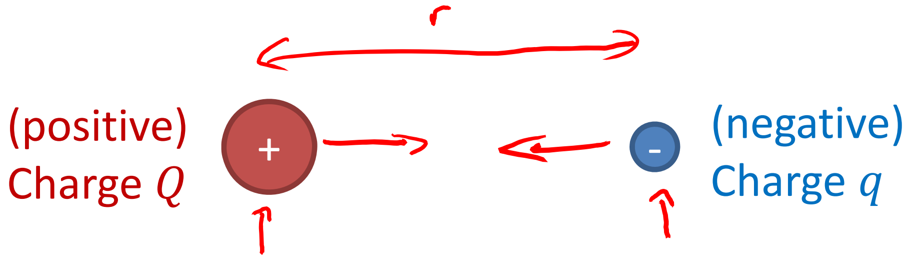
Charge and the electric field

About 200 years ago, people noticed that *charged* objects exerted a force on each other.



The force was observed to be:

1. Directed along a line between the two objects
2. Proportional to the product of the charges Qq
3. Proportional to the *inverse square* of the distance between them

We can write the force as a vector. In the coordinate system centred on the positive charge, we have

$$\mathbf{F} = \frac{Qq}{4\pi\varepsilon_0}\frac{1}{r^2}\hat{\mathbf{r}}$$

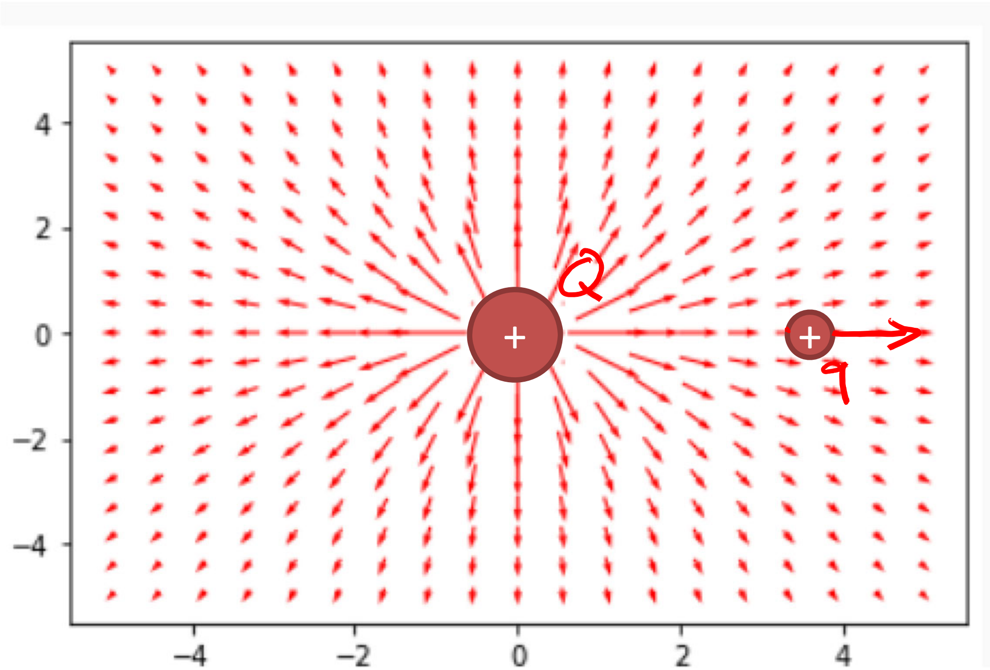
ε_0 is an experimentally-measured number, called the vacuum *permittivity:*

$$\varepsilon_0 = 8.854188 \times 10^{-12}\,\mathrm{A}^2\mathrm{s}^2/\mathrm{N}/\mathrm{m}^2$$

C^2

The *electric charge* is measured in *Coulombs* C.

The current understanding* is that small electric *charges* emit a *vector field*. This field in turn exerts a force on other charges.



$$\mathbf{F} = \frac{Qq}{4\pi\varepsilon_0}\frac{1}{r^2}\hat{\mathbf{r}}$$

$$= q\mathbf{E}$$

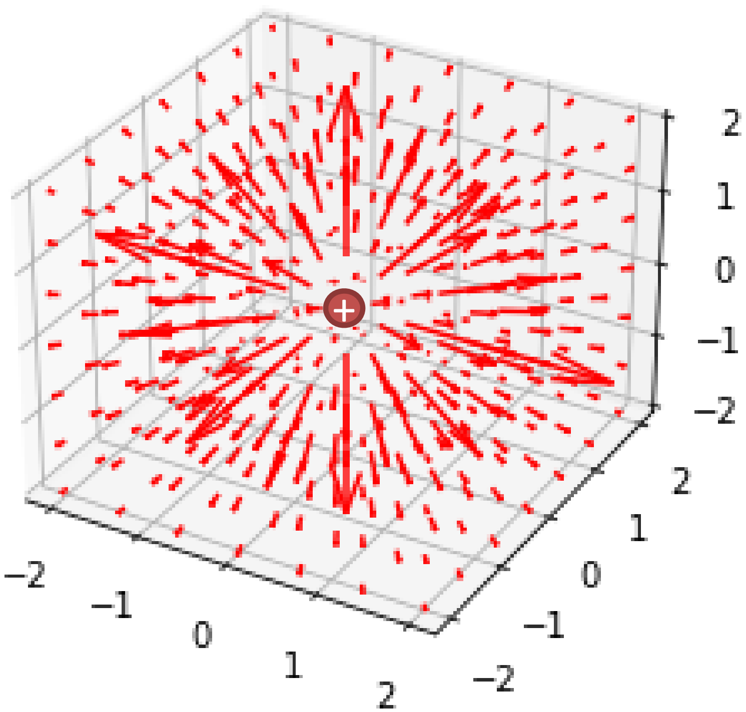
$$= q \underbrace{\frac{Q}{4\pi\varepsilon_0}\frac{1}{r^2}\hat{r}}$$

The <u>electric field</u> from a charge Q is

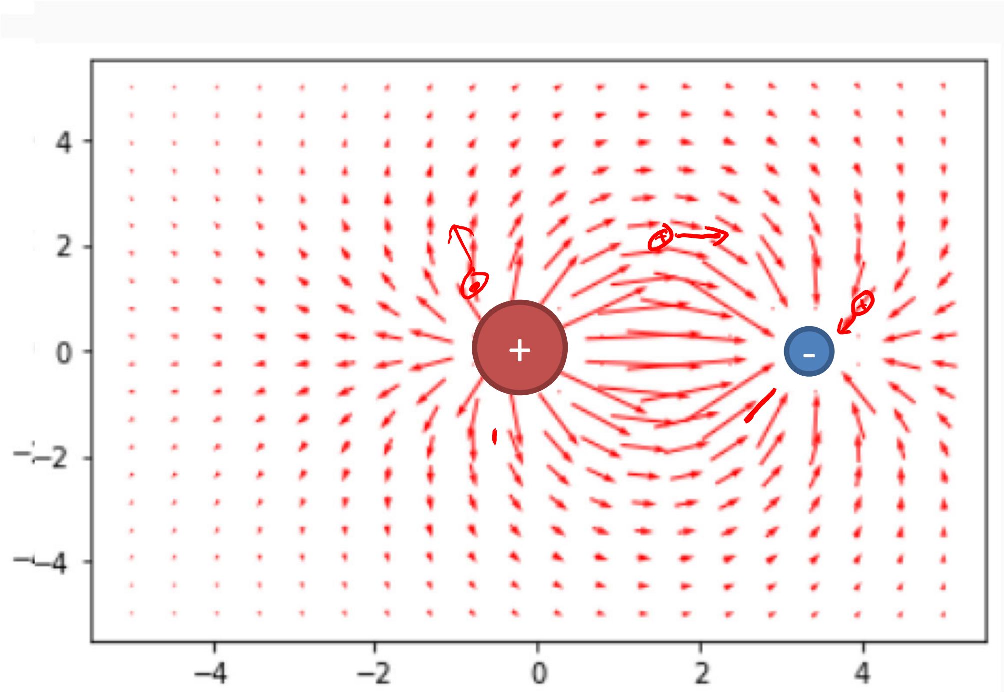
$$\boxed{\mathbf{E} = \frac{Q}{4\pi\varepsilon_0}\frac{1}{r^2}\hat{\mathbf{r}}}$$

*of classical theory; quantum theory has its own fields!

In 3D the electric field can look a bit more complicated:



Meanwhile, each of the *other charges* emits its own contribution to the field.

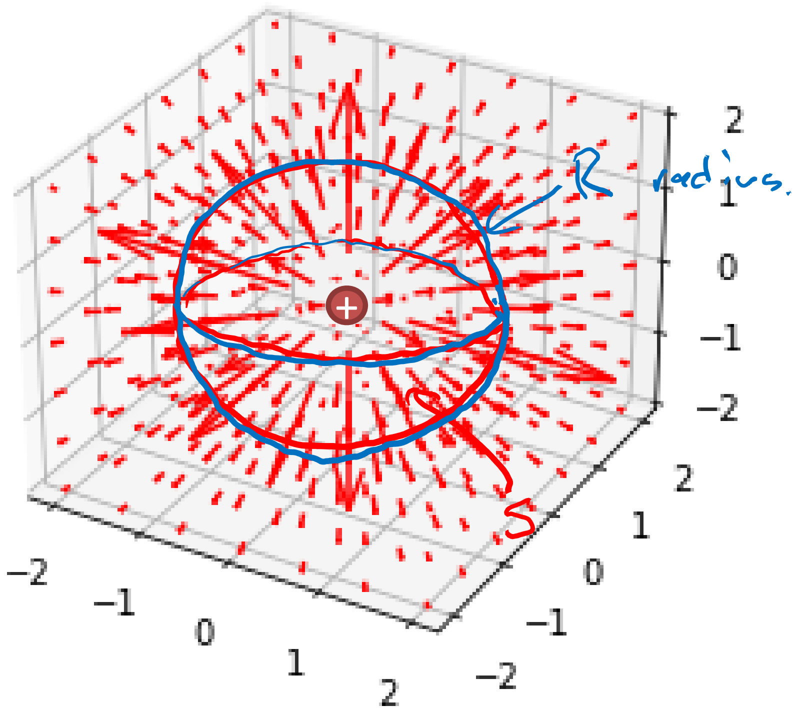


$$\mathbf{E} = \frac{Q_1}{4\pi\varepsilon_0}\frac{1}{|\mathbf{r}_1|^2}\hat{\mathbf{r}}_1 + \frac{Q_2}{4\pi\varepsilon_0}\frac{1}{|\mathbf{r}_2|^2}\hat{\mathbf{r}}_2$$

We consider a single "point charge" charge q, centred at the origin. The E field is

$$\mathbf{E} = \frac{q}{4\pi\varepsilon_0}\frac{1}{r^2}\hat{\mathbf{r}}$$

We now compute the *flux integral* of **E** over a spherical surface of radius R:



$d\underline{S} = \hat{\underline{r}}\, dS$

$$\iint_S \underline{E}\cdot d\underline{S} = \iint_S \frac{q}{4\pi\varepsilon_0}\frac{1}{r^2}\hat{\underline{r}}\cdot\hat{\underline{r}}\,dS$$

$r = R$

$$= \frac{q}{4\pi\varepsilon_0}\frac{1}{R^2}\iint_S \hat{\underline{r}}\cdot\hat{\underline{r}}\,dS$$

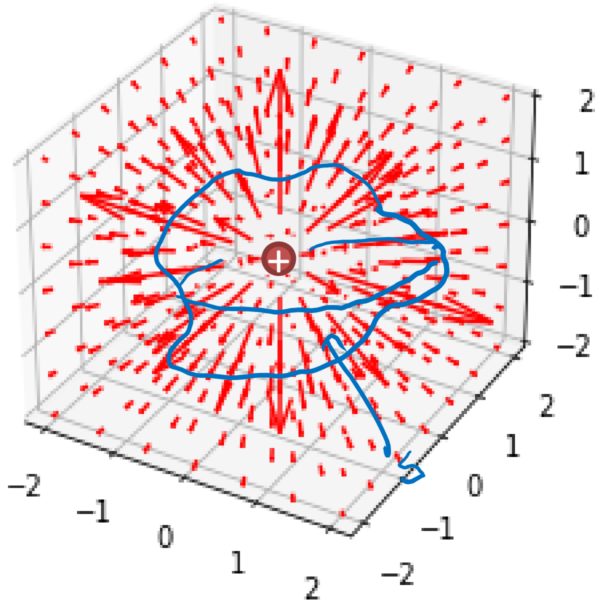
$$= \frac{q}{4\pi\varepsilon_0}\frac{1}{R^2}\underbrace{\iint_S dS}_{\text{Area of } S} = \frac{q}{\cancel{4\pi}\varepsilon_0}\frac{1}{\cancel{R^2}}\times\cancel{4\pi R^2} = \frac{q}{\varepsilon_0}.$$

$\Rightarrow$ doesn't depend on R

The resulting integral does not depend on the radius, or (it turns out) on the shape of the surface. Therefore we have

$$\underline{\iint_S \mathbf{E} \cdot \mathrm{d}\mathbf{S}} = \frac{Q_{\text{enc}}}{\varepsilon_0}$$

where S is *any surface enclosing the charge.*



What happens when we apply the divergence theorem?

$$\frac{Q_{enc}}{\varepsilon_0} = \iint_S \underset{\sim}{E} \cdot d\underset{\sim}{S} = \iiint_V (\underline{\nabla \cdot \underset{\sim}{E}})\, dV$$

~~But~~ we can write $Q_{enc} = \iiint_V \underline{\rho(\underset{\sim}{r})}\, dV$

charge density.

So we have found

$$\iint_S \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \iiint_V (\nabla \cdot \mathbf{E})\, \mathrm{d}V = \frac{Q_{\mathrm{enc}}}{\varepsilon_0}$$

But we can write the total enclosed charge in terms of the *charge density* $\rho(\boldsymbol{r})$:

$$\frac{Q_{enc}}{\varepsilon_0} = \iiint_V \frac{\rho(\underline{r})}{\varepsilon_0}\, dV \, .$$

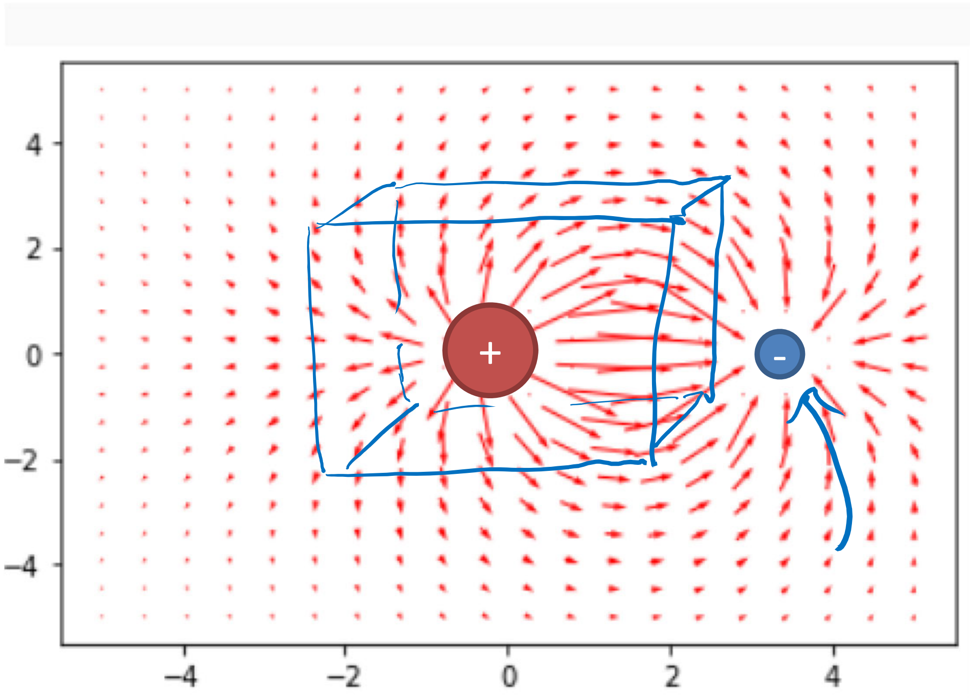
The only way this works for any volume V is if

$$\boxed{\nabla \cdot \mathbf{E} = \frac{\rho(\mathbf{r})}{\varepsilon_0}}$$

This is known as *Coulomb's law* and it is the first of Maxwell's equations.

What does Coulomb's law tell us?

$$\boxed{\nabla \cdot \mathbf{E} = \frac{\rho(\mathbf{r})}{\varepsilon_0}}$$



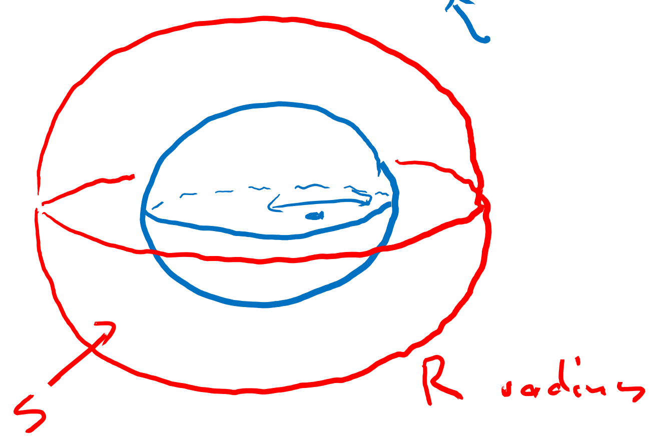
1. Positive charge is a *source* of electric field, while negative charge is a *sink.*

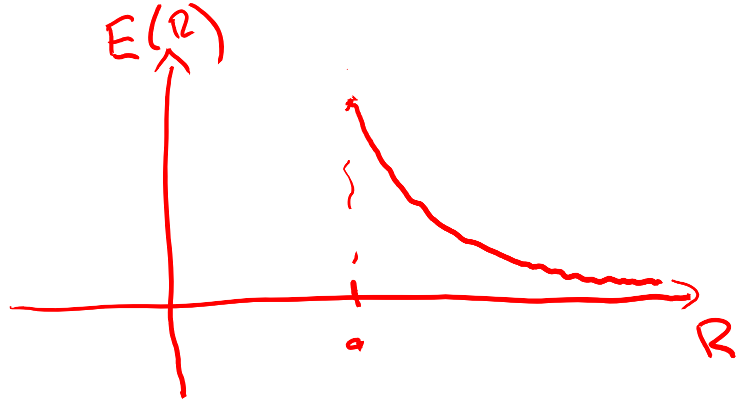
2. The flux integral

$$\boxed{\iint_S \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \frac{Q_{\mathrm{enc}}}{\varepsilon_0}}$$

only depends on the enclosed charge – charges outside the surface do not contribute anything.

Example: Compute the electric field for a sphere of radius a, having a uniform charge density ρ.





By symmetry: $\underset{\sim}{E} = E(r)\,\hat{\underset{\sim}{r}}$.

$$\iint_S \underset{\sim}{E}\cdot d\underset{\sim}{s} = \frac{Q_{enc}}{\varepsilon_0} = \frac{1}{\varepsilon_0}\iiint_{V_a} \rho\, dV = \frac{1}{\varepsilon_0}\rho \iiint_{V_a} dV = \frac{1}{\varepsilon_0}\rho\,\frac{4}{3}\pi a^3.$$

But

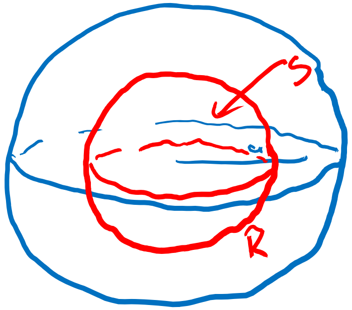
$$\iint_S \underset{\sim}{E}\cdot d\underset{\sim}{s} = \iint_S E(r)\,\hat{\underset{\sim}{r}}\cdot\hat{\underset{\sim}{r}}\, dS = \iint_S E(R)\, dS = E(R)\times 4\pi R^2$$

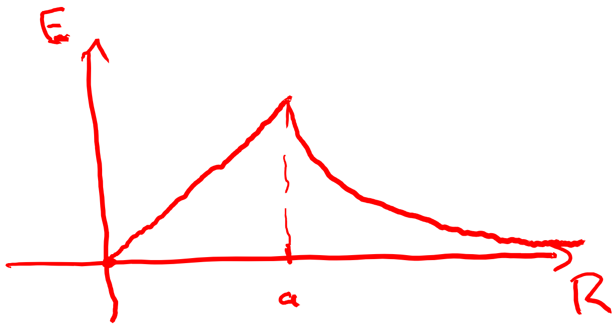
$r = R$

So

$$E(R)\times 4\pi R^2 = \frac{\rho}{\varepsilon_0}\frac{4}{3}\pi a^3 \Rightarrow E(R) = \frac{\rho}{\varepsilon_0}\frac{a^3}{R^2}.$$

What about inside?





Now

$$\iint_S \underset{\sim}{E} \cdot dS = \frac{Q_{enc}}{\varepsilon_0} = \frac{1}{\varepsilon_0} \iiint_{V_R} \rho \, dV = \frac{\rho}{\varepsilon_0} \iiint_{V_R} dV$$

$$= \frac{\rho}{\varepsilon_0} \frac{4}{3} \pi R^3 .$$

So

$$\iint_S E(r) \underset{\sim}{\hat{r}} \cdot \underset{\sim}{\hat{r}} \, dS = E(R) \iint_S dS = E(R) \times 4\pi R^2$$

Therefore

$$\underline{E(R)} \times \cancel{4}\cancel{\pi} R^2 = \frac{\rho}{\varepsilon_0} \frac{\cancel{4}}{3} \cancel{\pi} R^3 \Rightarrow E(R) = \frac{\rho}{\varepsilon_0} \frac{1}{3} R$$

What about the curl of the electric field?

Consider the field from a point charge:

$$\mathbf{E} = \frac{q}{4\pi\varepsilon_0}\frac{1}{r^2}\hat{\mathbf{r}} = E_r \hat{\underline{r}}$$

$$\nabla \times \underline{A} = \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}\left(A_\varphi \sin\theta\right) - \frac{\partial A_\theta}{\partial\varphi}\right)\hat{\mathbf{r}} + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\varphi} - \frac{\partial}{\partial r}\left(rA_\varphi\right)\right)\hat{\boldsymbol{\theta}} + \frac{1}{r}\left(\frac{\partial}{\partial r}\left(rA_\theta\right) - \frac{\partial A_r}{\partial\theta}\right)\hat{\boldsymbol{\varphi}}$$

Set $\underline{A} = \underline{E}$

$E_\varphi = 0$

$E_\theta = 0$.

So

$$\nabla \times \underline{E} = \frac{1}{r}\frac{1}{\sin\theta}\frac{\partial E_r}{\partial\varphi}\hat{\underline{\theta}} + \frac{1}{r}\left(-\frac{\partial E_r}{\partial\theta}\right)\hat{\underline{\varphi}}$$

But $\underline{E}$ doesn't depend on θ or φ, so

$$\nabla \times \underline{E} = 0.$$

So the Electric field is irrotational:

$$\nabla \times \mathbf{E} = \mathbf{0}$$

We can therefore express it as the gradient of a potential function:

$$\underset{\sim}{E} = \nabla \phi$$

By convention

$$\boxed{\mathbf{E} = -\nabla V}$$

$$\boxed{\nabla \cdot \underset{\sim}{E} = \frac{\rho}{\varepsilon_0}}$$

where V is known as the electrostatic potential. Taking the divergence leads to

$$\nabla \cdot (-\nabla V) = \frac{\rho}{\epsilon_0}$$

$$\boxed{\nabla^2 V = \frac{\rho}{\varepsilon_0}}$$

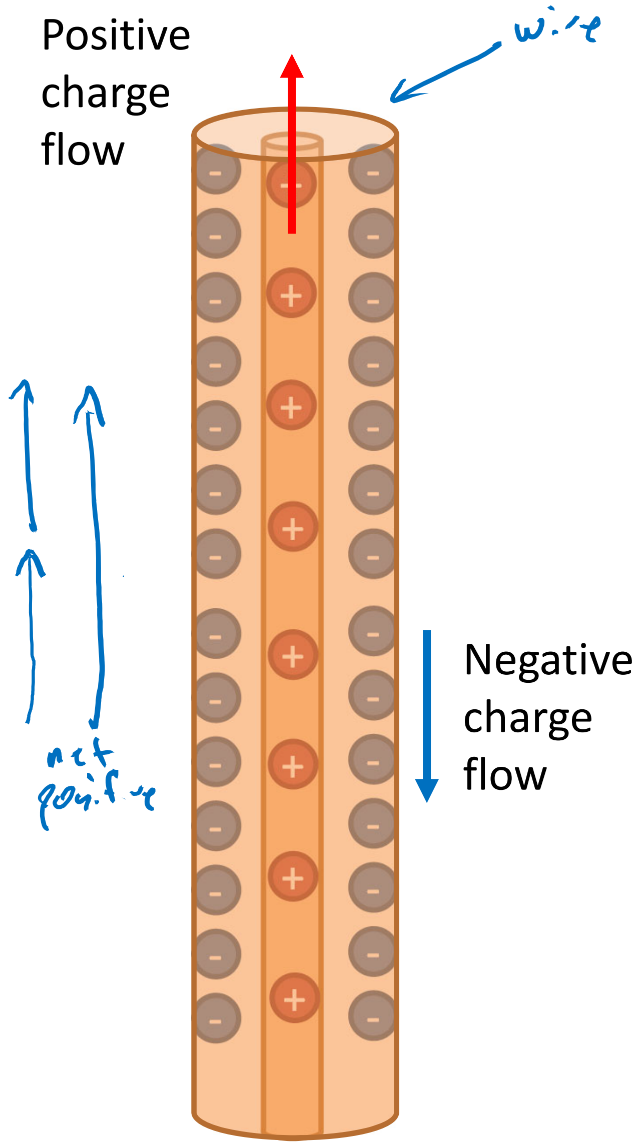
In free space,

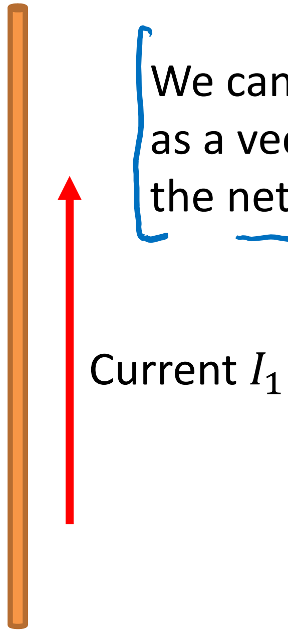
$$\nabla^2 V = 0$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) V = 0 .$$

Electric current and the magnetic field

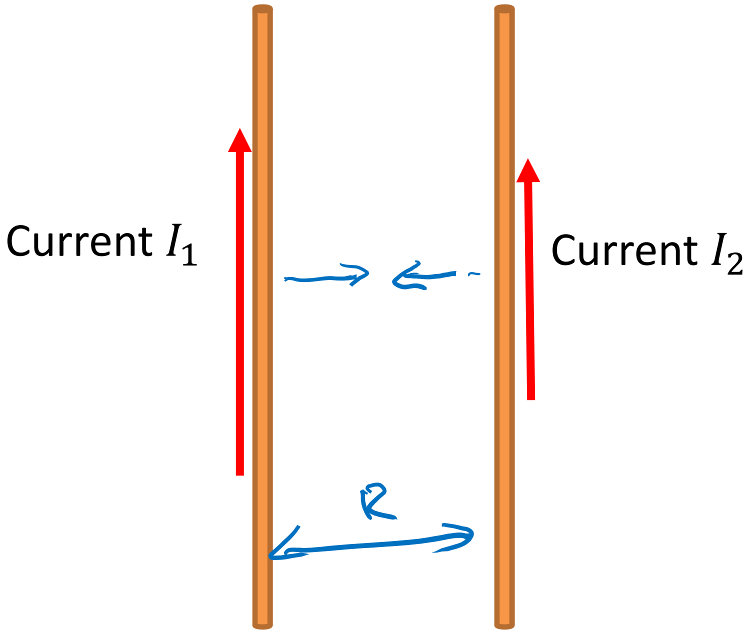
An electric current is a *moving line of charges, in which positive and negative charges are moving continuously in opposite directions*





We can depict the current as a vector $\boldsymbol{I}$, which gives the net flow of positive charge per second

Observation: electric currents exert forces on each other



The force per unit length is

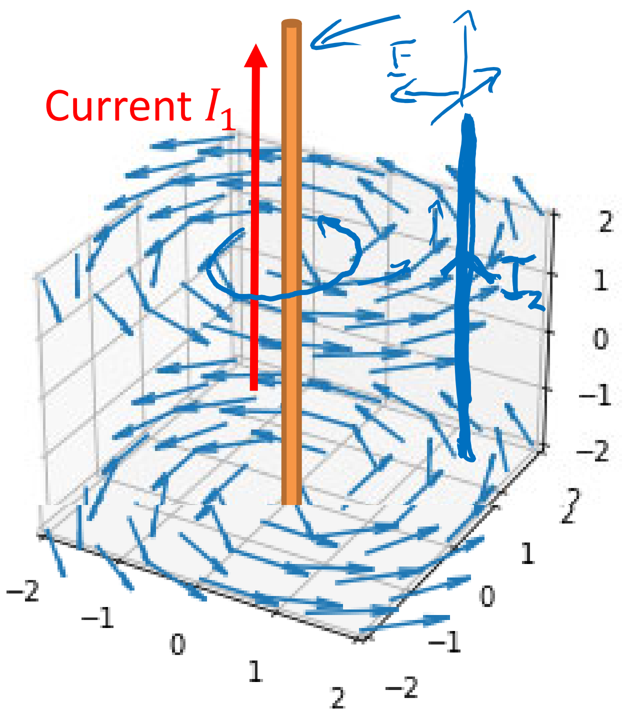
1. Directed along a line between the two currents

2. Proportional to the product of the currents $I_1 I_2$

3. *Inversely proportional* to the distance between them

$f \sim \frac{1}{R}$.

Each current emits a *magnetic field B, which circles around the z-axis:*



$$\boxed{\mathbf{B} = \frac{\mu_0}{2\pi}\frac{I_1}{r}\hat{\boldsymbol{\theta}}}$$

And the resulting force per unit length on *another current is*

$$\boxed{\mathbf{f} = \mathbf{I_2} \times \mathbf{B}}$$

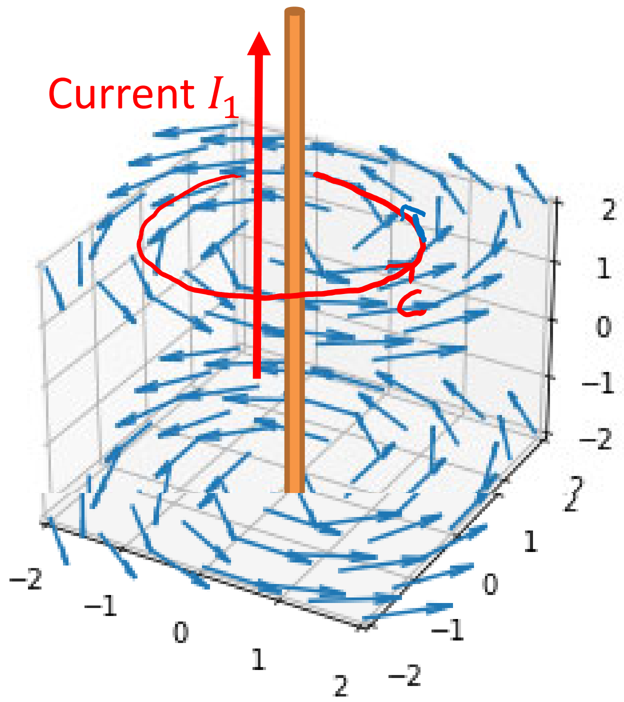
The constant μ_0 can be measured experimentally, and is called the *vacuum permeability*

$$\mu_0 = 1.256637 \times 10^{-6} \mathrm{N/A}^2$$

We now take the *vector line integral of*

$$\mathbf{B} = \frac{\mu_0}{2\pi}\frac{I_1}{r}\hat{\boldsymbol{\theta}}$$

along a loop of radius R enclosing the wire:



$$\oint_c \underset{\sim}{B} \cdot d\underset{\sim}{r} = \oint_c \frac{\mu_0}{2\pi}\frac{I_1}{r}\hat{\underset{\sim}{\theta}} \cdot d\underset{\sim}{r}$$

Now $\underbrace{d\underset{\sim}{r} = \hat{\underset{\sim}{\theta}}\, ds}$ ← arc-length

So

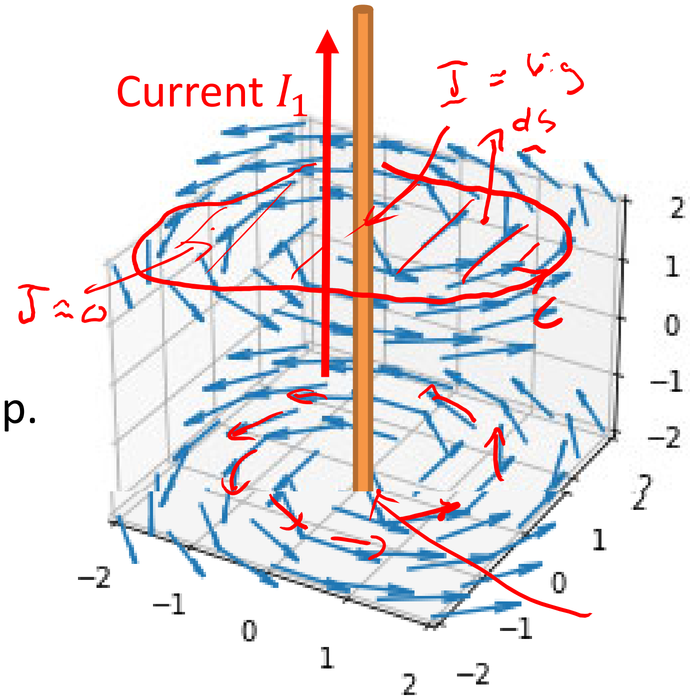
$$\oint_c \underset{\sim}{B} \cdot d\underset{\sim}{r} = \oint_c \frac{\mu_0}{2\pi}\frac{I_1}{R}\hat{\underset{\sim}{\theta}} \cdot \hat{\underset{\sim}{\theta}}\, ds$$

$$= \oint_c \frac{\mu_0}{2\pi R} I_1\, ds = \frac{\mu_0 I_1}{2\pi R}\underbrace{\oint_c ds}_{\text{Length of } c} = \frac{\mu_0 I_1}{\cancel{2\pi R}} \times \cancel{2\pi R} = \underline{\mu_0 I_1}.$$

So

$$\boxed{\oint \mathbf{B} \cdot \mathrm{d}\mathbf{r} = \mu_0 I_{\mathrm{enc}}}$$

Where I_{enc} is the enclosed charge. Note that this does not depend on the radius of the loop.



By Stokes' theorem

$$\oint_c \mathbf{B} \cdot \mathrm{d}\mathbf{r} = \iint_S \nabla \times \mathbf{B} \cdot \mathrm{d}\mathbf{S} = \mu_0 I_{\mathrm{enc}}$$

But we can also write the current in terms of *the current density* $\mathbf{J}(\boldsymbol{r})$:

$$I_{\mathrm{enc}} = \iint_S \mathbf{J} \cdot \mathrm{d}\mathbf{S}$$

Therefore:

$$\iint_S (\nabla \times \underset{\sim}{B}) \cdot d\underset{\sim}{S} = \mu_0 \iint_S \underset{\sim}{J} \cdot d\underset{\sim}{S} \implies \nabla \times \underset{\sim}{B} = \mu_0 \underset{\sim}{J}.$$

We have found an expression for the curl of the magnetic field:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

What about the divergence?

From the expression for a line current

$$\mathbf{B} = \frac{\mu_0}{2\pi}\frac{I_1}{r}\hat{\boldsymbol{\theta}}$$

The divergence is

$$\nabla \cdot \underset{\sim}{A} = \frac{1}{\rho}\frac{\partial(\rho A_\rho)}{\partial\rho} + \frac{1}{\rho}\frac{\partial A_\varphi}{\partial\varphi} + \frac{\partial A_z}{\partial z}$$

$$\nabla \cdot \underset{\sim}{B} = \frac{1}{r}\frac{\partial}{\partial r}(r B_r) + \frac{1}{r}\frac{\partial}{\partial\theta}(B_\theta) + \frac{\partial B_z}{\partial z}.$$

$$= \frac{1}{r}\frac{\partial}{\partial\theta}B_\theta = \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{\mu_0}{2\pi}\frac{I_1}{r}\right)$$

$$= 0.$$

We therefore have the following equations for the magnetic field:

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

$\nabla \cdot \underset{\sim}{E} = \frac{\rho}{\varepsilon_0}$

$\nabla \times \underset{\sim}{E} = 0$

Maxwell's equations

What have we found so far?

For the electric field

$$\nabla \cdot \mathbf{E} = \frac{\rho(\mathbf{r})}{\varepsilon_0}$$

$$\nabla \times \mathbf{E} = \mathbf{0}$$

For the magnetic field

$$\nabla \cdot \mathbf{B} = 0$$

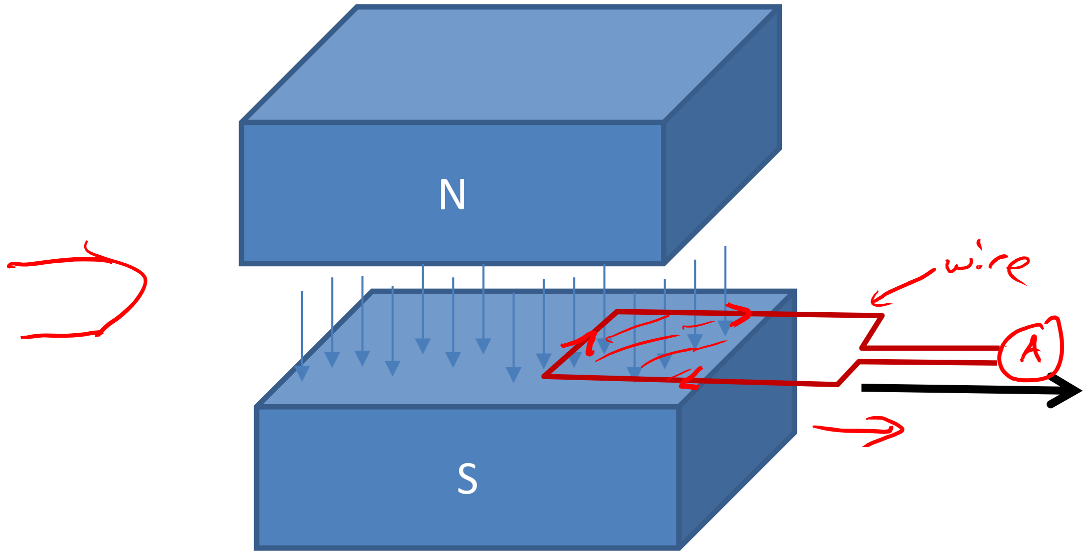
$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

The constants ε_0 and μ_0 are well established from the measurement of optical forces on charges and currents:

$$\mu_0 = 1.256637 \times 10^{-6} \mathrm{N/A^2}$$

$$\varepsilon_0 = 8.854188 \times 10^{-12} \mathrm{A^2 s^2/N/m^2}$$

Experiments by Faraday showed that a *changing magnetic field*
Produced a force that acted in *exactly the same way as an electric field.*



This showed that the electric field and the magnetic field were *coupled together:*

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

That is: *a changing magnetic field causes rotation of the electric field*

These equations now look unbalanced.

$$\nabla \cdot \mathbf{E} = \frac{\rho(\mathbf{r})}{\varepsilon_0} \qquad \nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \boxed{\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \dots}$$

In 1873, Maxwell proposed a modification of the curl equation for the magnetic field

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

What are the consequences of this?

Maxwell's equations:

$$\nabla \cdot \mathbf{E} = \frac{\rho(\mathbf{r})}{\varepsilon_0} \qquad \nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

Consider an electric and magnetic field pair in free space,
So that $\rho = 0$ and $\boldsymbol{J} = \mathbf{0}$ everywhere. Maxwell's equations are

$$\nabla \cdot \mathbf{E} = 0 \qquad \nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

Take the curl of the third equation:

$$\nabla \times (\nabla \times E) = \nabla \times \left(-\frac{\partial B}{\partial t} \right)$$

$$= -\frac{\partial}{\partial t}(\nabla \times B)$$

$$= -\frac{\partial}{\partial t}\left(\mu_0 \varepsilon_0 \frac{\partial E}{\partial t}\right)$$

$$= -\mu_0 \varepsilon_0 \frac{\partial^2 E}{\partial t^2}.$$

But $\nabla \times (\nabla \times E) = -\nabla^2 E + \nabla(\nabla \cdot E)$.

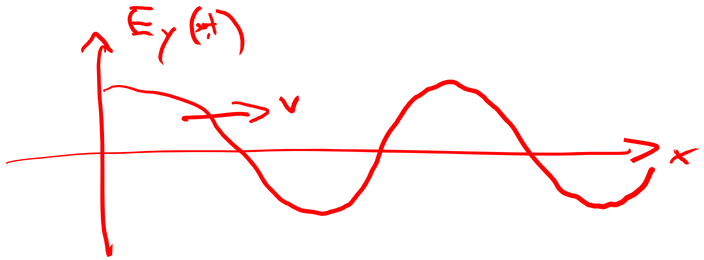
$$= -\nabla^2 E.$$

So

$$\nabla^2 E = -\mu_0 \varepsilon_0 \frac{\partial^2 E}{\partial t^2}.$$

So

$$\nabla^2\mathbf{E} = \mu_0\varepsilon_0 \frac{\partial^2\mathbf{E}}{\partial t^2}$$



This is a *wave equation* (we will do this in a few weeks).

It has solutions which are waves with a velocity

$$v = \frac{1}{\sqrt{\mu_0\varepsilon_0}}$$

$$\mu_0 = 1.256637 \times 10^{-6}\mathrm{N/A^2}$$
$$\varepsilon_0 = 8.854188 \times 10^{-12}\mathrm{A^2s^2/N/m^2}$$

$$= \frac{1}{\sqrt{1.26 \times 10^{-6} \times 8.85 \times 10^{-12}}} = \frac{1}{\sqrt{11 \times 10^{-18}}} = \frac{1}{\sqrt{11}} \times 10^{9}$$

$$= 0.3 \times 10^{9} = \underline{3 \times 10^{8}\ \mathrm{ms^{-1}}}.$$

Plot of E-field of an electromagnetic plane wave: