Charge and the electric field

About 200 years ago, people noticed that *charged* objects exerted a force on each other.

The force was observed to be:

1. Directed along a line between the two objects

2. Proportional to the product of the charges Qq

3. Proportional to the *inverse square* of the distance between them

We can write the force as a vector. In the coordinate system centred on the positive charge, we have

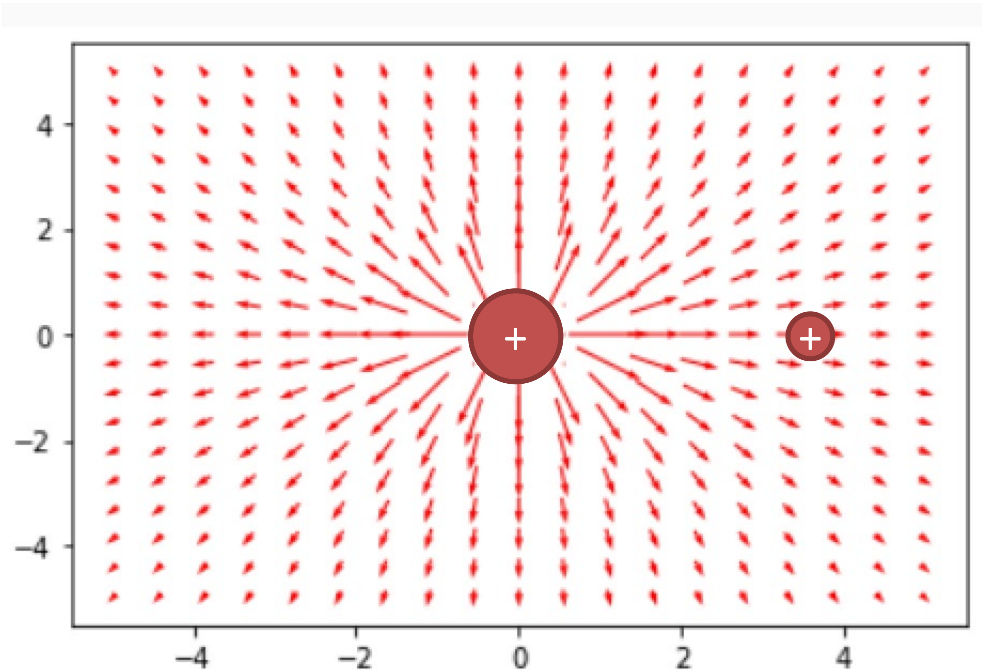
$$\mathbf{F} = \frac{Qq}{4\pi\varepsilon_0}\frac{1}{r^2}\hat{\mathbf{r}}$$

ε_0 is an experimentally-measured number, called the vacuum *permittivity:*

$$\varepsilon_0 = 8.854188 \times 10^{-12} \mathrm{A}^2\mathrm{s}^2/\mathrm{N}/\mathrm{m}^2$$

The *electric charge* is measured in *Coulombs* C.

The current understanding* is that small electric *charges* emit a *vector field*. This field in turn exerts a force on other charges.



$$\mathbf{F} = \frac{Qq}{4\pi\varepsilon_0}\frac{1}{r^2}\hat{\mathbf{r}}$$

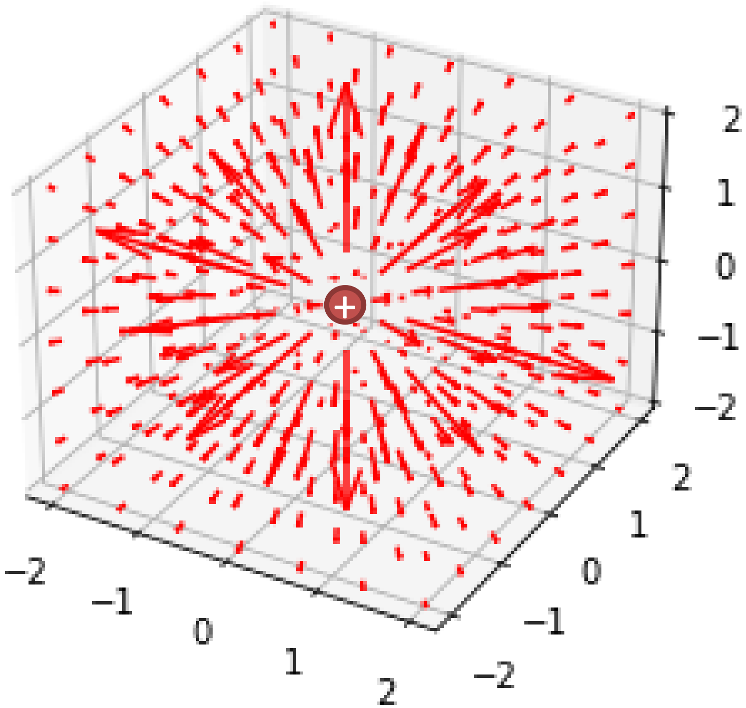
$$= q\mathbf{E}$$

The electric field from a charge Q is

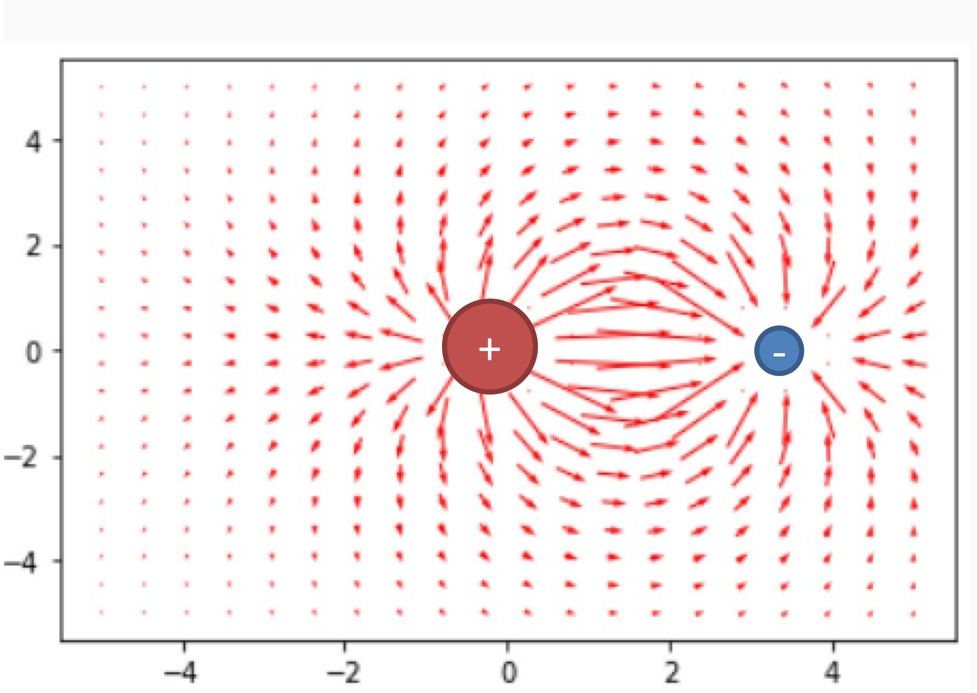
$$\mathbf{E} = \frac{Q}{4\pi\varepsilon_0}\frac{1}{r^2}\hat{\mathbf{r}}$$

*of classical theory; quantum theory has its own fields!

In 3D the electric field can look a bit more complicated:



Meanwhile, each of the *other charges* emits its own contribution to the field.

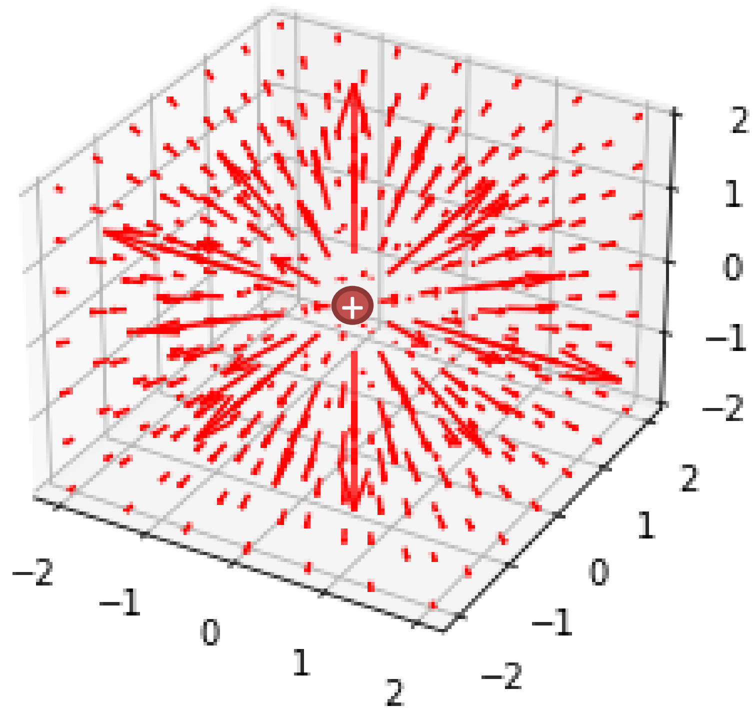


$$\mathbf{E} = \frac{Q_1}{4\pi\varepsilon_0}\frac{1}{|\mathbf{r}_1|^2}\hat{\mathbf{r}_1} + \frac{Q_2}{4\pi\varepsilon_0}\frac{1}{|\mathbf{r}_2|^2}\hat{\mathbf{r}_2}$$

We consider a single “point charge” charge q, centred at the origin. The E field is

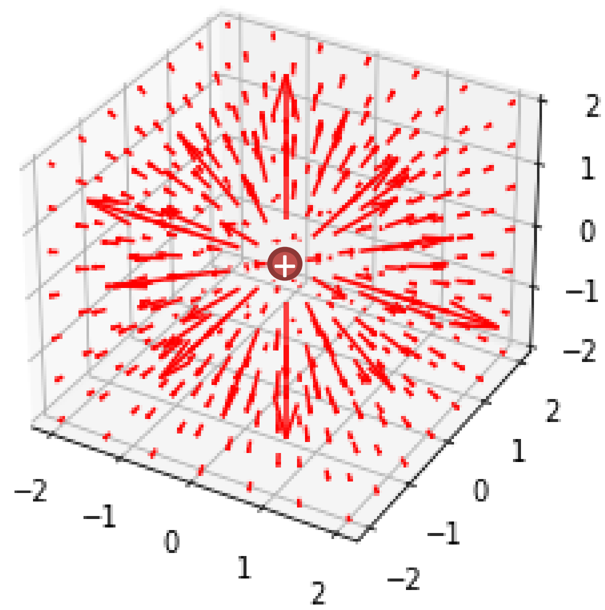
$$\mathbf{E} = \frac{q}{4\pi\varepsilon_0}\frac{1}{r^2}\hat{\mathbf{r}}$$

We now compute the *flux integral* of **E** over a spherical surface of radius R:



The resulting integral does not depend on the radius, or (it turns out) on the shape of the surface. In general we have

$$\iint_S \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \frac{Q_{\mathrm{enc}}}{\varepsilon_0}$$



where S is *any surface enclosing the charge.*

What happens when we apply the divergence theorem?

So we have found

$$\iint_S \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \iiint_V (\nabla \cdot \mathbf{E})\, \mathrm{d}V = \frac{Q_{\mathrm{enc}}}{\varepsilon_0}$$

But we can write the total enclosed charge in terms of the *charge density* $\rho(\boldsymbol{r})$:

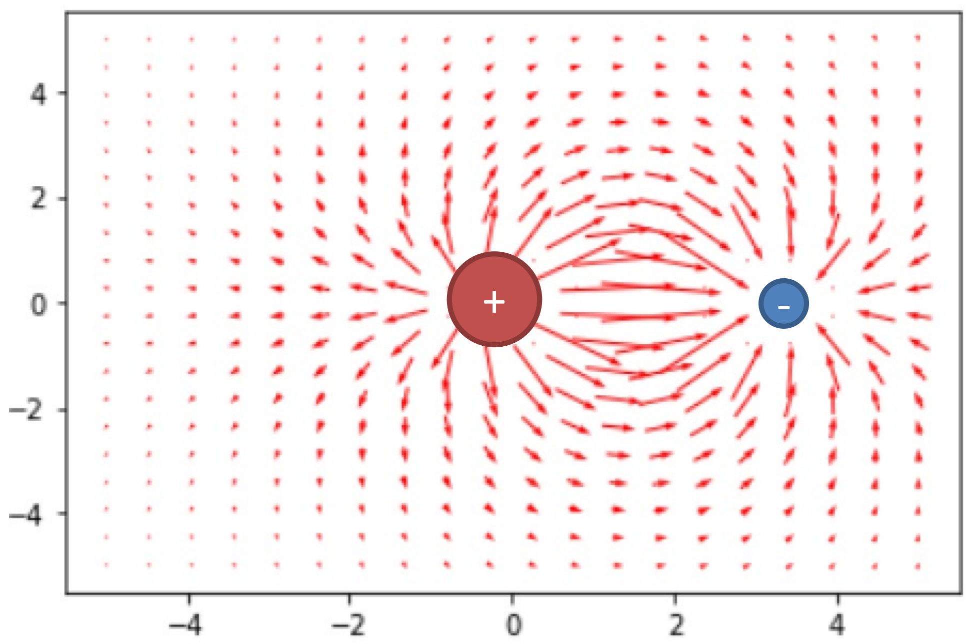
The only way this works for any volume V is if

$$\nabla \cdot \mathbf{E} = \frac{\rho(\mathbf{r})}{\varepsilon_0}$$

This is known as *Coulomb's law* and it is the first of Maxwell's equations.

What does Coulomb's law tell us?

$$\nabla \cdot \mathbf{E} = \frac{\rho(\mathbf{r})}{\varepsilon_0}$$



1. Positive charge is a *source* of electric field, while negative charge is a *sink.*

2. The flux integral

$$\iint_S \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \frac{Q_{\text{enc}}}{\varepsilon_0}$$

only depends on the enclosed charge – charges outside the surface do not contribute anything.

Example: Compute the electric field for a sphere of radius R, having a uniform charge density ρ.

Example: A continuous line of charge can be represented by a *line charge density (charge per unit length)* $\lambda(\boldsymbol{r})$. Compute the Electric field for an infinite line charge aligned along the z axis.

What about the curl of the electric field?

Consider the field from a point charge:

$$\mathbf{E} = \frac{q}{4\pi\varepsilon_0}\frac{1}{r^2}\hat{\mathbf{r}}$$

So the Electric field is irrotational:

$$\nabla \times \mathbf{E} = \mathbf{0}$$

We can therefore express it as the gradient of a potential function:

By convention

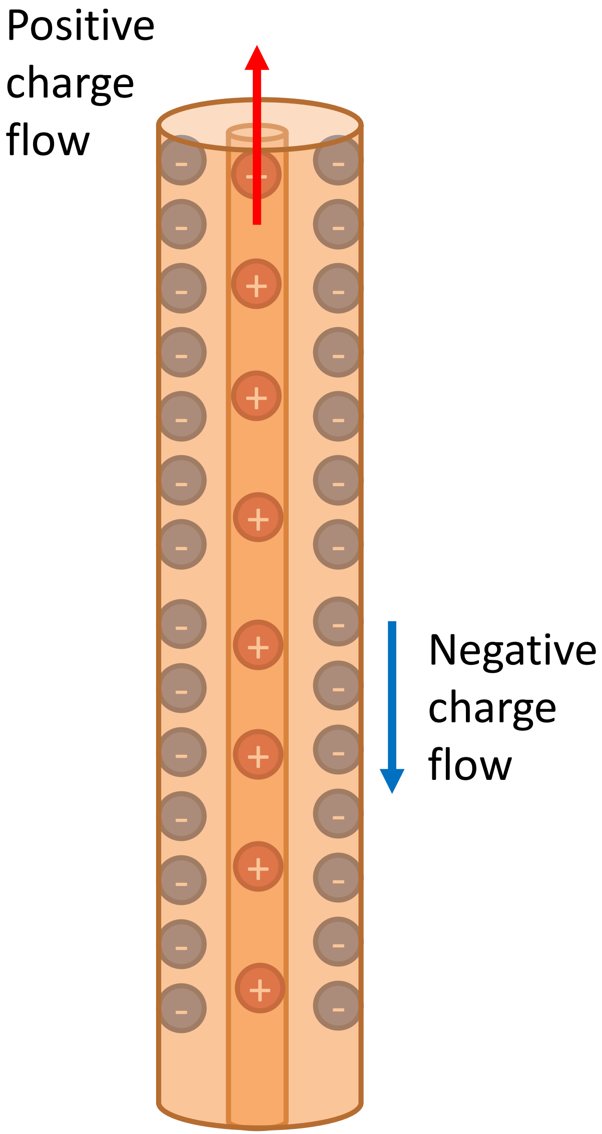
$$\mathbf{E} = -\nabla V$$

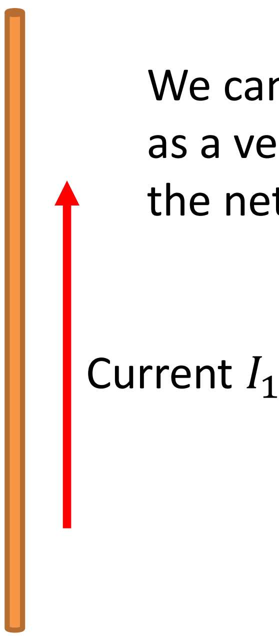
where V is known as the electrostatic potential. Taking the divergence leads to

$$\nabla \cdot (-\nabla V) = \frac{\rho}{\epsilon_0}$$

Electric current and the magnetic field

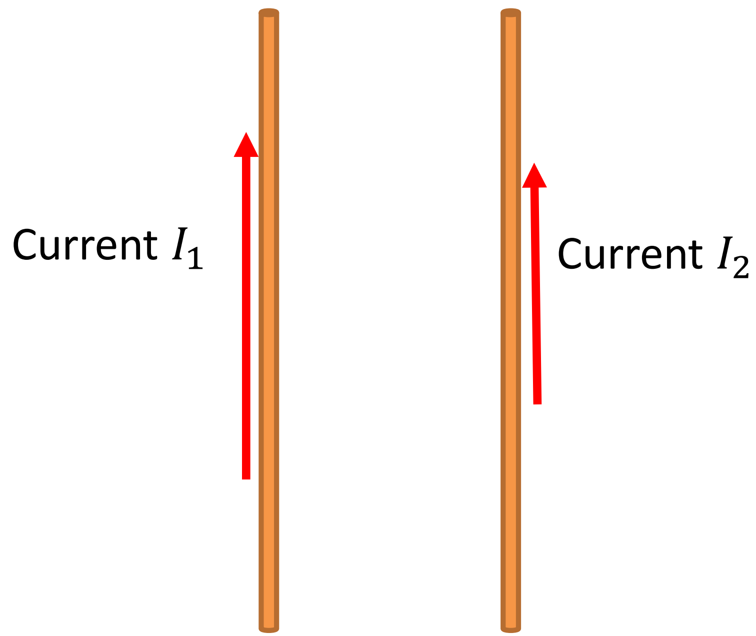
An electric current is a *moving line of charges, in which positive and negative charges are moving continuously in opposite directions*





We can depict the current as a vector $\boldsymbol{I}$, which gives the net flow of positive charge per second

Observation: electric currents exert forces on each other



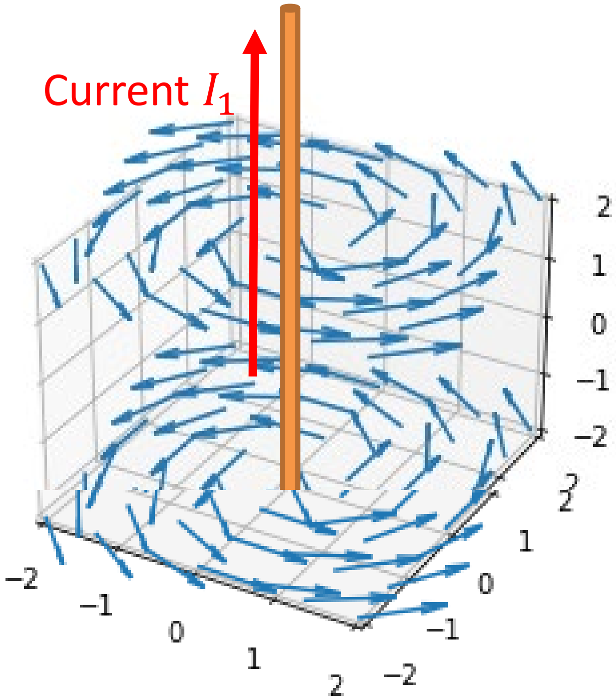
The force per unit length is

1. Directed along a line between the two currents

2. Proportional to the product of the currents $I_1 I_2$

3. *Inversely proportional* to the distance between them

Each current emits a *magnetic field B, which circles around the z-axis:*



$$\mathbf{B} = \frac{\mu_0}{2\pi}\frac{I_1}{r}\hat{\boldsymbol{\theta}}$$

And the resulting force per unit length on *another current is*

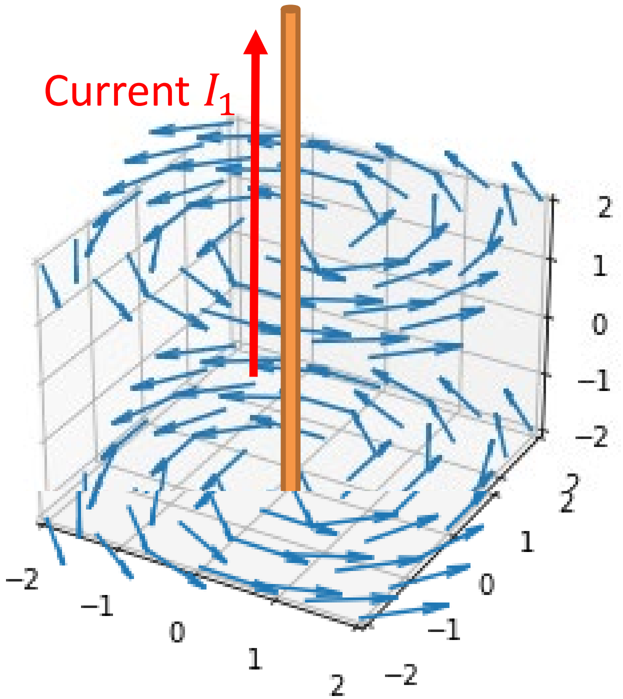
$$\mathbf{f} = \mathbf{I_2} \times \mathbf{B}$$

The constant μ_0 can be measured experimentally, and is called the *vacuum permeability*

$$\mu_0 = 1.256637 \times 10^{-6} \mathrm{N}/\mathrm{A}^2$$

We now take the *vector line integral of*

$$\mathbf{B} = \frac{\mu_0}{2\pi}\frac{I_1}{r}\hat{\boldsymbol{\theta}}$$

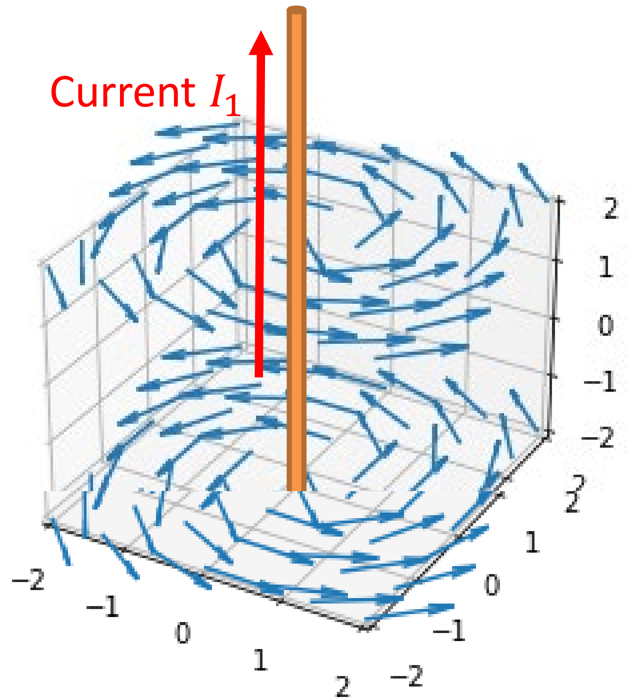


along a loop of radius R enclosing the wire:

So

$$\oint \mathbf{B} \cdot \mathrm{d}\mathbf{r} = \mu_0 I_{\mathrm{enc}}$$

Where I_{enc} is the enclosed charge. Note
that this does not depend on the radius of the loop.



By Stokes' theorem

$$\oint \mathbf{B} \cdot \mathrm{d}\mathbf{r} = \iint_S \nabla \times B \cdot \mathrm{d}\mathbf{S} = \mu_0 I_{\mathrm{enc}}$$

But we can also write the current in terms of *the current density* $\boldsymbol{J}(\boldsymbol{r})$:

$$I_{\mathrm{enc}} = \iint_S \mathbf{J} \cdot \mathrm{d}\mathbf{S}$$

Therefore:

We have found an expression for the curl of the magnetic field:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

What about the divergence?

From the expression for a line current

$$\mathbf{B} = \frac{\mu_0}{2\pi} \frac{I_1}{r} \hat{\boldsymbol{\theta}}$$

The divergence is

We therefore have the following equations for the magnetic field:

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

Maxwell’s equations

What have we found so far?

For the electric field

$$\nabla \cdot \mathbf{E} = \frac{\rho(\mathbf{r})}{\varepsilon_0}$$

$$\nabla \times \mathbf{E} = \mathbf{0}$$

For the magnetic field

$$\nabla \cdot \mathbf{B} = 0$$

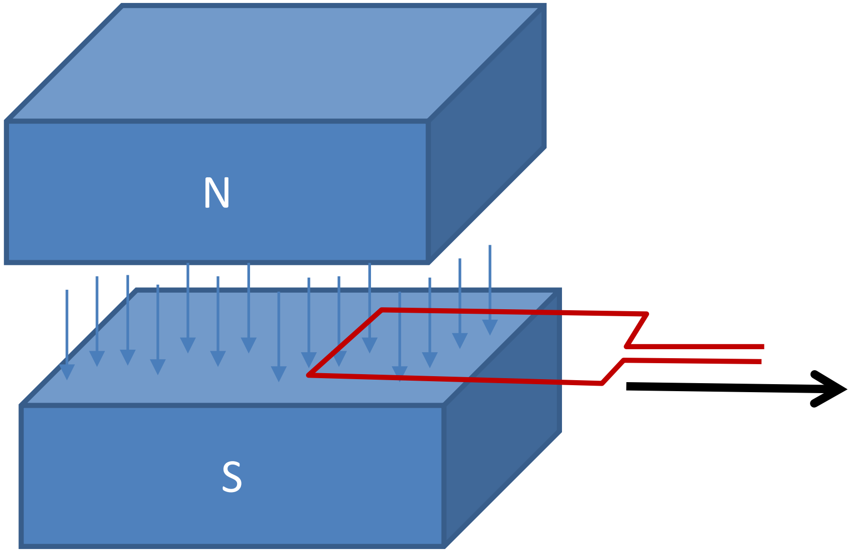
$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

The constants ε_0 and μ_0 are well established from the measurement of optical forces on charges and currents:

$$\mu_0 = 1.256637 \times 10^{-6} \mathrm{N/A}^2$$

$$\varepsilon_0 = 8.854188 \times 10^{-12} \mathrm{A}^2\mathrm{s}^2/\mathrm{N/m}^2$$

Experiments by Faraday showed that a *changing magnetic field*
Produced a force that acted in *exactly the same way as an electric field.*



This showed that the electric field and the magnetic field were *coupled together:*

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

That is: *a changing magnetic field causes rotation of the electric field*

These equations now look unbalanced.

$$\nabla \cdot \mathbf{E} = \frac{\rho(\mathbf{r})}{\varepsilon_0} \qquad \nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

In 1873, Maxwell proposed a modification of the curl equation for the magnetic field

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

What are the consequences of this?

Maxwell's equations:

$$\nabla \cdot \mathbf{E} = \frac{\rho(\mathbf{r})}{\varepsilon_0} \qquad \nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

Consider an electric and magnetic field pair in free space,
So that $\rho = 0$ and $\boldsymbol{J} = \mathbf{0}$ everywhere. Maxwell's equations are

$$\nabla \cdot \mathbf{E} = 0 \qquad \nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

Take the curl of the third equation:

So

$$\nabla^2 \mathbf{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

This is a *wave equation* (we will do this in a few weeks).

It has solutions which are waves with a velocity

$$v = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$$

$$\mu_0 = 1.256637 \times 10^{-6} \mathrm{N/A}^2$$

$$\varepsilon_0 = 8.854188 \times 10^{-12} \mathrm{A}^2 \mathrm{s}^2/\mathrm{N}/\mathrm{m}^2$$

Plot of E-field of an electromagnetic plane wave: