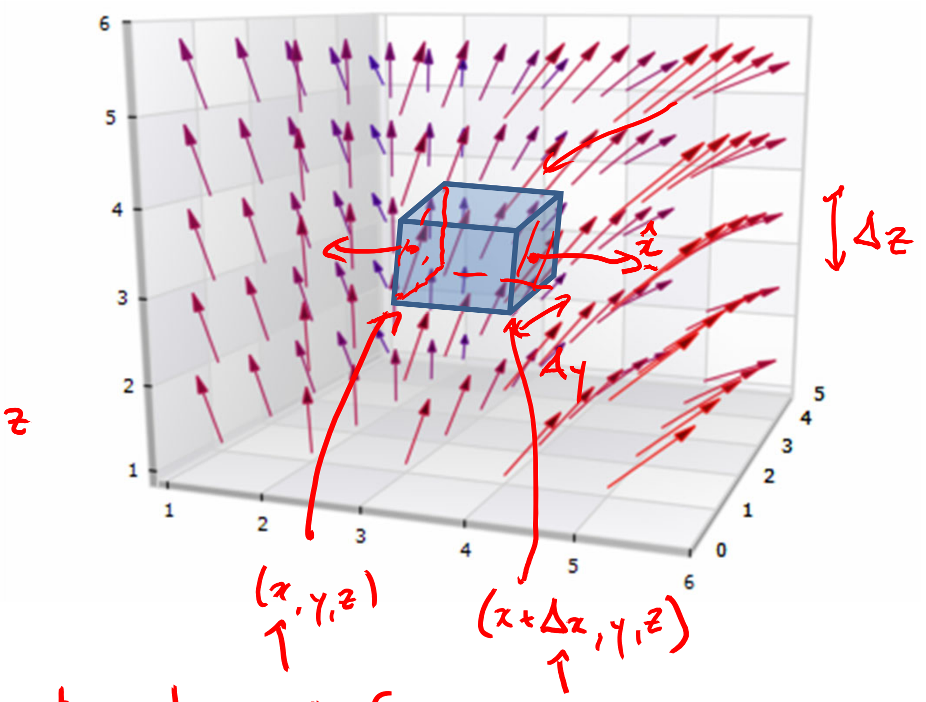
Where ΔS is the area of the loop C and $\hat{\mathbf{n}}$ is the unit normal vector to this area element.

(From the end of the week on line integrals)

We can define the *divergence* of a vector field as a *surface integral over a volume:*

Consider a vector field F, and draw *a small box* in 3D, with side-lengths Δx, Δy and Δz.

The surface integral of F over the surface S of the box is



$$\frac{1}{\Delta V}\iint_S \mathbf{F}\cdot \mathbf{d}S = \frac{1}{\Delta V}\iint \underline{\underset{\sim}{F}(x+\Delta x, y, z)}\cdot \underset{\sim}{\hat{x}}\, dy\, dz$$

$$-\frac{1}{\Delta V}\iint \underset{\sim}{F}(x, y, z)\cdot \underset{\sim}{\hat{x}}\, dy\, dz$$

+ integrals over the other 4 faces.

$$\approx \left[F_x(x+\Delta x, y, z)\frac{\Delta y\,\Delta z}{\Delta V} - F_x(x, y, z)\frac{\Delta y\,\Delta z}{\Delta V}\right] \leftarrow \simeq \Delta x .$$

$$\Delta V = \Delta x\,\Delta y\,\Delta z$$

$$+ F_y(x, y+\Delta y, z)\frac{\Delta x\,\Delta z}{\Delta V} - F_y(x, y, z)\frac{\Delta x\,\Delta z}{\Delta V}$$

$$+ F_z(x, y, z+\Delta z)\frac{\Delta x\,\Delta y}{\Delta V} - F_z(x, y, z)\frac{\Delta x\,\Delta y}{\Delta V} .$$

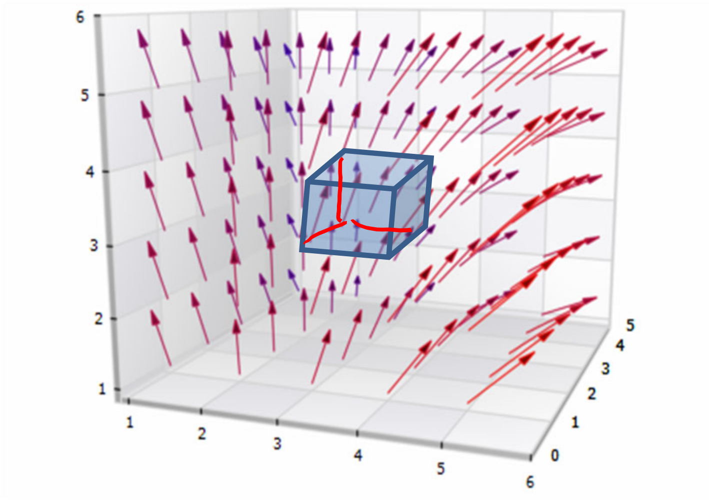
So, when the box is sufficiently small,

$$\frac{\partial F_x}{\partial x}$$

$$\underline{\frac{1}{\Delta V}\iint_S \mathbf{F}\cdot\mathbf{d}S} \approx \left.\begin{array}{l} \dfrac{F_x(x+\Delta x,y,z)-F_x(x,y,z)}{\Delta x} \\ +\dfrac{F_y(x,y+\Delta y,z)-F_y(x,y,z)}{\Delta y} \\ +\dfrac{F_z(x,y,z+\Delta z)-F_z(x,y,z)}{\Delta z} \end{array}\right]$$

In the limit as $\Delta V \rightarrow 0$, we have

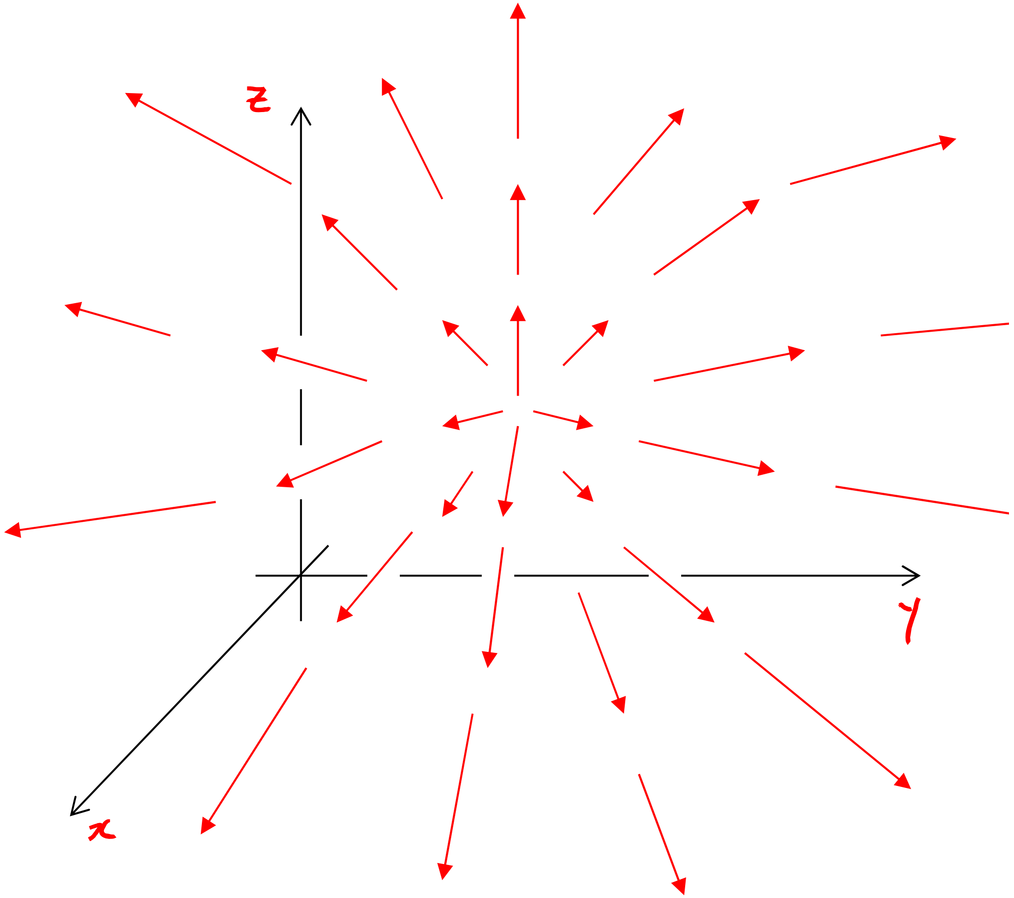
$$\lim_{\Delta V\to 0}\iint_S \mathbf{F}\cdot\mathbf{d}S = \nabla\cdot\mathbf{F}$$



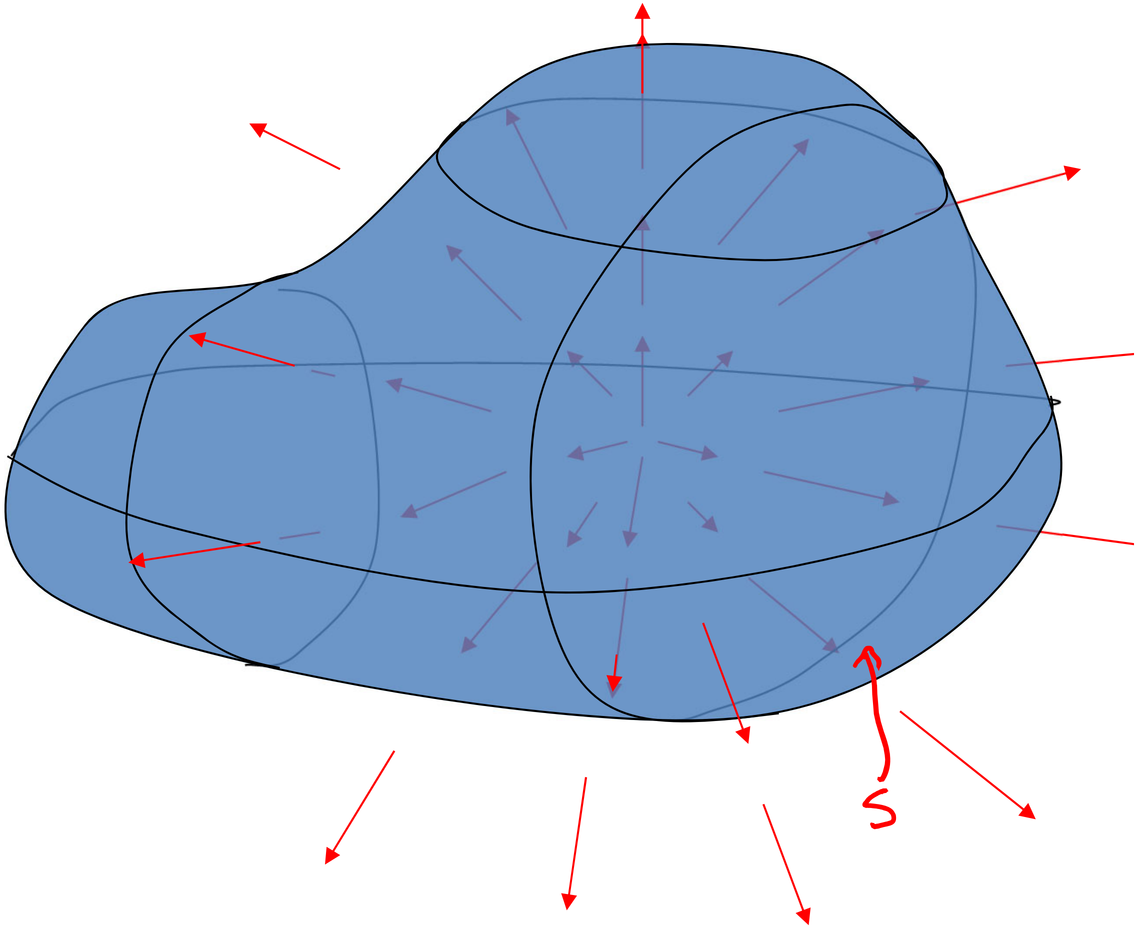
That is, the divergence at a point
is the limit of the flux integral over a small
surface surrounding that point.

The Divergence Theorem
(a.k.a. Gauss's theorem)

We consider a vector field $\mathbf{F}$ in 3D…



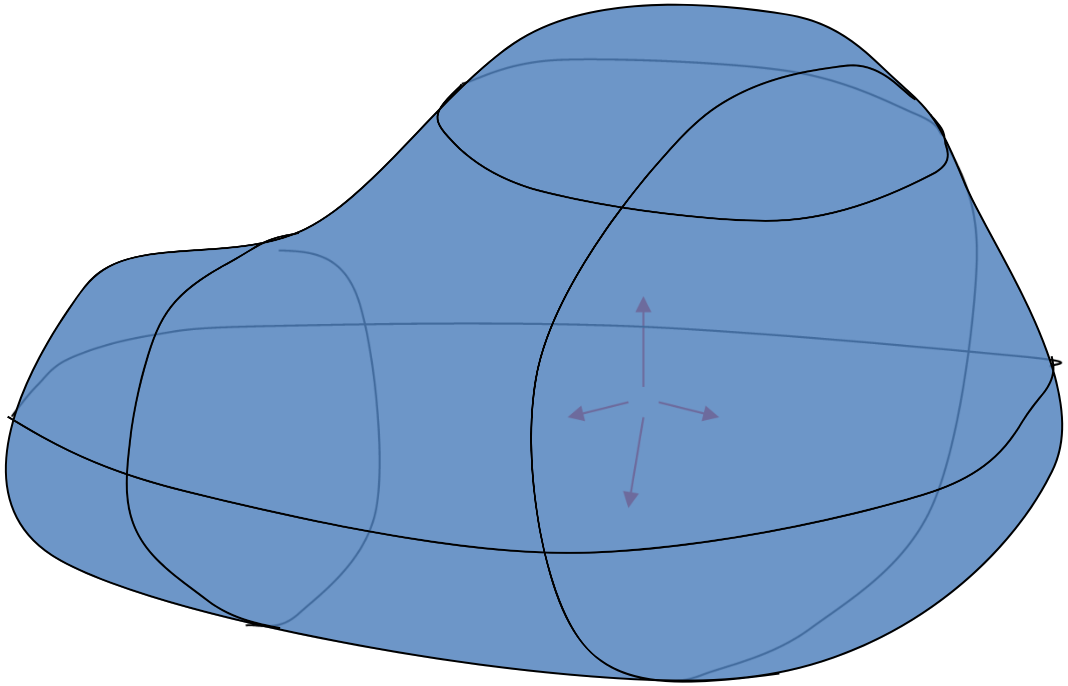
We consider a vector field $\mathbf{F}$ in 3D, defined in the interior and on the boundary S of some volume V.



We consider a vector field $\mathbf{F}$ in 3D, defined in the interior and on the boundary S of some volume V.

The divergence of $\mathbf{F}$ is a scalar field, which we can integrate over the V:

$$\iiint_V \underline{\nabla \cdot} F dV$$



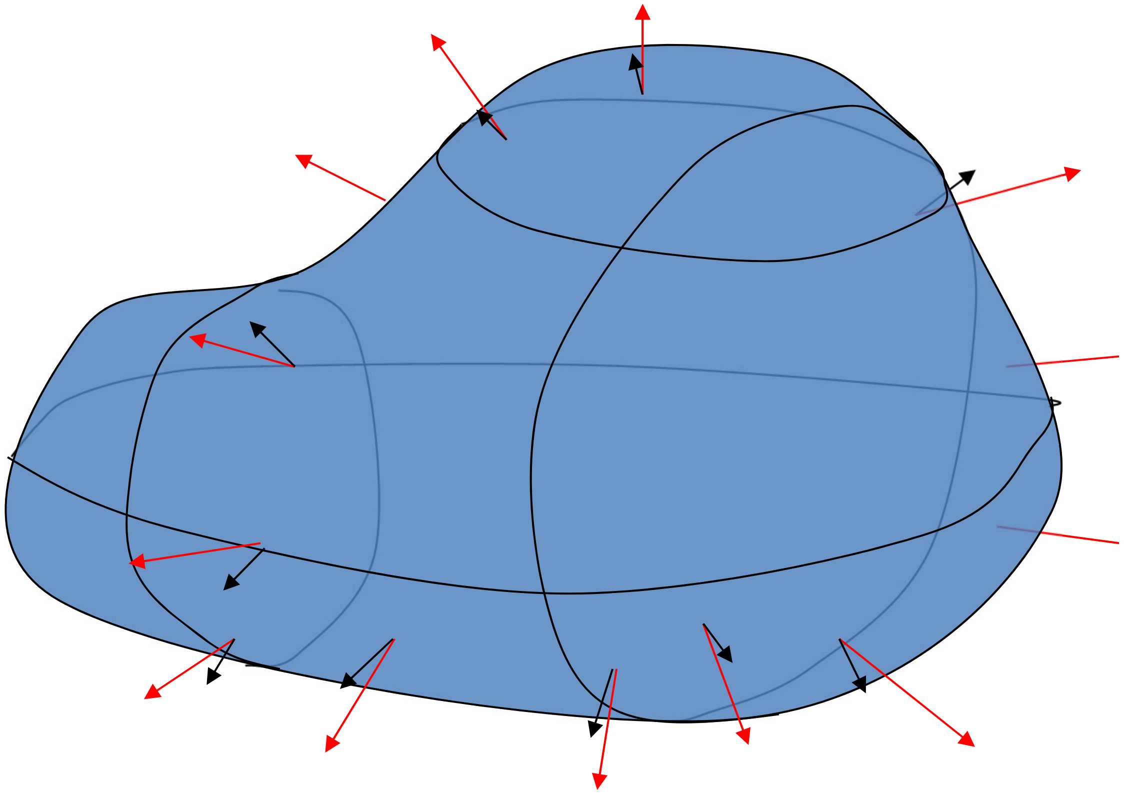
We consider a vector field **F** in 3D, defined in the interior and on the boundary S of some volume V.

The divergence of **F** is a scalar field, which we can integrate over the V:

$$\iiint_V \underline{\nabla} \cdot F dV$$

We can also integrate the flux of **F** over the surface S:

$$\underline{\iint_S \mathbf{F} \cdot \mathbf{d}S}$$

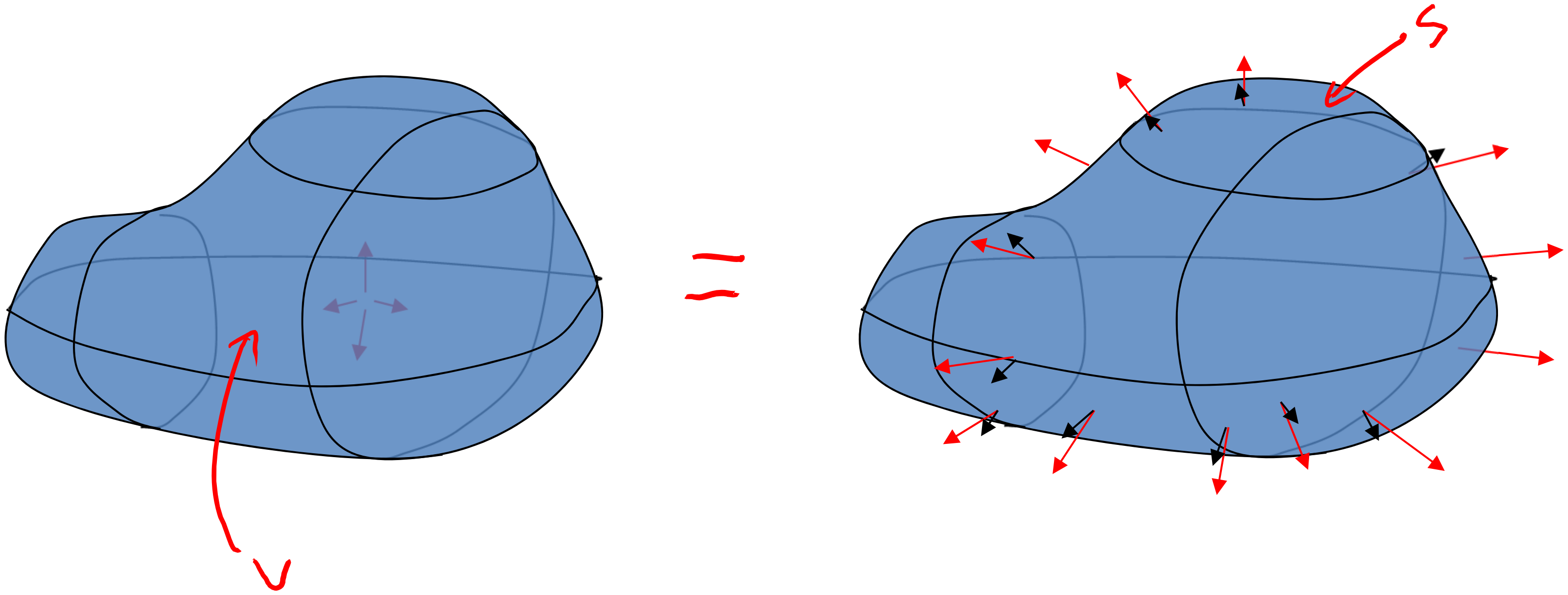


The divergence theorem states that these two quantities are *equal*.

The divergence theorem:

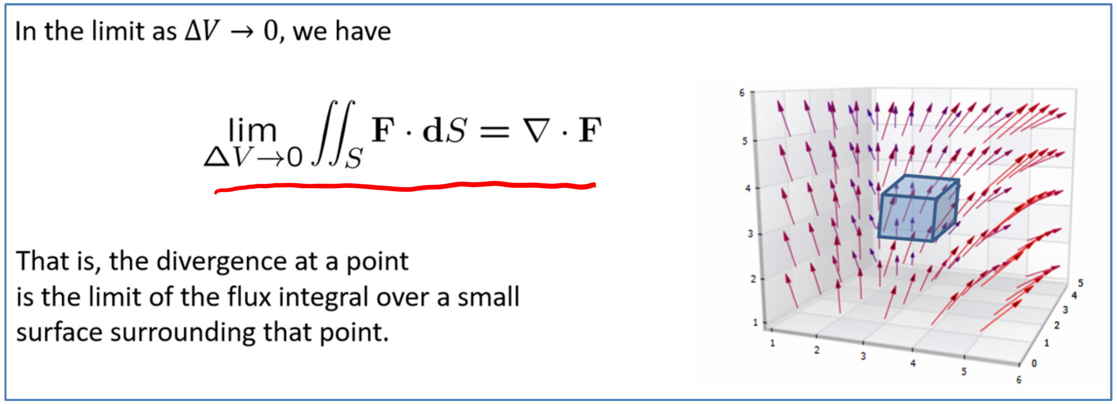
The integral of a divergence of a vector field over a volume is equal to the flux integral over the bounding surface.

$$\iiint_V \nabla \cdot F dV \quad = \iint_S \mathbf{F} \cdot \mathbf{d}S$$



Why does this work?

Recall the *"alternative definition"* of divergence:



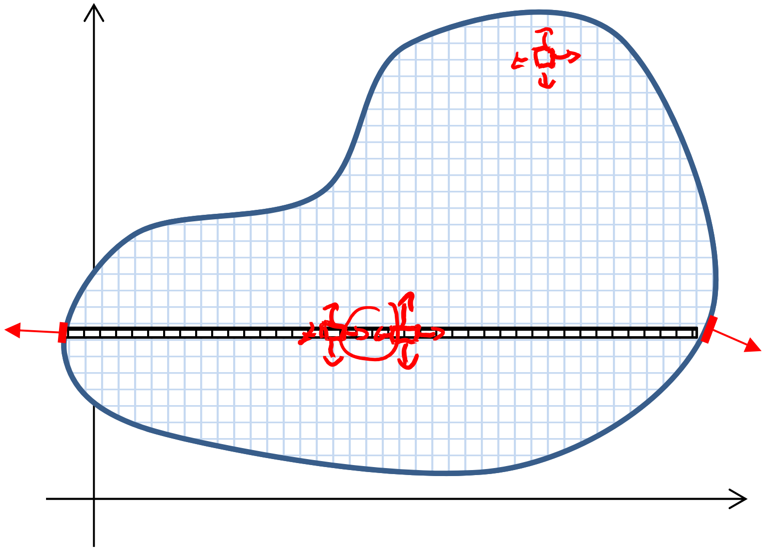
In the limit as $\Delta V \to 0$, we have

$$\underline{\lim_{\Delta V \to 0} \iint_S \mathbf{F} \cdot \mathbf{d}S = \nabla \cdot \mathbf{F}}$$

That is, the divergence at a point is the limit of the flux integral over a small surface surrounding that point.

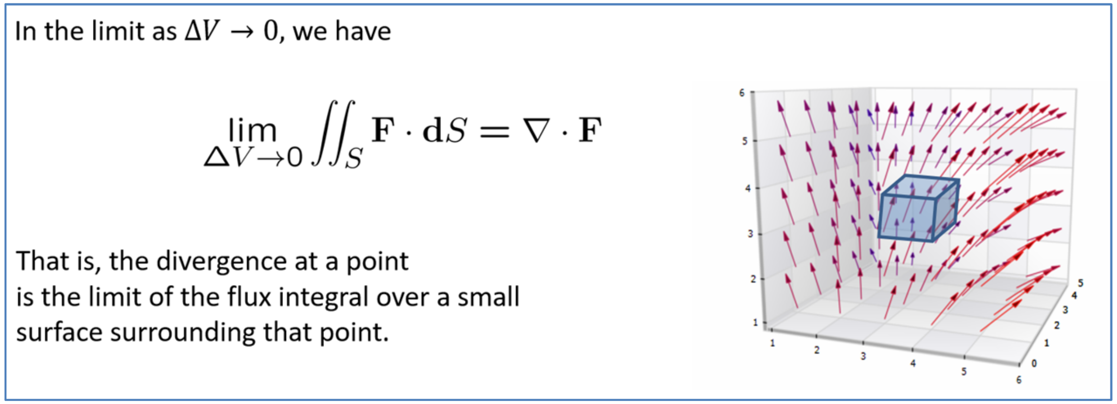
The volume integral is the sum of the surface fluxes over all the interior boxes

The internal sides "cancel out", leaving only the contribution from the edges



Why does this work?

Recall the *“alternative definition”* of divergence:

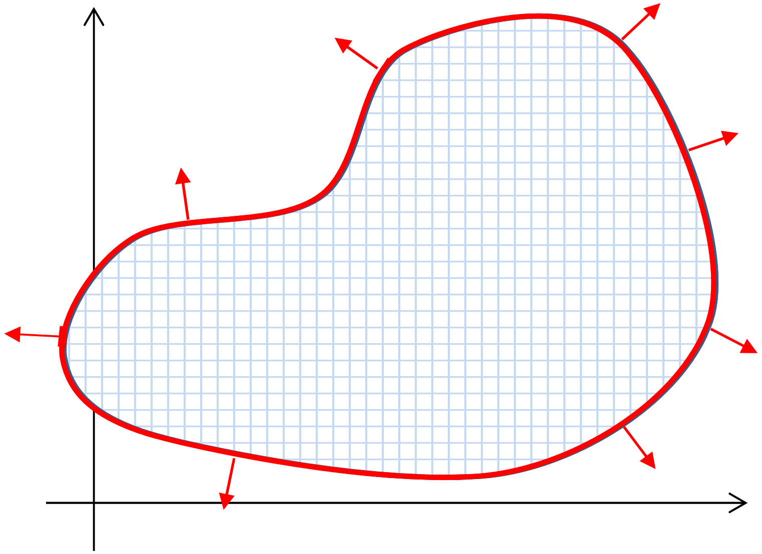


In the limit as $\Delta V \to 0$, we have

$$\lim_{\Delta V \to 0} \iint_S \mathbf{F} \cdot \mathbf{d}S = \nabla \cdot \mathbf{F}$$

That is, the divergence at a point
is the limit of the flux integral over a small
surface surrounding that point.

The volume integral is the sum of the surface fluxes over all the interior boxes

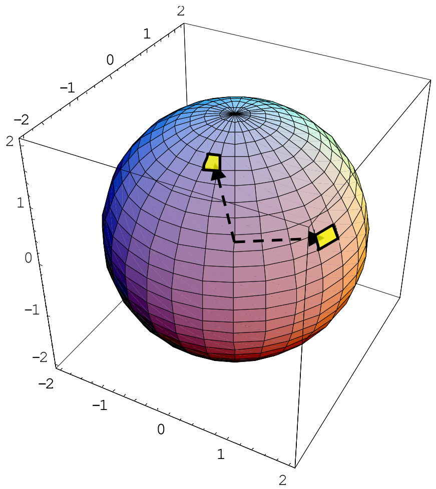


The internal sides “cancel out”, leaving only the contribution from the edges

Example: Use the divergence theorem to calculate the flux of

$$\mathbf{F} = \langle \underline{1 + 2x}, 3y, -z \rangle$$

out of the unit sphere centred at the origin.



$$\oiint_S \underset{\sim}{F} \cdot d\underset{\sim}{S} = \iiint_V \nabla \cdot \underset{\sim}{F} \, dV$$

$$= \iiint_V \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \right) dV$$

$$= \iiint (2 + 3 - 1) \, dV$$

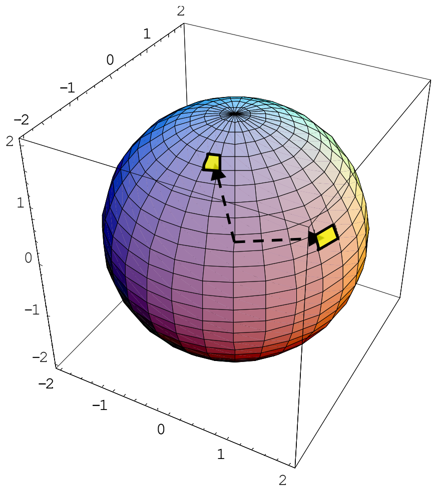
$$= 4 \iiint_V 1 \, dV = 4 \times \text{Volume of sphere}$$

$$= 4 \times \frac{4}{3} \pi \times 1^3 = \frac{16\pi}{3} .$$

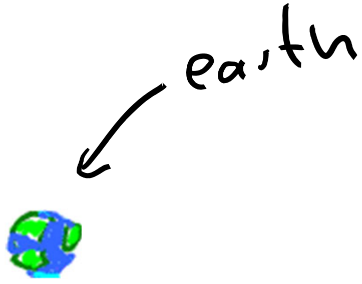
Example: Use the divergence theorem to calculate the flux of

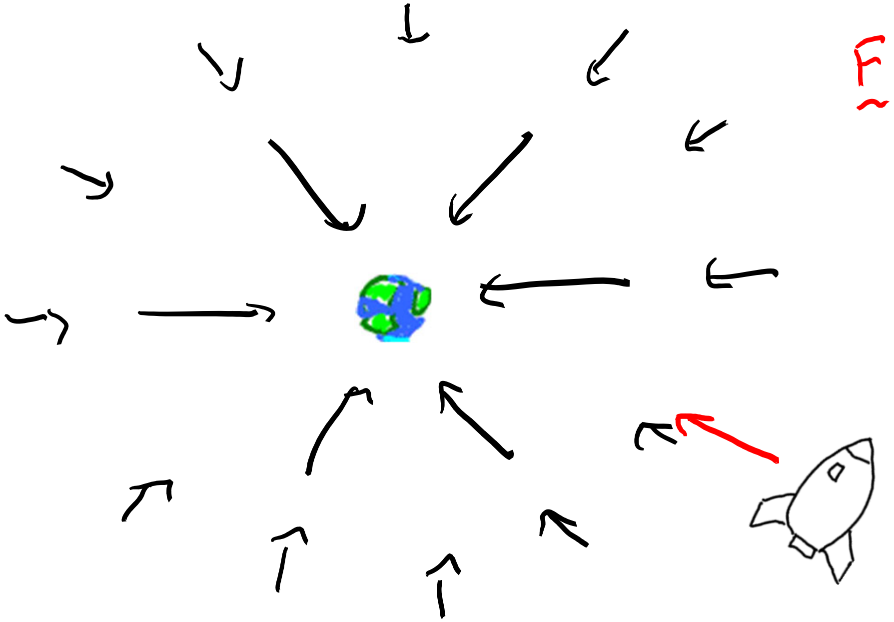
$$\mathbf{F} = \langle 1 - x^2, -y^2, z \rangle$$

out of the unit sphere centred at the point <2, 1, 4>.



Example: Gravity





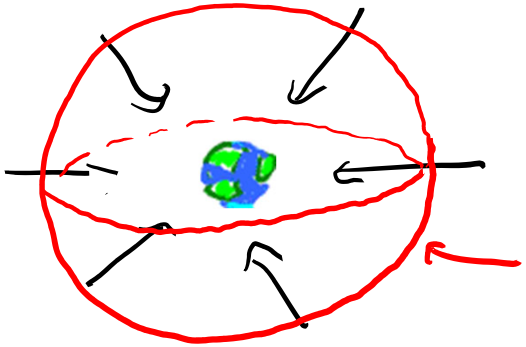
$$\underset{\sim}{F} = f(r)\,\underset{\sim}{\hat{r}}$$

Newton's Law:

force

$$\nabla\cdot\underset{\sim}{F} = -G\rho(x,y,z)$$

constant density.



S radius R

$$M = \iiint_V \rho(x,y,z)\,dV$$

The divergence theorem states

$$\iint_S \underset{\sim}{F}\cdot d\underset{\sim}{S} = \iiint_V \underbrace{\nabla\cdot\underset{\sim}{F}}\,dV = \iiint_V(-G\rho)\,dV = -G\iiint_V \rho\,dV$$

$$= \underline{-GM}.$$

But $\underset{\sim}{F} = f(r)\hat{\underset{\sim}{r}}$, and $d\underset{\sim}{S} = \hat{\underset{\sim}{r}}\,dS$, so

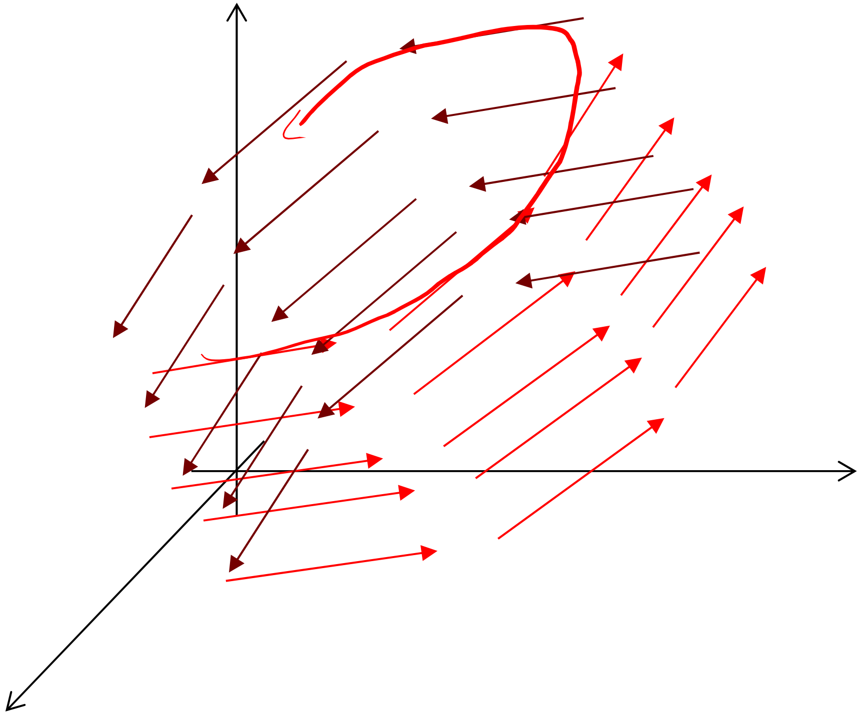
on S, r = R.

$$\iint_S \underset{\sim}{F}\cdot d\underset{\sim}{S} = \iint_S f(r)\hat{\underset{\sim}{r}}\cdot\hat{\underset{\sim}{r}}\,dS = \iint_S f(r)\,dS = f(R)\underbrace{\iint_S dS}_{\text{Area} = 4\pi R^2}$$

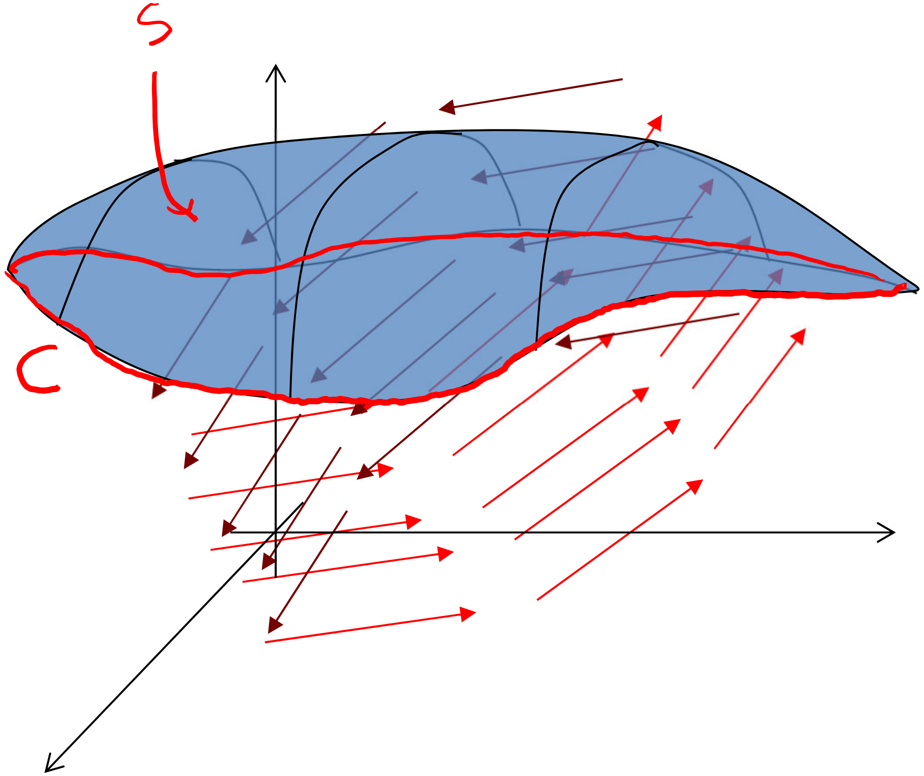
$$= \underline{f(R)\times 4\pi R^2}. \Rightarrow f(R) = \frac{-MG}{4\pi R^2}.$$

Stokes’ theorem

We consider a vector field $\mathbf{F}$ in 3D...



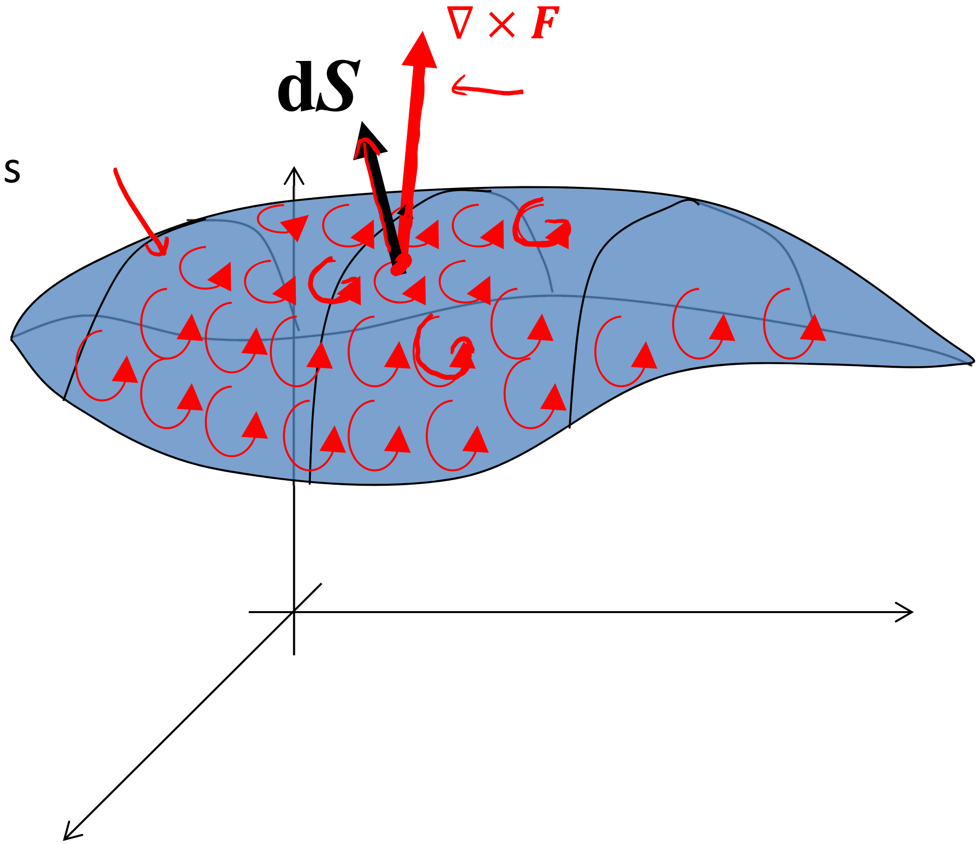
We consider a vector field $\mathbf{F}$ in 3D,
and a surface S with a closed boundary C.



We consider a vector field **F** in 3D,
and a surface S with a closed boundary C.

The flux integral of the curl $\nabla \times \mathbf{F}$ through S is

$$\iint_S \underline{(\nabla \times \mathbf{F}) \cdot \mathrm{d}S}$$



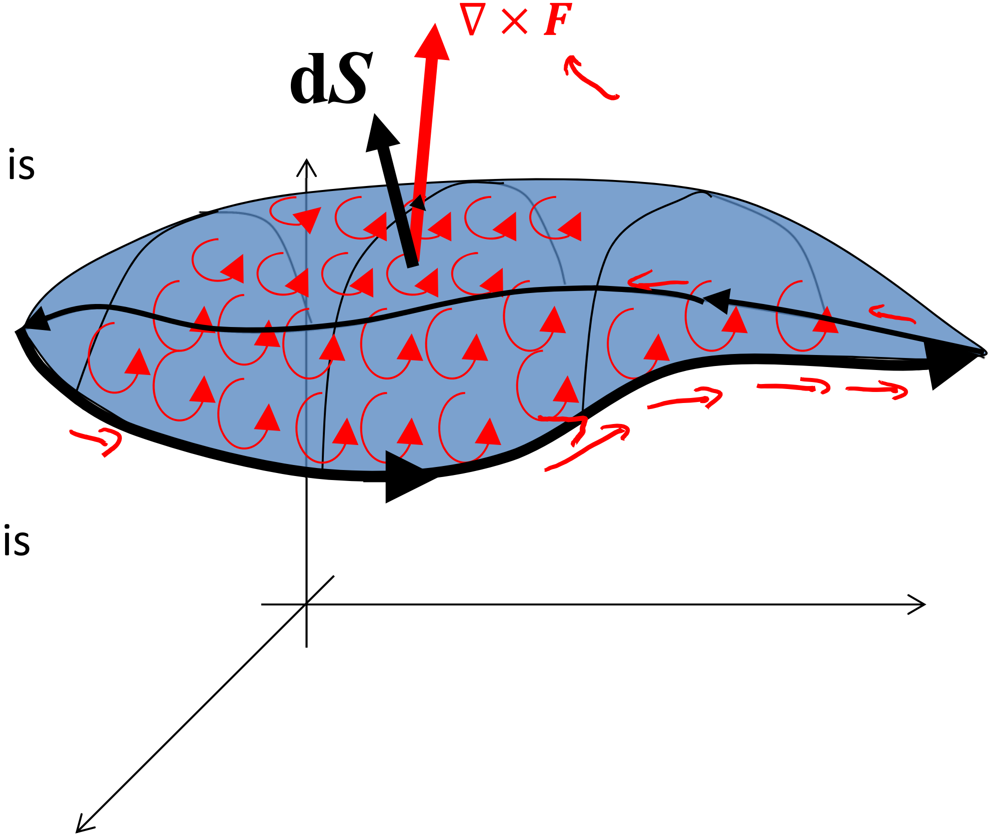
We consider a vector field **F** in 3D,
and a surface S with a closed boundary C.

The flux integral of the curl $\nabla \times \mathbf{F}$ through S is

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{d}S$$

The line integral around the boundary of S is

$$\oint_C \mathbf{F} \cdot \mathbf{d}r$$

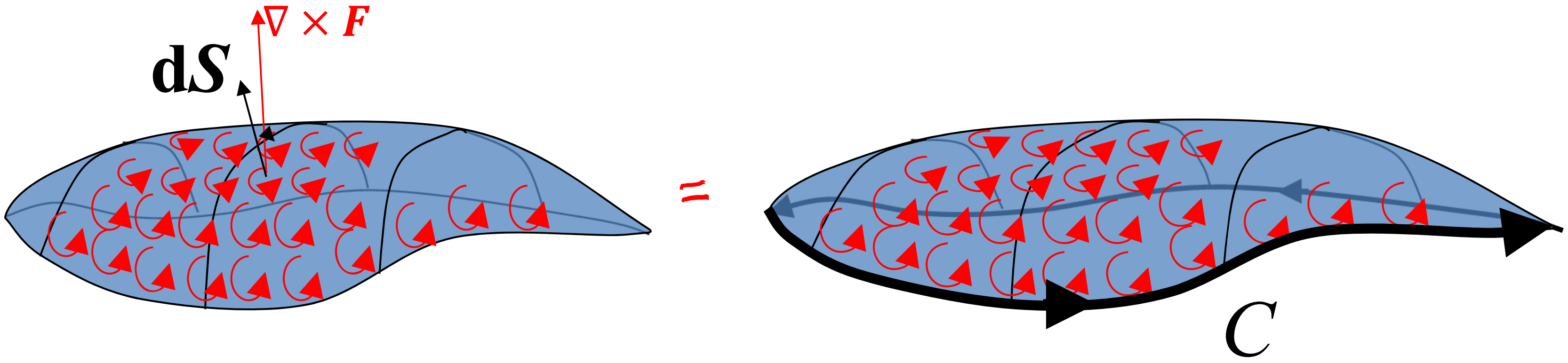


Stoke's theorem says that these two quantities are *equal*.

Stokes theorem:

The integral of the curl of a vector field over a surface is equal to the line integral around the edge of the surface.

$$\iint_S (\underline{\nabla \times \mathbf{F}}) \cdot \mathbf{d}S \quad = \underline{\oint_C \mathbf{F} \cdot \mathbf{d}r}$$



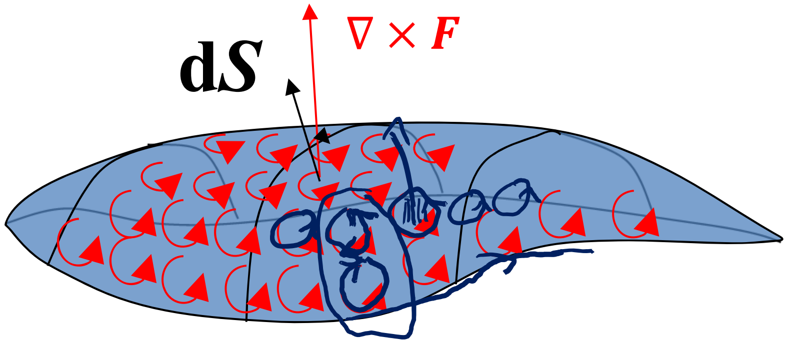
Why?

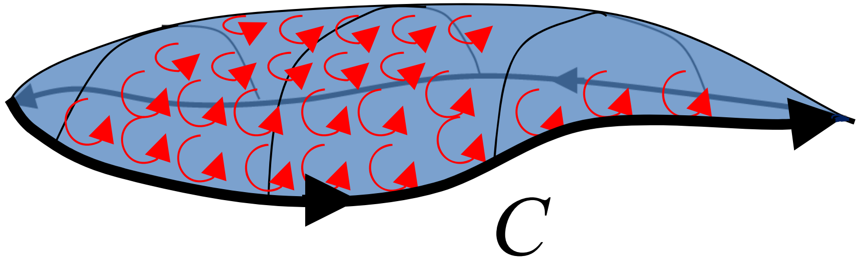
Recall that the curl is just the
Line integral around a loop:

Alternative definition of the curl:

$$\underline{\nabla \times \mathbf{F}} = \hat{\mathbf{n}} \underline{\lim_{\Delta S \to 0} \frac{1}{\Delta S} \oint_C \mathbf{F} \cdot d\mathbf{r}}$$

Where ΔS is the area of the loop C and $\hat{\mathbf{n}}$ is the unit normal vector to this area element.

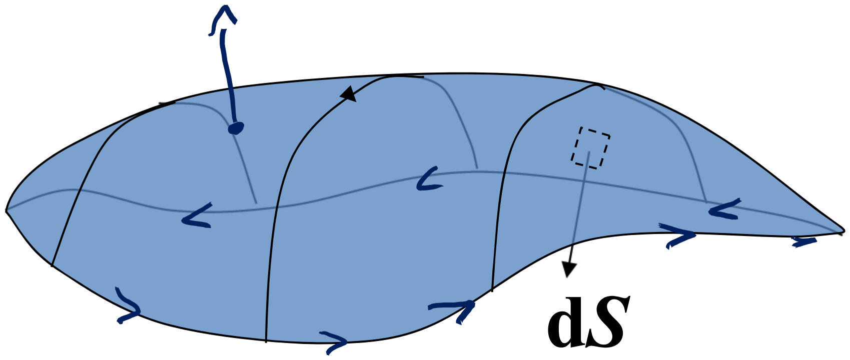


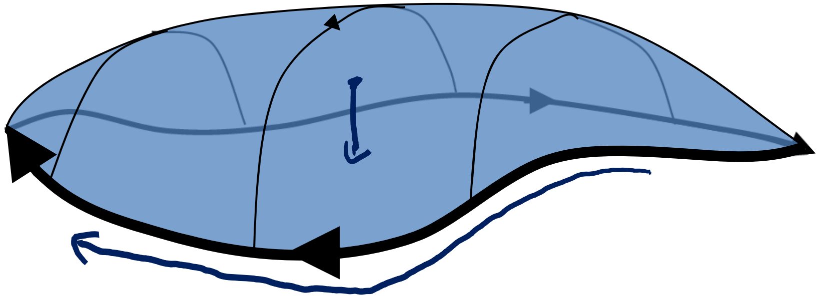


In the integral over the surface, the interior loops cancel out, leaving the line integral around the boundary.

Important thing to be aware of: Stokes' theorem assumes that the Surface and the Curve are both *oriented* in the same way.

That is: the *normal to the surface must point in the same direction as is traversed by the curve, according to the right-hand rule.*

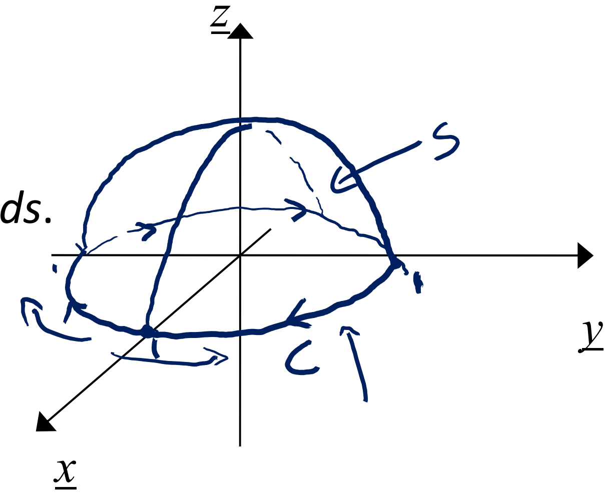




Example: Use Stokes' theorem to calculate the flux integral of the curl of the field

$$\underline{\mathbf{F} = \langle -y, x, 0\rangle} \leftarrow$$

on the upper half of the unit sphere, oriented *downwards*.



Using Stokes' theorem,

$$\iint_S (\nabla \times \underset{\sim}{F}) \cdot d\underset{\sim}{S} = \underbrace{\oint_C \underset{\sim}{F} \cdot d\underset{\sim}{r}}$$

Parametrize C:

$$\underset{\sim}{r}(t) = \langle \cos(-t), \sin(-t), 0 \rangle$$

$$= \langle \underbrace{\cos t}_{x}, \underbrace{-\sin t}_{y}, \underbrace{0}_{z} \rangle \quad \text{where} \quad 0 \le t \le 2\pi$$

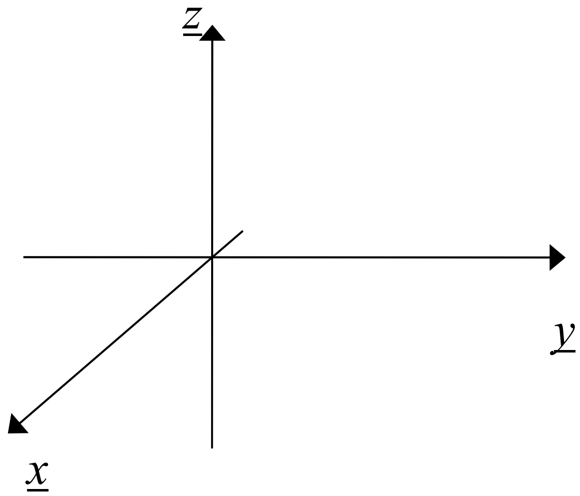
So

$$\frac{d\underset{\sim}{r}}{dt} = \langle -\sin t, -\cos t, 0 \rangle$$

So

$$\oint_C \underset{\sim}{F} \cdot d\underset{\sim}{r} = \int_0^{2\pi} \underset{\sim}{F} \cdot \frac{d\underset{\sim}{r}}{dt}\, dt = \int_0^{2\pi} \langle \sin t, \cos t, 0 \rangle \cdot \langle -\sin t, -\cos t, 0 \rangle\, dt$$

$$= \int_0^{2\pi} (-\sin^2 t - \cos^2 t)\, dt = -\int_0^{2\pi} dt = -2\pi.$$

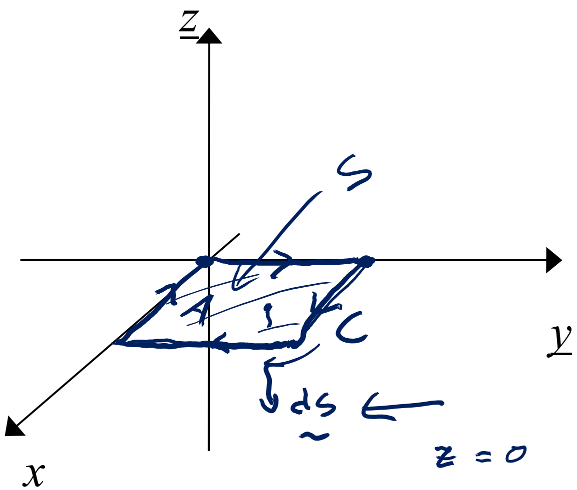

z
y
x

Example: Use Stokes' theorem to calculate

$$\underline{\int_C \mathbf{F}\cdot d\mathbf{r}}$$

where

$$\mathbf{F} = -y^2\hat{\mathbf{i}} + \frac{1}{2}x^2\hat{\mathbf{j}} + zx\hat{\mathbf{k}}$$



$z = 0$

$g(x,y) = 0$

and C is the square with vertices <0,0,0>, <0,2,0>, <2,2,0>, <2,0,0>, traversed in the negative sense.

From Stokes' theorem

$$\oint_C \underset{\sim}{F}\cdot d\underset{\sim}{r} = \iint_S (\nabla\times\underset{\sim}{F})\cdot d\underset{\sim}{S}$$

$d\underset{\sim}{S} = \langle 0,0,-1\rangle\, dx\,dy$

Now $\nabla\times\underset{\sim}{F} = \begin{vmatrix} \hat{\underset{\sim}{i}} & \hat{\underset{\sim}{j}} & \hat{\underset{\sim}{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^2 & \frac{1}{2}x^2 & zx \end{vmatrix}$

$= \hat{\underset{\sim}{i}}(0-0) - \hat{\underset{\sim}{j}}(z-0) + \hat{\underset{\sim}{k}}(x+2y)$

$$= \iint_S \langle 0,-z,x+2y\rangle\cdot\langle 0,0,-1\rangle\, dx\,dy$$

$$= \iint_A 0 + 0 - (x+2y)\, dx\,dy = \iint_A (x+2y)\, dx\,dy$$

So

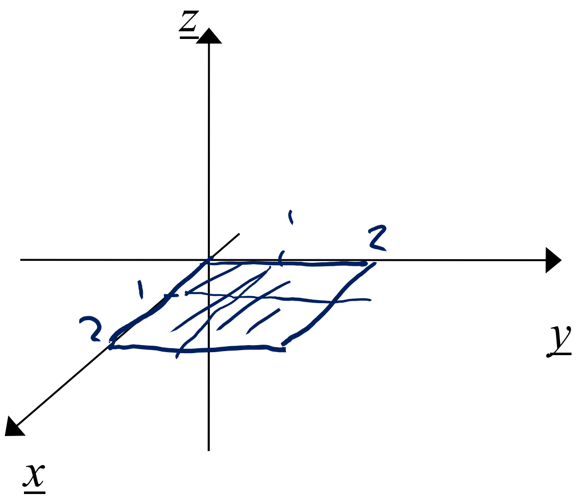
$$\iint_A (x + 2y)\, dx\, dy$$

$$= \underbrace{\iint_A x \; dx\, dy} + 2 \underbrace{\iint_A y \, dx\, dy}$$

$$= \bar{x} \times (\text{Area of } A) + 2 \underset{\swarrow}{\bar{y}} \times (\text{Area of } A)$$

$$= 1 \times 4 + 2 \times 1 \times 4$$

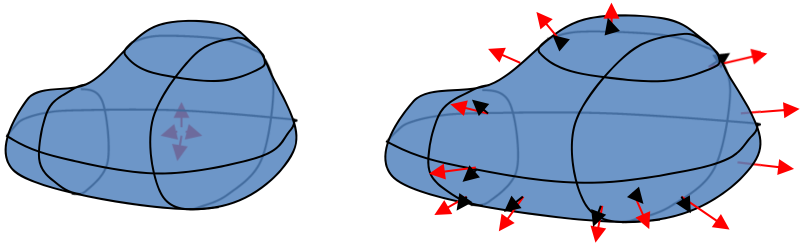
$$= 4 + 8 = 12$$



Integral theorems overview

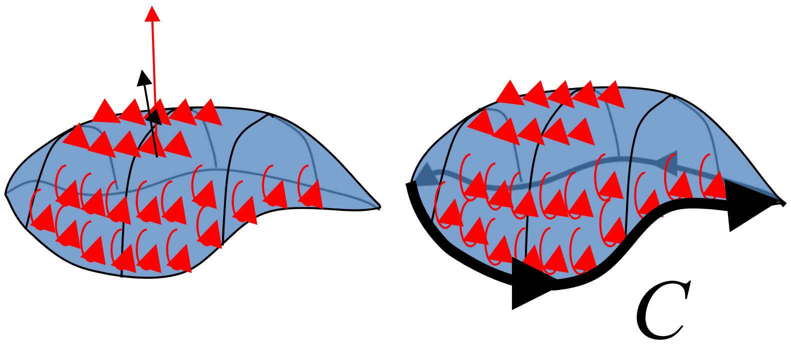
1. The divergence theorem

$$\iiint_V \nabla \cdot F\,dV = \iint_S \mathbf{F}\cdot \mathbf{d}S$$



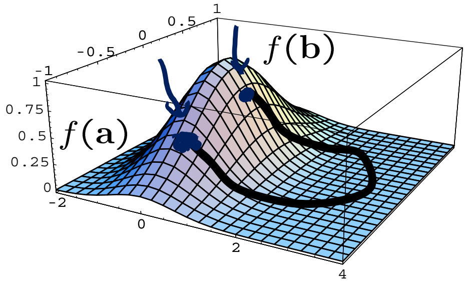
2. Stokes' theorem

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{d}S = \oint_C \mathbf{F}\cdot \mathbf{d}r$$



3. The fundamental theorem

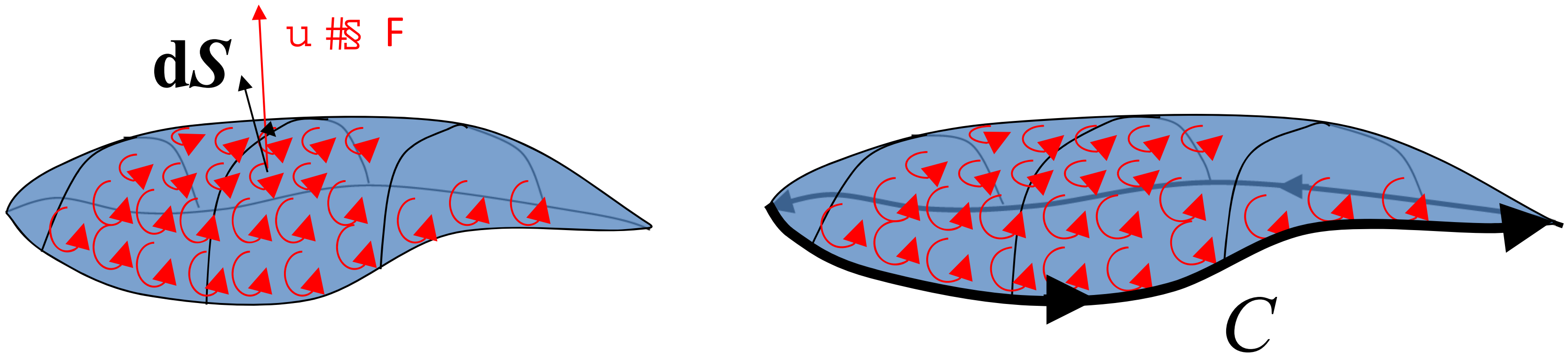
$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{b}) - f(\mathbf{a})$$



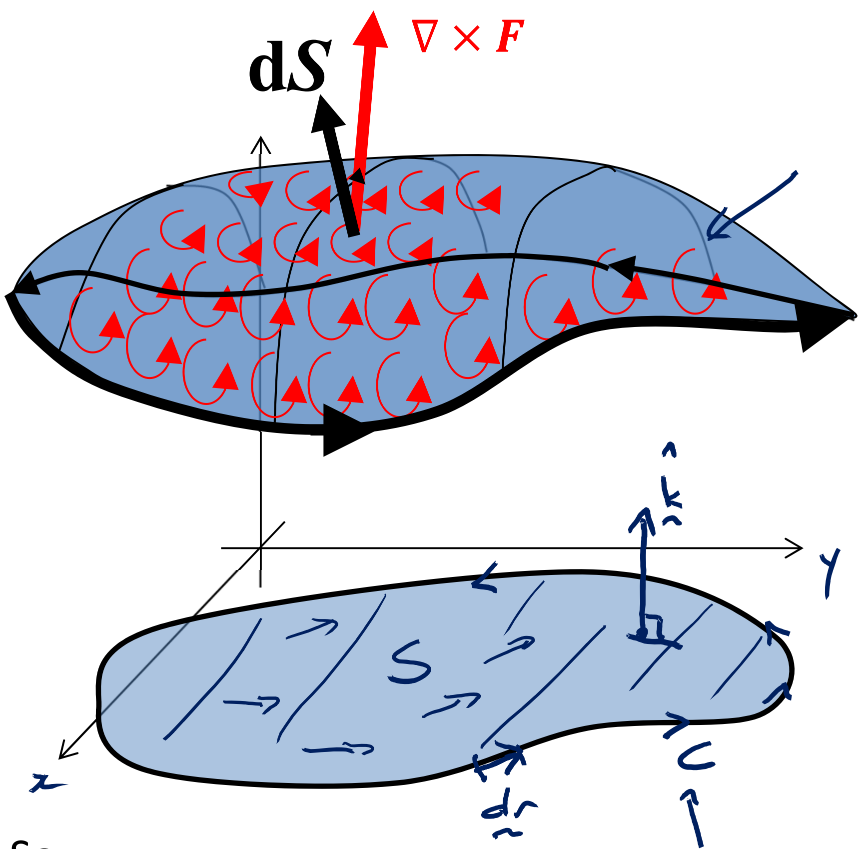
Green's theorem in the plane

Recall Stokes' theorem:

$$\iint_S \underline{(\nabla \times \mathbf{F})} \cdot \mathbf{d}S = \oint_C \underline{\mathbf{F} \cdot \mathbf{d}r}$$



When we project this theorem onto the x-y plane, we obtain an important theorem called Green's theorem.



On the plane,

$$\underline{d\mathbf{S} = \hat{\mathbf{k}}dxdy}$$

$$\underline{\mathbf{F}} = < \underline{F_x(x,y), F_y(x,y), 0} >$$

$$\underline{d\mathbf{r} = < dx, dy, 0 >}$$

$$\nabla \times \underset{\sim}{F} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & 0 \end{vmatrix}$$

So:

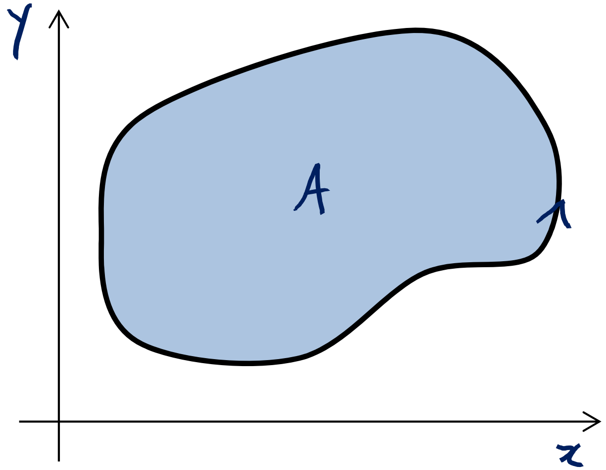
$$\underbrace{\iint_S (\nabla \times \underset{\sim}{F}) \cdot d\underset{\sim}{S}} = \iint_S < -\frac{\partial F_y}{\partial z}, -\frac{\partial F_x}{\partial z}, \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} > \cdot <0,0,1> dxdy$$

$$= \left[\underset{\sim}{i}\left(0 - \frac{\partial F_y}{\partial z}\right) - \underset{\sim}{j}\left(0 - \frac{\delta F_x}{\partial z}\right) + \underset{\sim}{k}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) \right]$$

$$= \iint_A \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) dxdy = \oint_C \underset{\sim}{F} \cdot \underset{\underset{\langle dx, dy, 0 \rangle}{\uparrow}}{d\underset{\sim}{r}} = \oint_C F_x dx + \oint_C F_y dy$$

So we have

$$\underline{\oint_C \overset{\checkmark}{F_x}\mathrm{dx} + \overset{\checkmark}{F_y}\mathrm{dy} = \int_A \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) \mathrm{d}x\mathrm{dy}}$$



Since F_x and F_y can be any function at all, we often write this as

$$\boxed{\oint_C \underline{P(x,y)}\mathrm{dx} + \underline{Q(x,y)}\mathrm{dy} = \int_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}x\mathrm{dy}}$$

Where $\underline{P(x,y)}$ and $\underline{Q(x,y)}$ are analytic functions in the x,y plane.

This identity is known as Green’s theorem in the plane, and it is extremely important in, for example, complex analysis.