Integral theorems

Recall: We could define the *curl* in terms of a line integral around a loop:

Alternative definition of the curl:

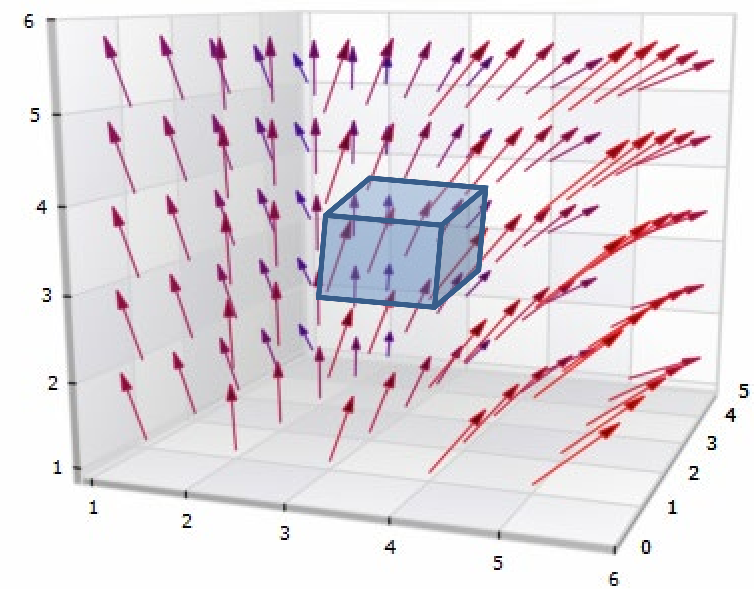
$$\nabla \times \mathbf{F} = \hat{\mathbf{n}} \lim_{\Delta S \to 0} \frac{1}{\Delta S} \oint_C \mathbf{F} \cdot d\mathbf{r}$$

Where ΔS is the area of the loop C and **n** is the unit normal vector to this area element.

(From the end of the week on line integrals)

We can define the *divergence* of a vector field as a *surface integral over a volume:*

Consider a vector field F, and draw *a small box* in 3D, with side-lengths Δx, Δy and Δz.



The surface integral of F over the surface S of the box is

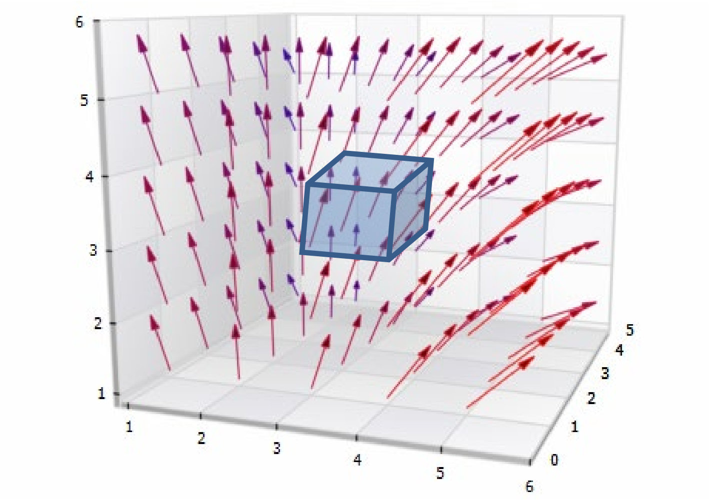
$$\iint_S \mathbf{F} \cdot \mathbf{d}S =$$

So, when the box is sufficiently small,

$$\frac{1}{\Delta V}\iint_S \mathbf{F}\cdot\mathbf{d}S \approx \frac{F_x(x+\Delta x,y,z)-F_x(x,y,z)}{\Delta x} + \frac{F_y(x,y+\Delta y,z)-F_y(x,y,z)}{\Delta y} + \frac{F_z(x,y,z+\Delta z)-F_z(x,y,z)}{\Delta z}$$

In the limit as $\Delta V \to 0$, we have

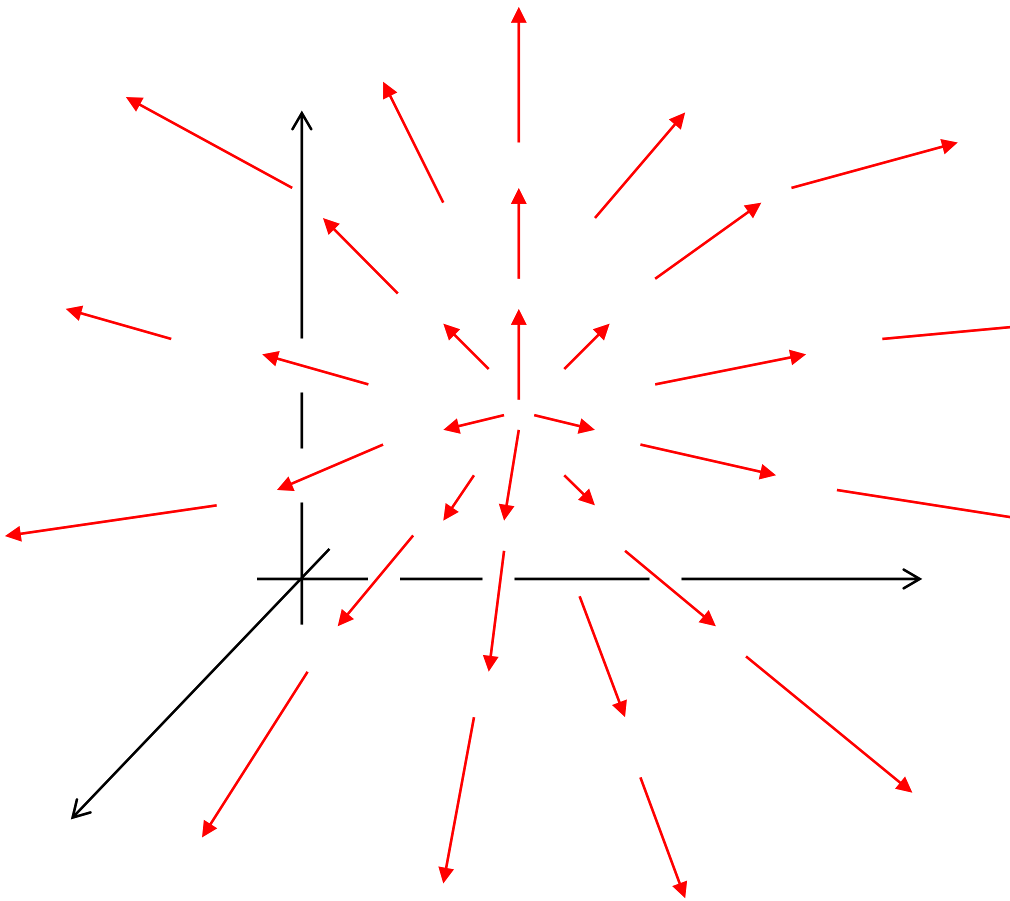
$$\lim_{\Delta V\to 0}\iint_S \mathbf{F}\cdot\mathbf{d}S = \nabla\cdot\mathbf{F}$$



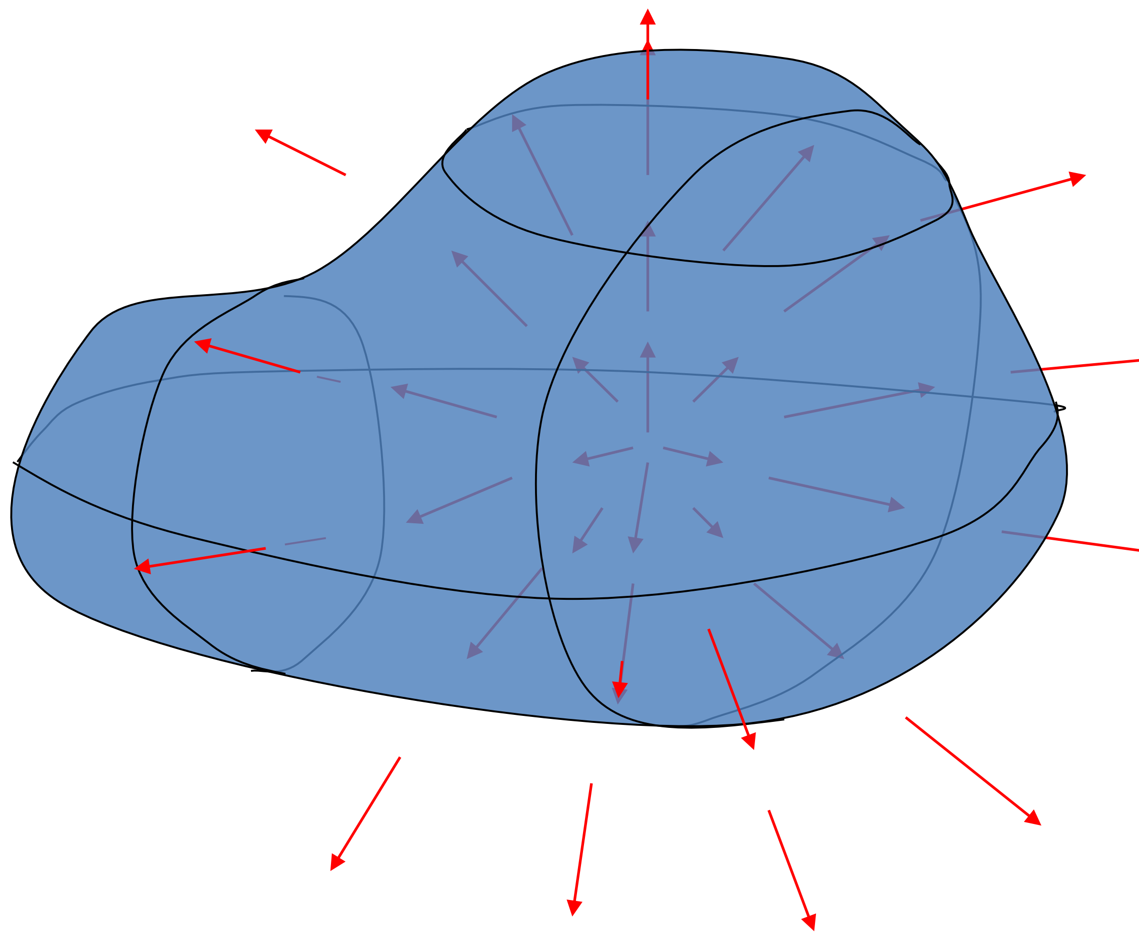
That is, the divergence at a point
is the limit of the flux integral over a small
surface surrounding that point.

The Divergence Theorem
(a.k.a. Gauss's theorem)

We consider a vector field $\mathbf{F}$ in 3D…



We consider a vector field $\mathbf{F}$ in 3D, defined in the interior and on the boundary S of some volume V.



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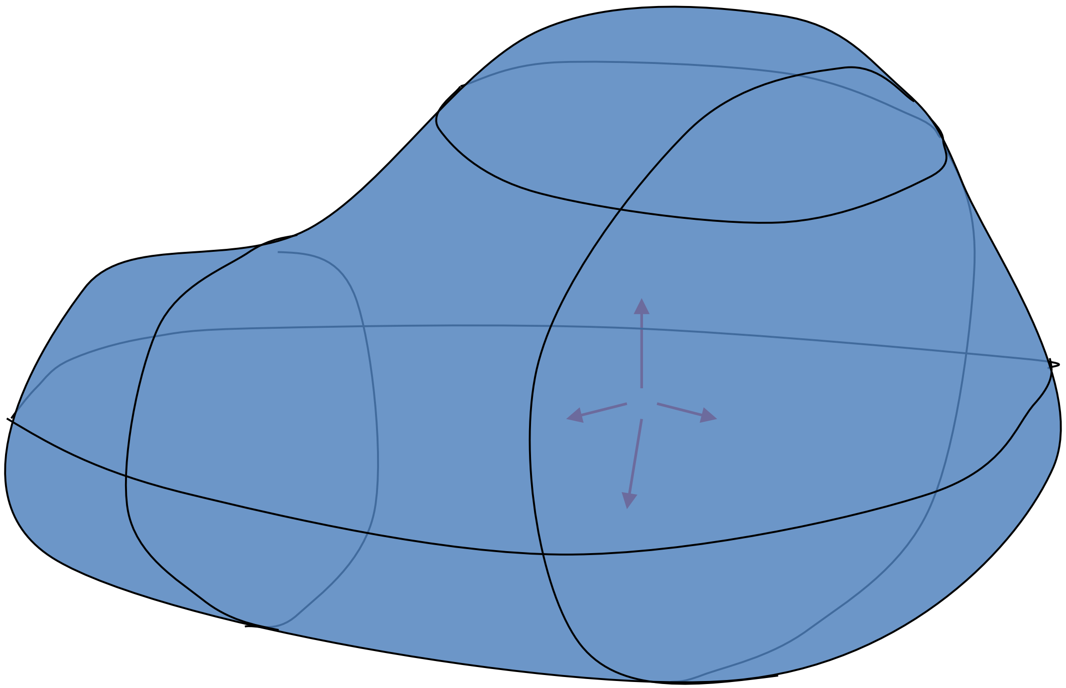

The divergence of $\mathbf{F}$ is a scalar field, which we can integrate over the V:

$$\iiint_V \nabla \cdot F dV$$

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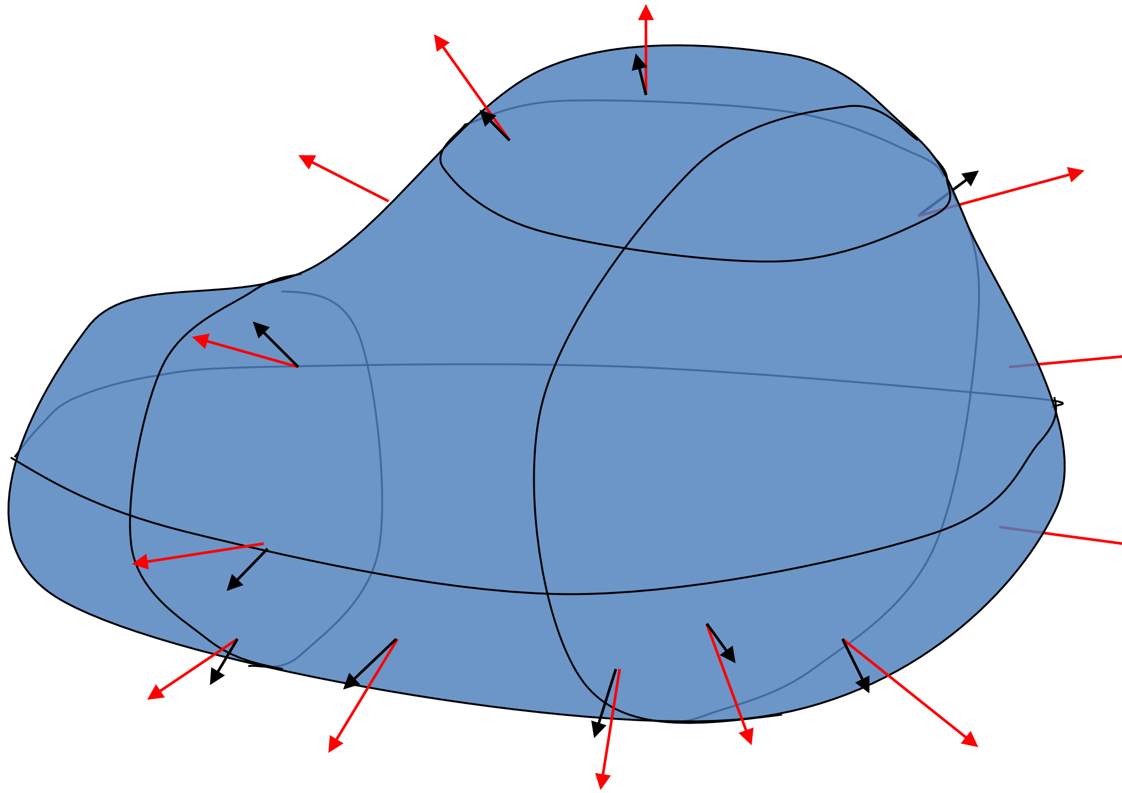


The divergence of $\mathbf{F}$ is a scalar field, which we can integrate over the V:

$$\iiint_V \nabla \cdot F dV$$

We can also integrate the flux of $\mathbf{F}$ over the surface S:

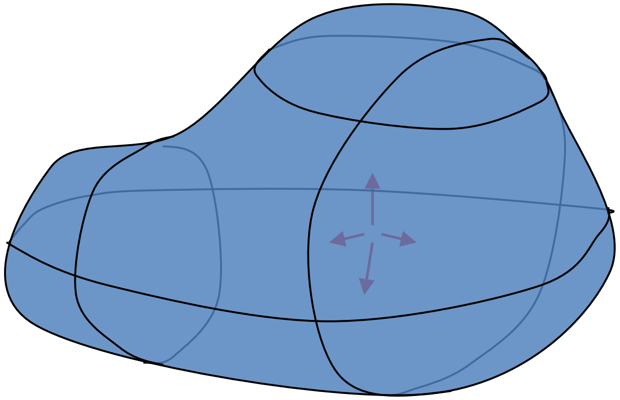
$$\iint_S \mathbf{F} \cdot \mathbf{d}S$$

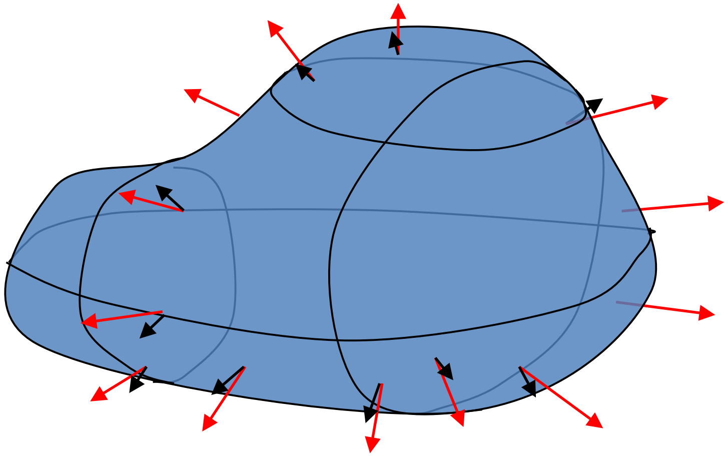
The divergence theorem states that these two quantities are *equal*.

The divergence theorem:

The integral of a divergence of a vector field over a volume is equal to the flux integral over the bounding surface.

$$\iiint_V \nabla \cdot F dV \quad = \iint_S \mathbf{F} \cdot \mathbf{d}S$$





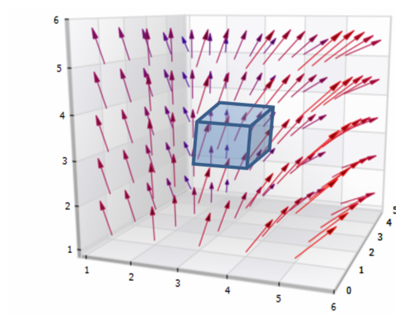
Why does this work?

Recall the *“alternative definition”* of divergence:

In the limit as $\Delta V \to 0$, we have

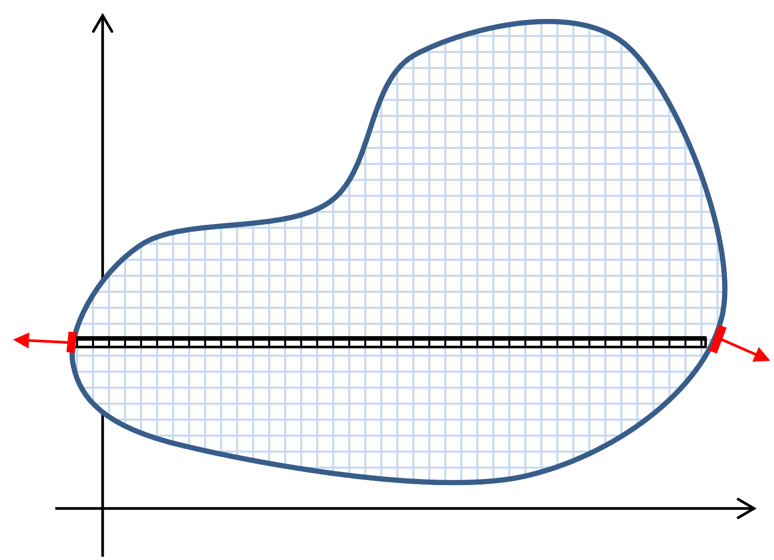
$$\lim_{\Delta V \to 0} \iint_S \mathbf{F} \cdot \mathbf{d}S = \nabla \cdot \mathbf{F}$$

That is, the divergence at a point
is the limit of the flux integral over a small
surface surrounding that point.



The volume integral is the sum of the surface fluxes over all the interior boxes

The internal sides “cancel out”, leaving only the contribution from the edges



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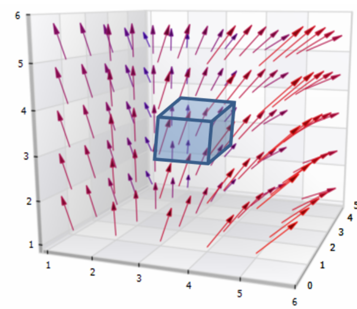

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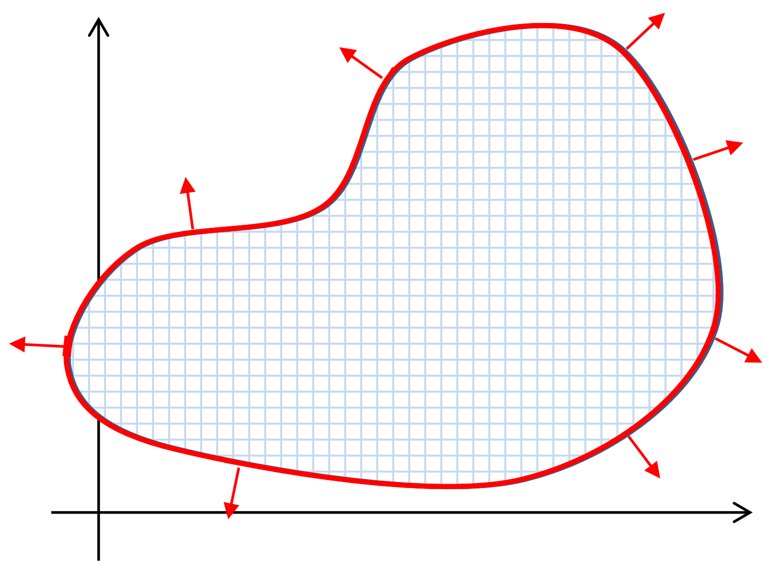
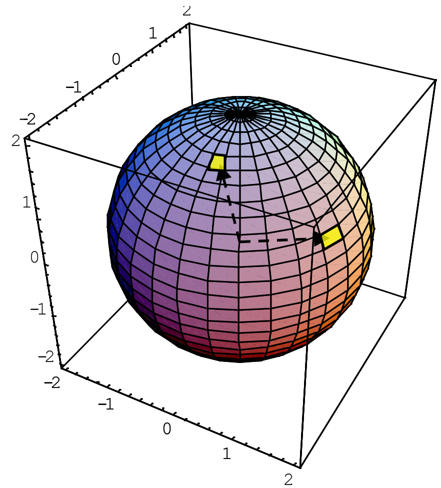


The internal sides "cancel out", leaving only the contribution from the edges

Example: Use the divergence theorem to calculate the flux of

$$\mathbf{F} = \langle 1 + 2x, 3y, -z \rangle$$

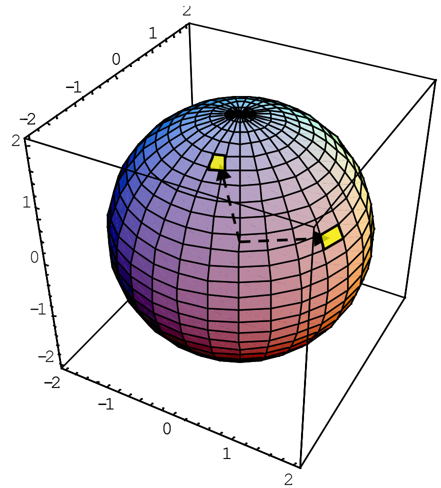
out of the unit sphere centred at the origin.



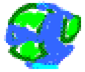
Example: Use the divergence theorem to calculate the flux of

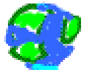
$$\mathbf{F} = \langle 1 - x^2, -y^2, z \rangle$$

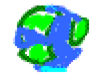
out of the unit sphere centred at the point <2, 1, 4>.



Example: Gravity

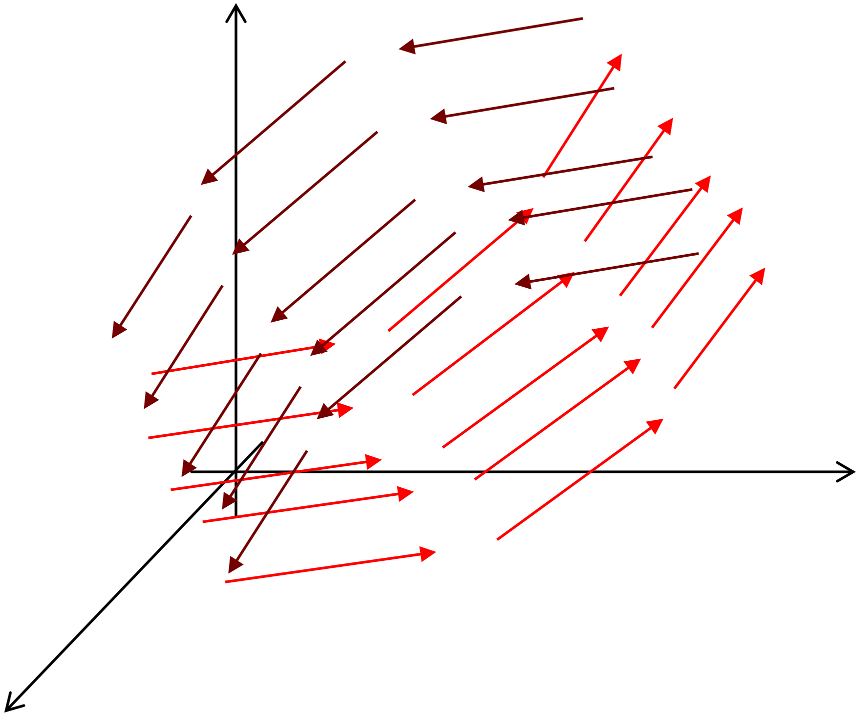




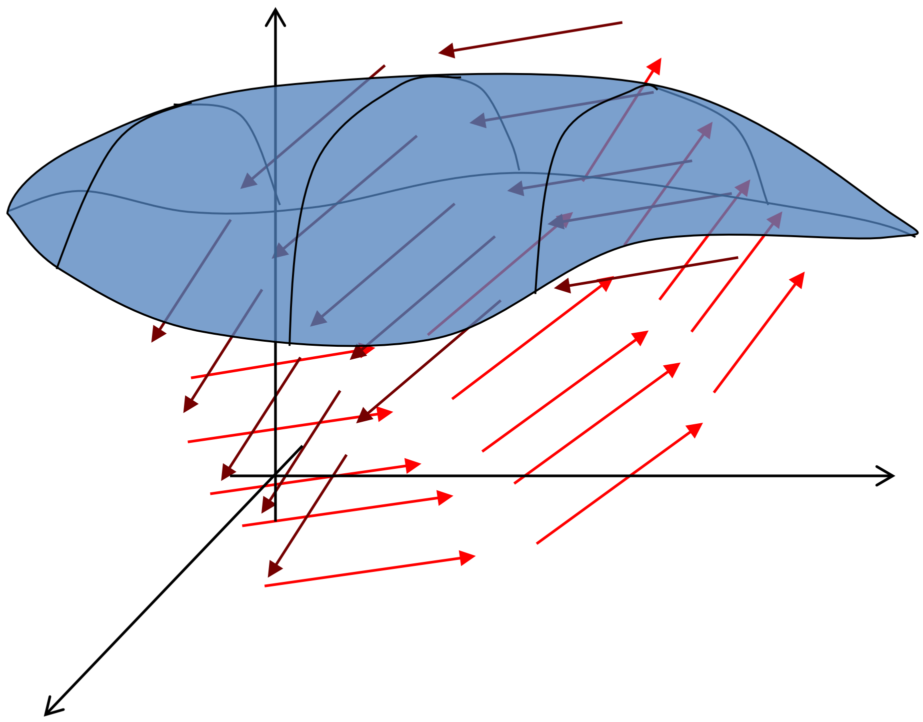



Stokes’ theorem

We consider a vector field $\mathbf{F}$ in 3D…



We consider a vector field $\mathbf{F}$ in 3D,
and a surface S with a closed boundary C.



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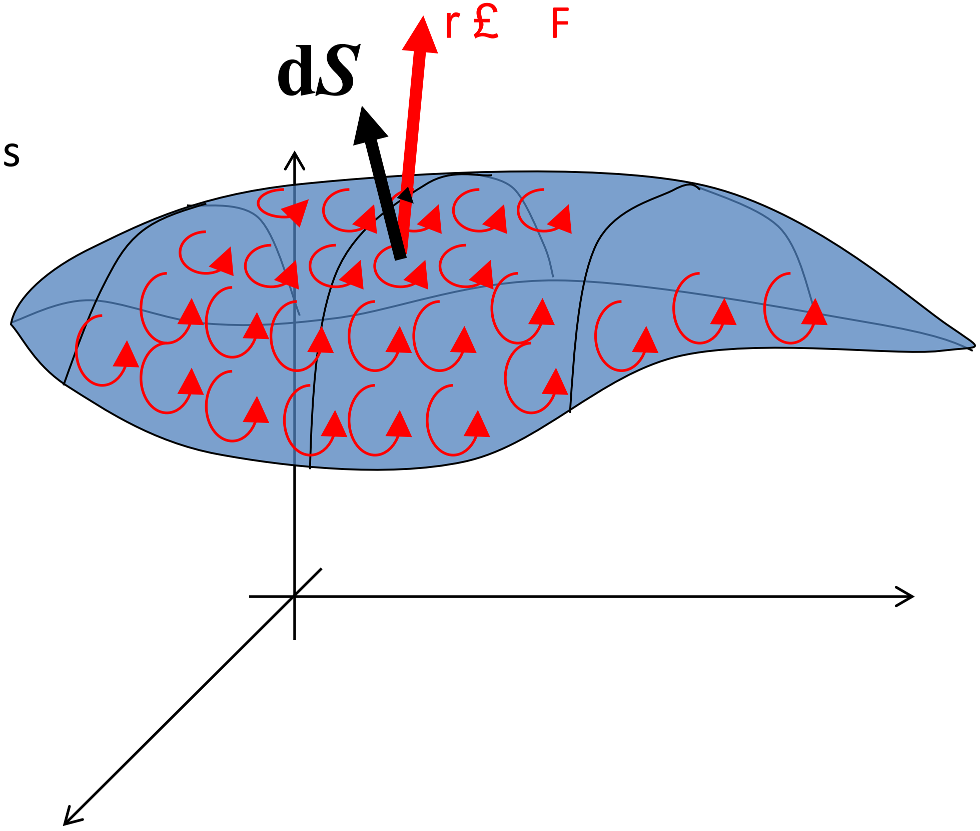

The flux integral of the curl $\nabla \times \mathbf{F}$ through S is

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{d}S$$

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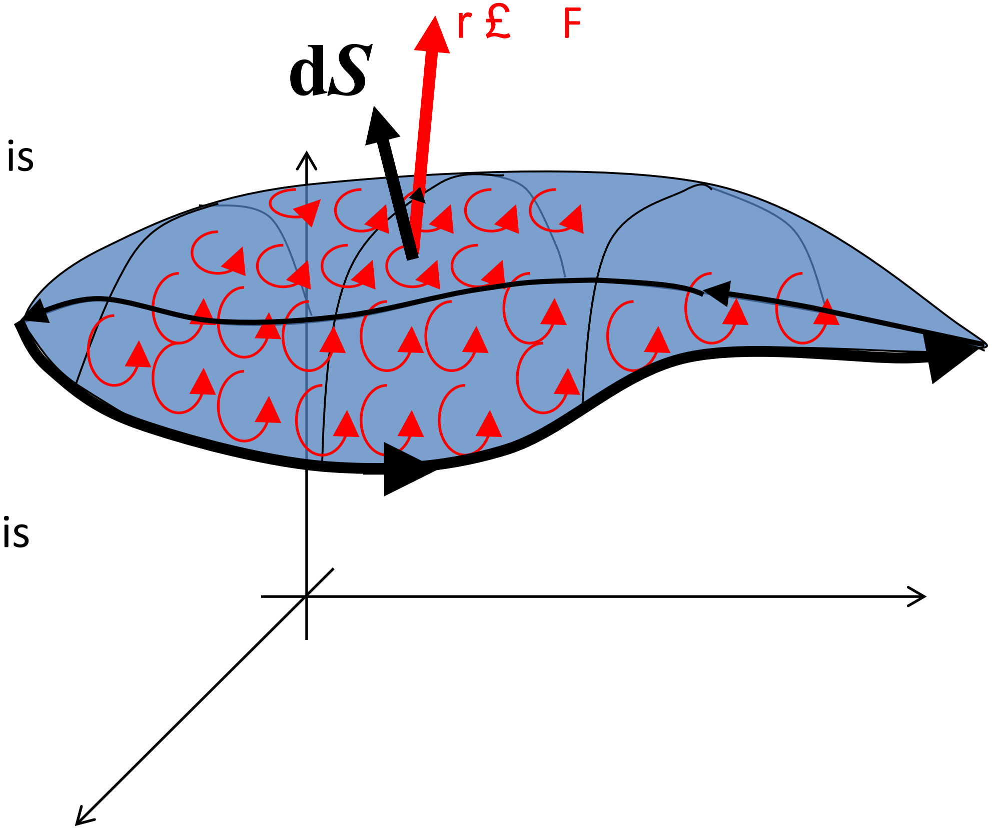


The flux integral of the curl $\nabla \times \mathbf{F}$ through S is

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{d}S$$

The line integral around the boundary of S is

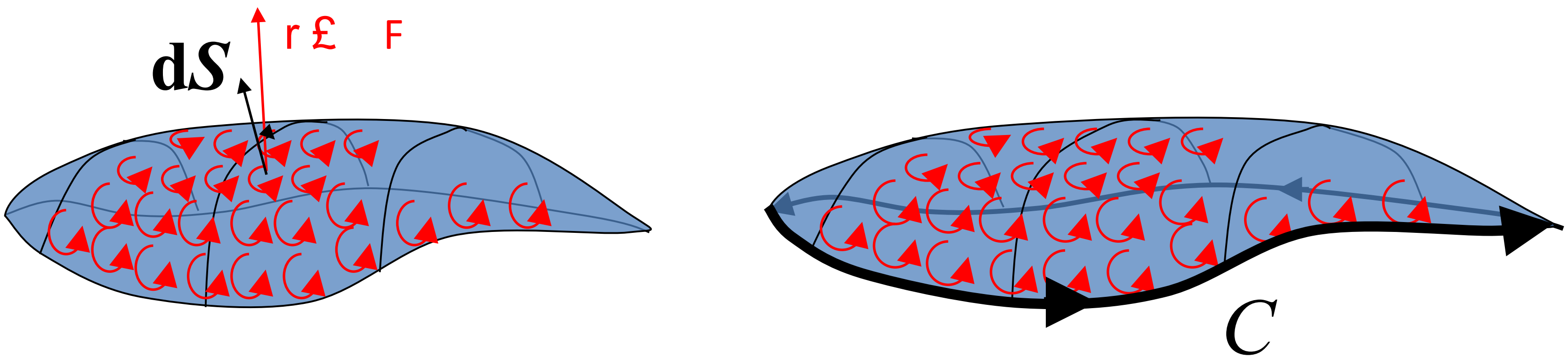
$$\oint_C \mathbf{F} \cdot \mathbf{d}r$$

Stoke’s theorem says that these two quantities are *equal*.

Stokes theorem:

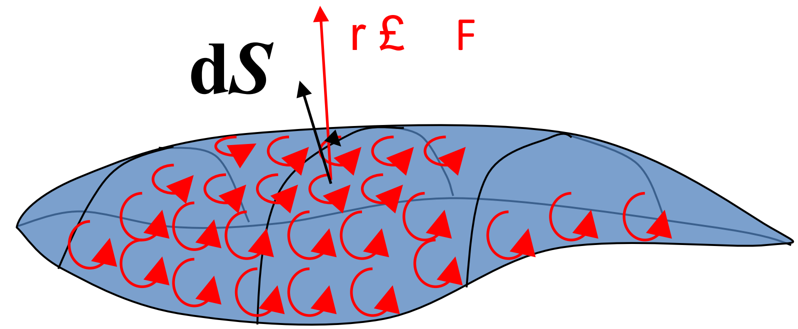
The integral of the curl of a vector field over a surface is equal to the line integral around the edge of the surface.

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{d}S \;=\; \oint_C \mathbf{F} \cdot \mathbf{d}r$$



Why?

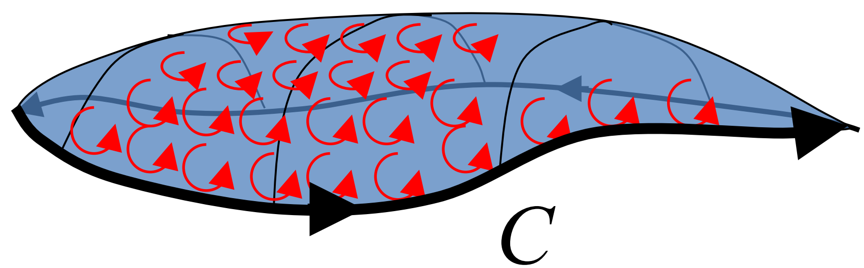
Recall that the curl is just the
Line integral around a loop:



Alternative definition of the curl:

$$\nabla \times \mathbf{F} = \hat{\mathbf{n}} \lim_{\Delta S \to 0} \frac{1}{\Delta S} \oint_C \mathbf{F} \cdot d\mathbf{r}$$

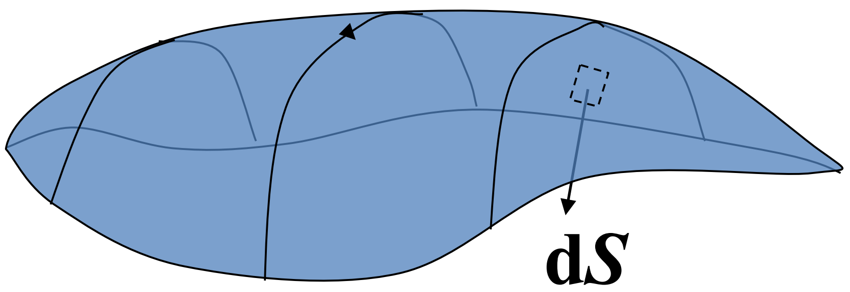
Where ΔS is the area of the loop C and **n** is the unit normal vector to this area element.

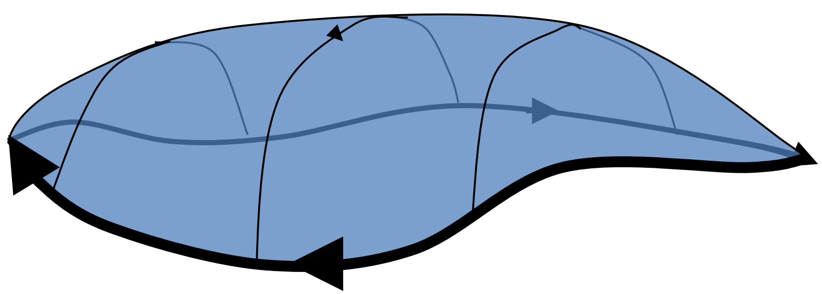


In the integral over the surface, the interior loops cancel out, leaving the line integral around the boundary.

Important thing to be aware of: Stokes' theorem assumes that the Surface and the Curve are both *oriented* in the same way.

That is: the *normal to the surface must point in the same direction as is traversed by the curve, according to the right-hand rule.*

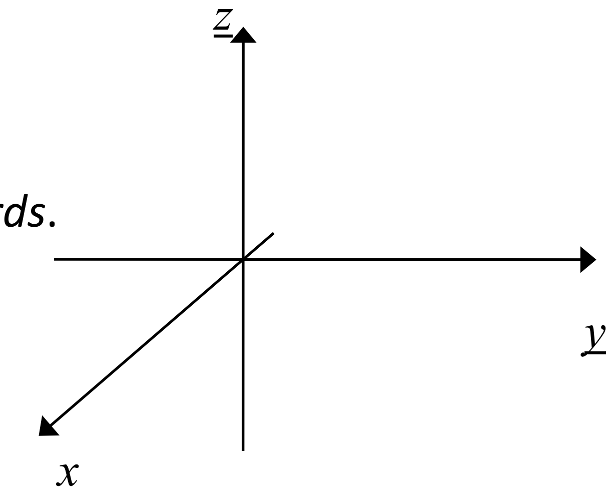


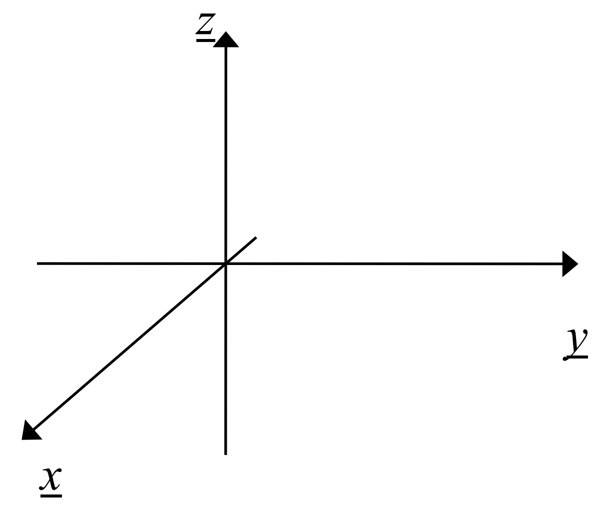


Example: Use Stokes' theorem to calculate the flux integral of the curl of the field

$$\mathbf{F} = \langle y, x, 0 \rangle$$

on the upper half of the unit sphere, oriented *downwards*.

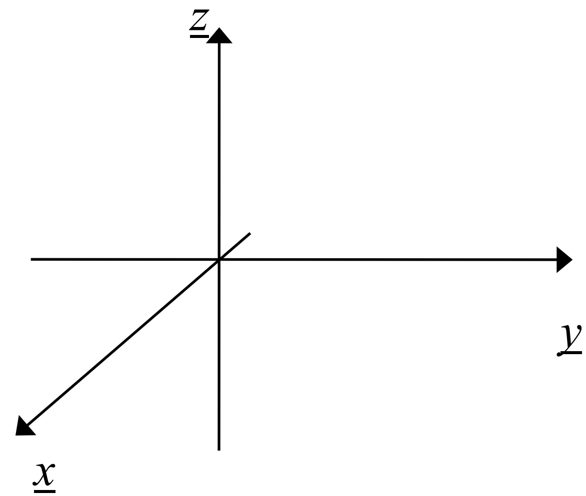




z
y
x

Example: Use Stokes' theorem to calculate

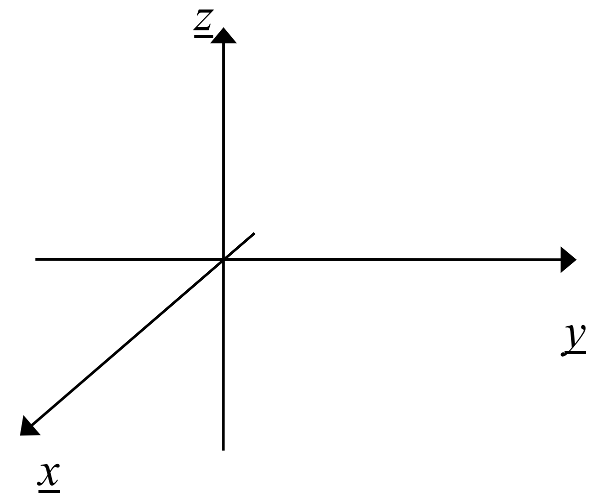
$$\int_C \mathbf{F} \cdot d\mathbf{S}$$



where

$$\mathbf{F} = -y^2\hat{\mathbf{i}} + \frac{1}{2}x^2\hat{\mathbf{j}} + zx\hat{\mathbf{k}}$$

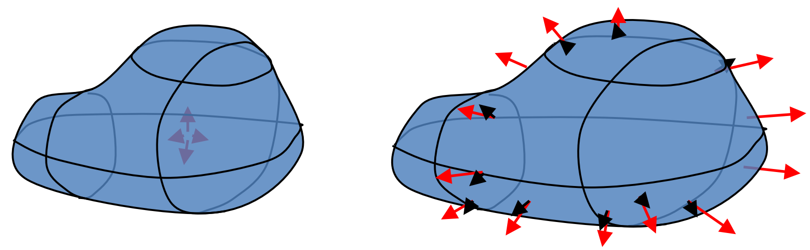
and C is the square with vertices <0,0,0>, <0,2,0>, <2,2,0>, <2,0,0>, traversed in the negative sense.


z
y
x

Integral theorems overview

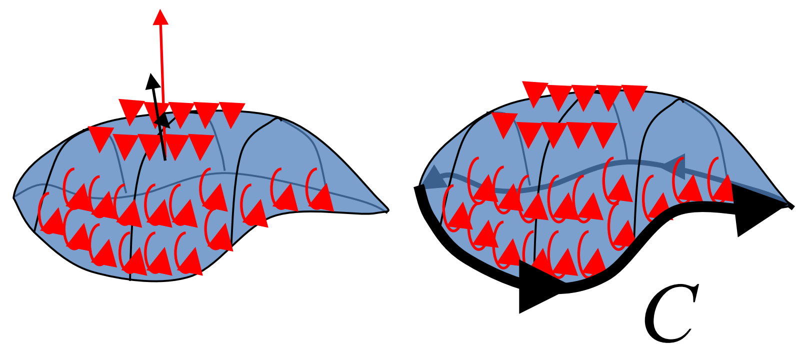
1. The divergence theorem

$$\iiint_V \nabla \cdot F dV = \iint_S \mathbf{F} \cdot \mathbf{d}S$$



2. Stokes' theorem

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{d}S = \oint_C \mathbf{F} \cdot \mathbf{d}r$$



3. The fundamental theorem

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{b}) - f(\mathbf{a})$$