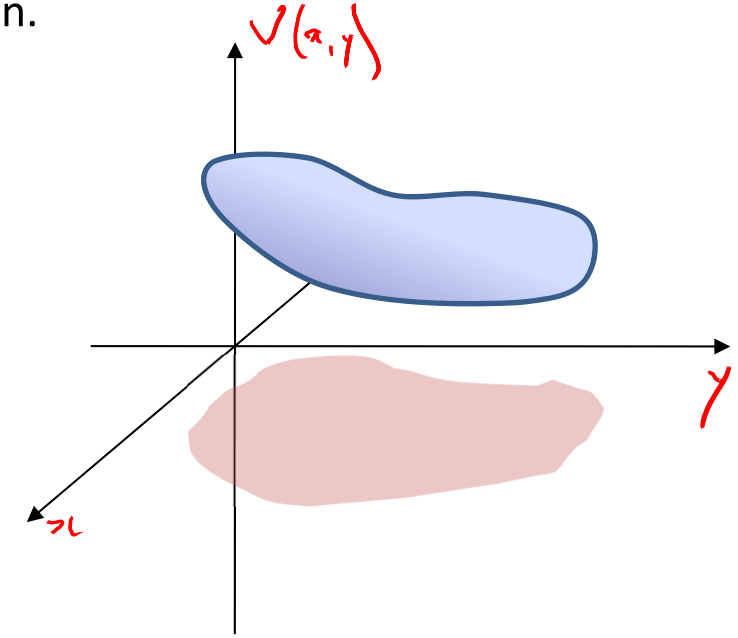
Second order partial differential equations (in 2D)

A <u>Partial Differential Equation</u> expresses a relationship between the derivatives of a multi-variable function on a domain.



E.g. The Laplace equation for the electric potential

$$\nabla^2 V = 0$$

In 2D is

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

We can represent a PDE as an equation involving a *differential operator:*

$$\mathcal{L}\psi = 0$$

$$\mathcal{L} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

To solve a PDE we need to fund a function that fits the PDE itself, as well as any *boundary conditions*.

Classification of 2nd order PDEs

Partial differential equations can be *linear* or *nonlinear.*

A linear PDE has the property that you can add multiples of any two solutions to get another solution.

i.e. If ψ_1 and ψ_2 are solutions to

$$\mathcal{L}\psi = 0$$

Then

$$\psi_{\text{new}} = a\psi_1 + b\psi_2$$

is also a solution.

$$\mathcal{L}\psi_{new} = 0$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(a\psi_1 + b\psi_2) = 0$$

$$a\left(\frac{\partial^2 \psi_1}{\partial x^2} + \frac{\partial^2 \psi_1}{\partial y^2}\right) + b\left(\frac{\partial^2 \psi_2}{\partial x^2} + \frac{\partial^2 \psi_2}{\partial y^2}\right) = 0$$

We can recognize non-linear PDEs in that they often "mix up" or multiply derivatives together. E.g.

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial \psi}{\partial y}\psi = 0$$

$$\left(\frac{\partial \psi}{\partial x}\right)^2 + \frac{\partial^2 \psi}{\partial y^2} = 0$$

In two dimensions, the most general form of a linear 2nd-order PDE is

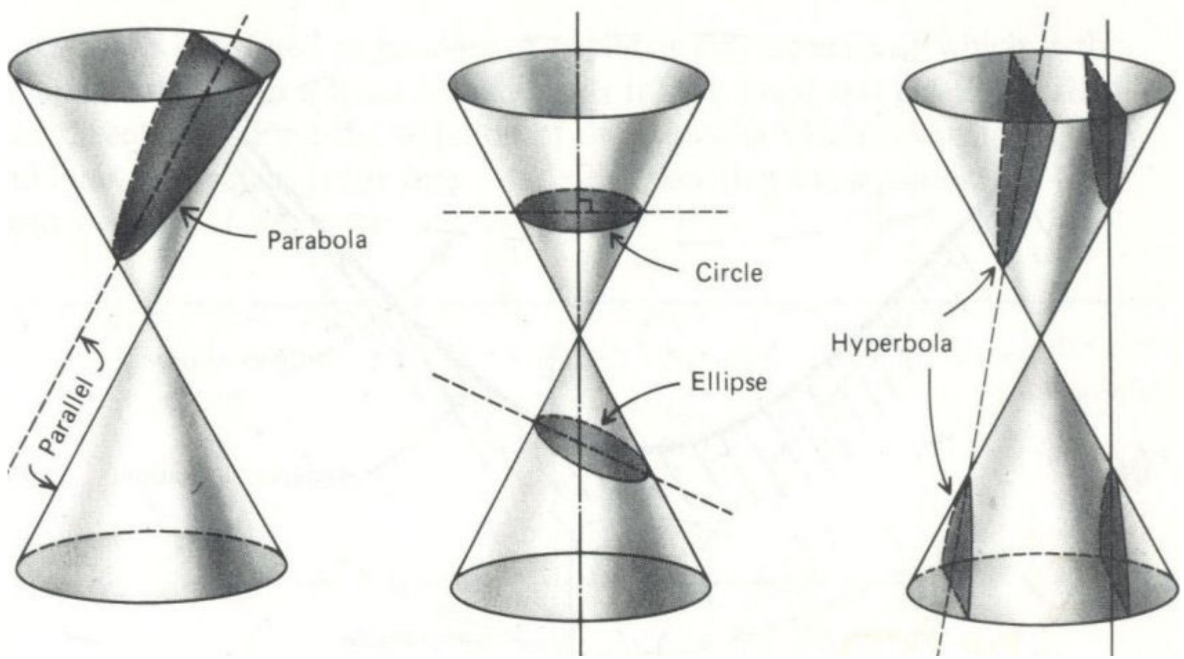
$$A(x,y)\frac{\partial^2 u}{\partial x^2} + 2B(x,y)\frac{\partial^2 u}{\partial x \partial y} + C(x,y)\frac{\partial^2 u}{\partial y^2} = F(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y})$$

where F is a linear function. We restrict ourselves for the moment to the case of *constant coefficients:*

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} = F(x,y,u,u_x,u_y)$$

Depending on A, B and C, the PDE falls into one of three categories:

1. $B^2 - 4\,AC > 0$: Hyperbolic
2. $B^2 - 4AC < 0$: Elliptic
3. $B^2 - 4AC = 0$: Parabolic



"Canonical" Examples:

The wave equation:

$$\frac{\partial^2 \psi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} = 0$$

hyperbolic

Laplace's equation

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

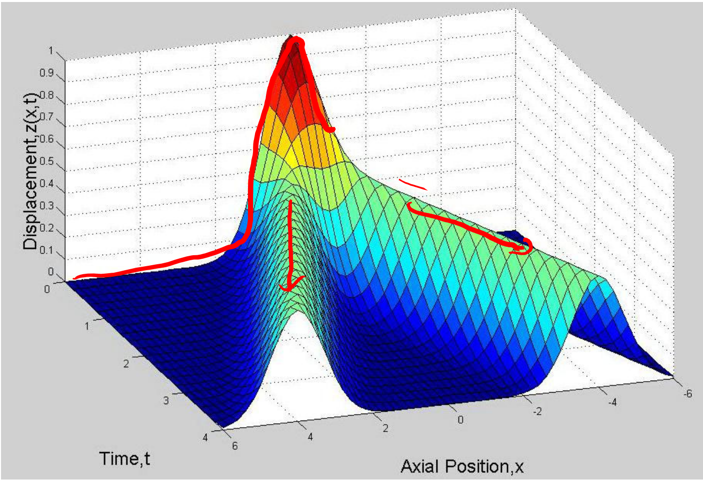
elliptic

The heat equation:

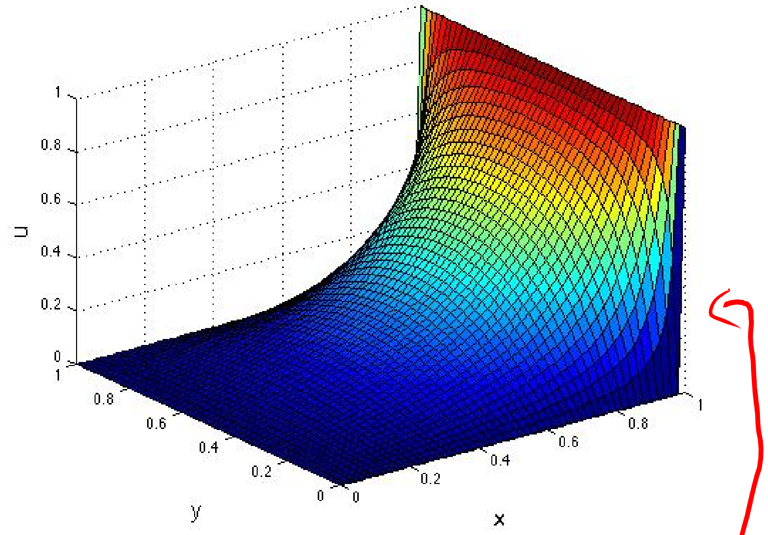
$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{\kappa}\frac{\partial u}{\partial t}$$

parabolic

General characteristics of solutions to the different types of equations

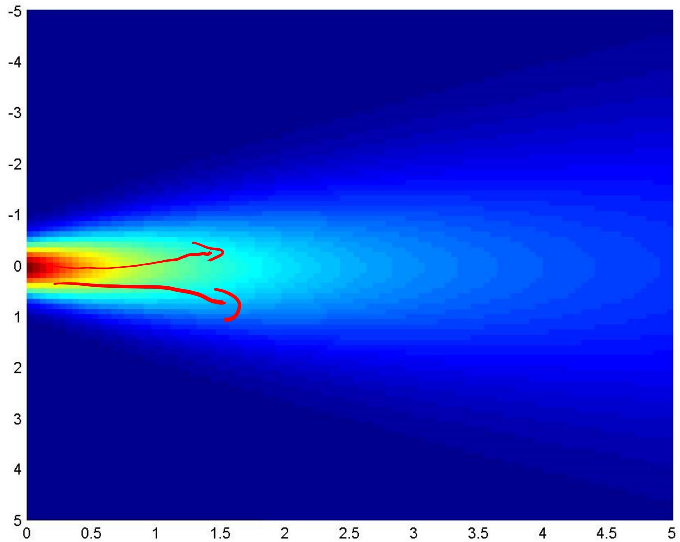


Hyperbolic:
Propagation of signals



Elliptic: as smooth as possible

Parabolic:
Spreading out

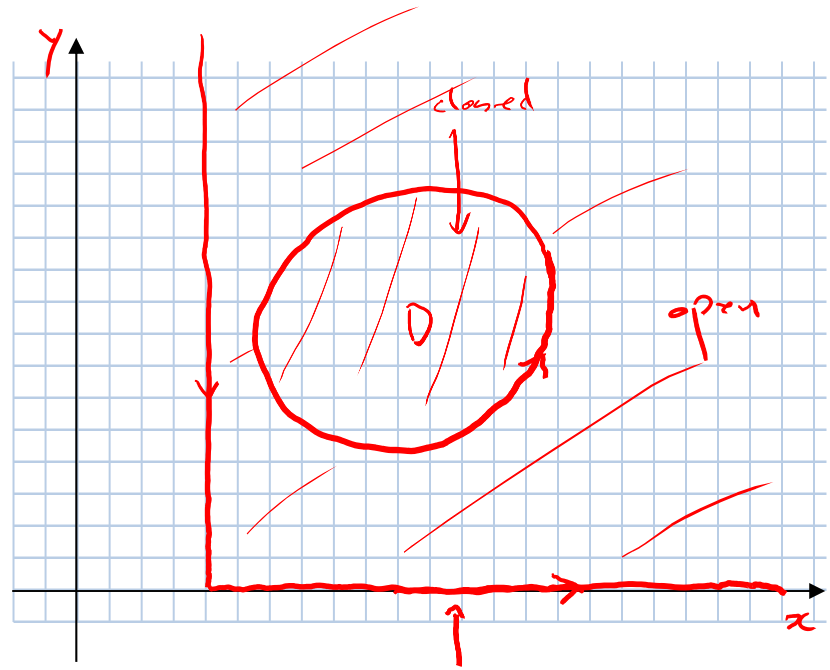


Boundary conditions for 2D PDEs

In two dimensions, a boundary line can be *parameterized*

$$\mathbf{r}(\mathbf{s}) = (x(s), y(s))$$

A boundary curve can be *open* or *closed*.



The unit normal vector to the boundary is

$$\hat{\mathbf{n}} = \left(\frac{dy}{ds}, -\frac{dx}{ds}\right) / \sqrt{x'(s)^2 + y'(s)^2}$$

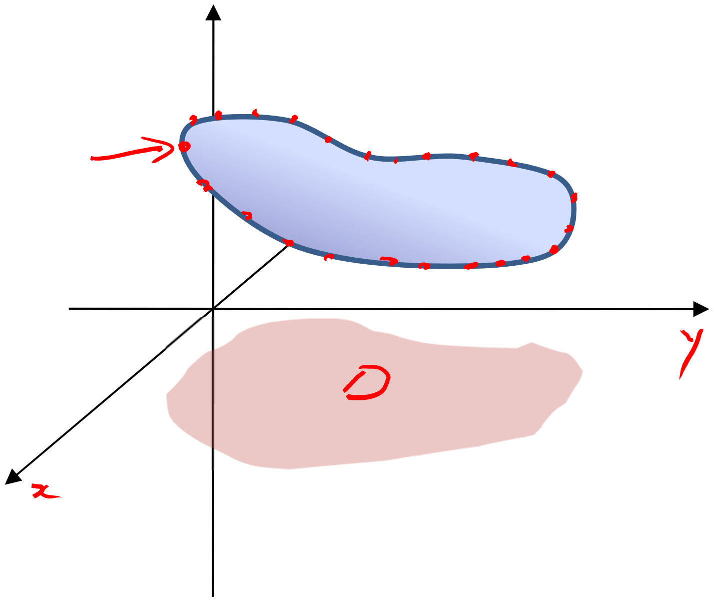
The derivative of the solution $\psi(x, y)$ normal to the boundary is

$$\underbrace{\frac{\partial \psi}{\partial n}} := \hat{n} \cdot \nabla \psi \; .$$

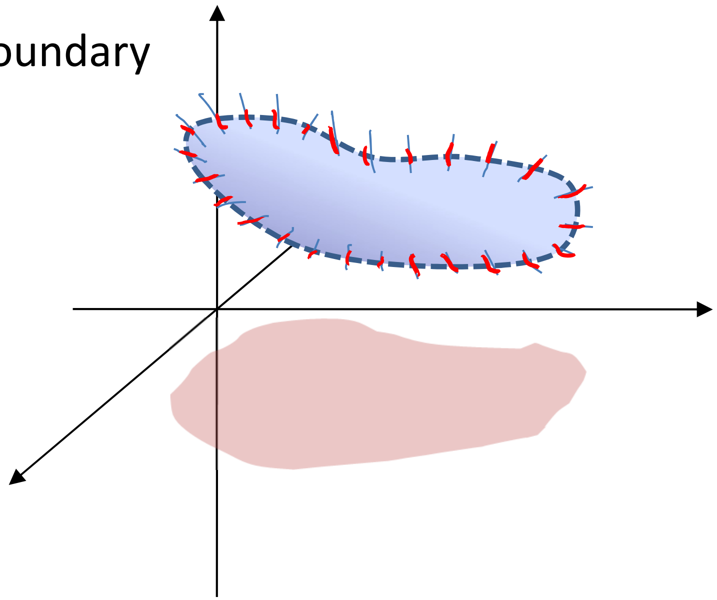
derivative in the normal direction.

Types of boundary conditions:

1. Dirichlet conditions:
   Specify the *value of the solution* on the boundary



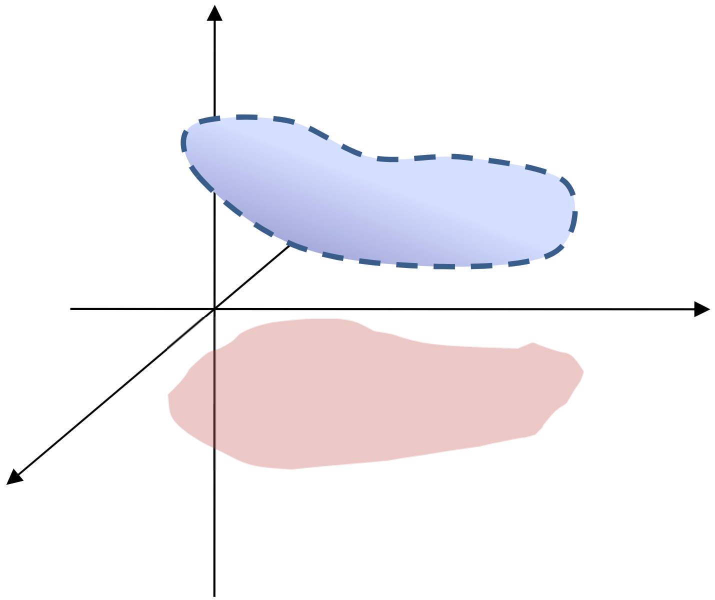
2. Neumann conditions:
   Specify the *normal derivative* of the solution on the boundary



If the value or derivative is set to *zero,* these are known as *homogeneous* Dirichlet or Neumann conditions.

3. Mixed conditions:
   Specify some ratio of the value and the derivative on the boundary

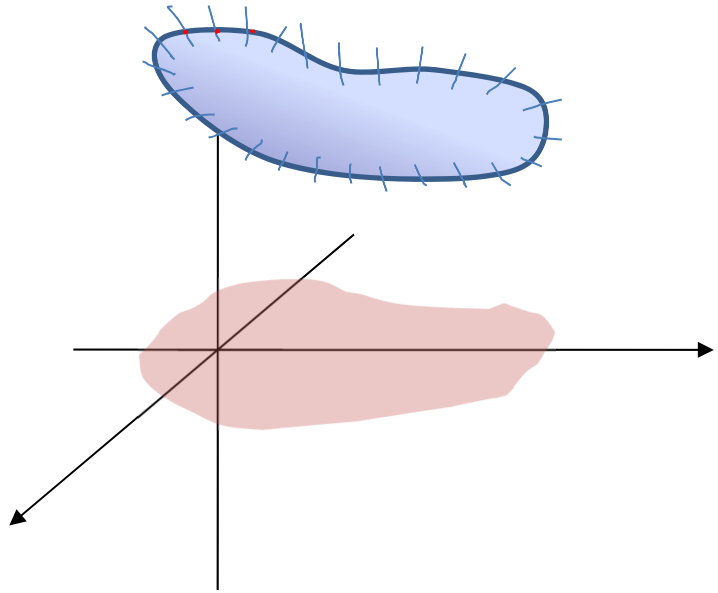
$$\alpha \psi + \beta \frac{\partial \psi}{\partial n} = 0$$



4. Cauchy conditions:
   Specify *both* the normal derivative and the value on the boundary

$$\psi$$

$$\frac{\partial \psi}{\partial n}$$



The following table shows what boundary conditions are needed for a given problem:

	Boundary	Elliptic	Parabolic	Hyperbolic
Dirichlet/ Neumann/ mixed BCs	Open	Insufficient	Sufficient; unique solution	Insufficient
	Closed	Sufficient, unique solution	Overspecified	Solution not unique
Cauchy BCs	Open	Sufficient, unique (but unstable)	Overspecified	Sufficient, unique
	Closed	Overspecified	Overspecified	Overspecified

Methods of Solution of PDEs

1. Find a series of functions that fit the PDE in the interior, then combine these to match the boundary conditions

2. Represent the entire solution as an integral over the boundary

In the next section, we will use the *Sturm-Liouville theory* from last week to construct a general approach to Method 1.

Solving Partial Differential Equations using Separation of Variables

Canonical examples:

The heat equation:

$$\kappa \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$ $\leftarrow$ parabolic

Laplace's equation

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$ $\leftarrow$ elliptic $\leftarrow$

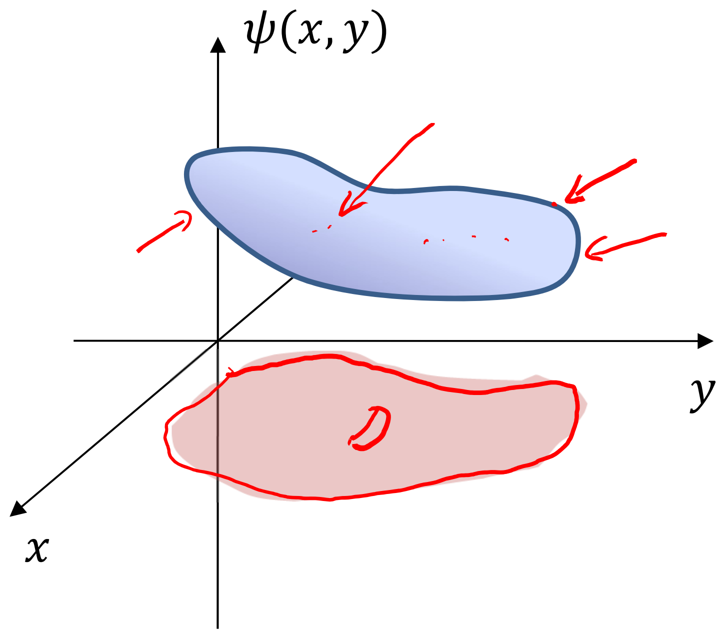
The wave equation:

$$\nabla^2 \psi - \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} = 0$$ $\leftarrow$ hyperbolic.

Consider Laplace's equation in 2D:

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

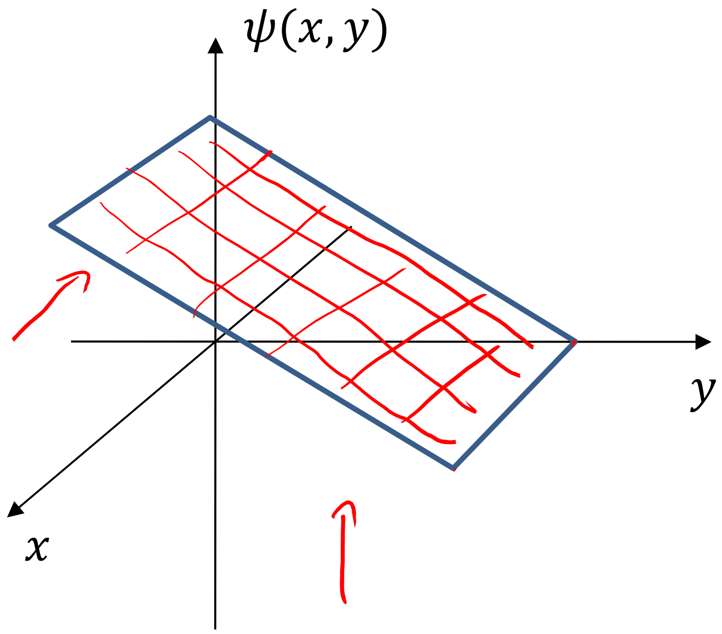
The solution is determined *uniquely* in some domain D
If we specify the value of ψ on the edge of the domain.

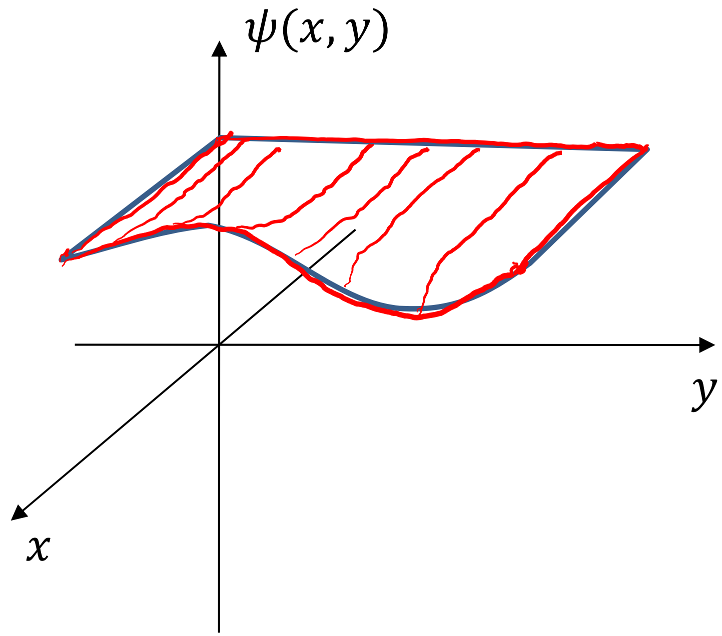


Solutions to Laplace's equation

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

have *no local maxima or minima.* The solutions are therefore "as smooth as possible, while still fitting the boundary conditions".





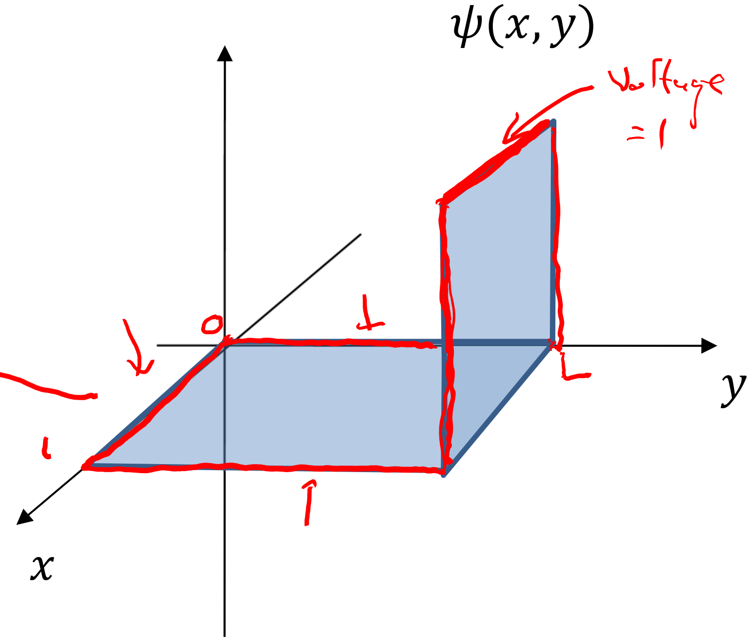
We will now find the solution to

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

On the domain $D = \{(x, y) | 0 \le x \le 1, \overset{0}{\cancel{-L}} \le y \le L\}$
with the boundary conditions

y = 0

$$\psi(x, 0) = 0,$$
$$\psi(x, L) = 1,$$
$$\psi(0, y) = 0,$$
$$\psi(1, y) = 0$$



We first substitute the <u>Separation Ansatz:</u>

Let $\psi(x, y) = X(x) Y(y)$

Sub into Laplace's equation:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) X Y = 0$$

$$\frac{\partial^2}{\partial x^2}(XY) + \frac{\partial^2}{\partial y^2}(XY) = 0$$

$$Y \frac{\partial^2 X}{\partial x^2} + X \frac{\partial^2 Y}{\partial y^2} = 0$$

$$Y X'' + X Y'' = 0$$

Divide by XY:

$$\frac{X'' \cancel{Y}}{\cancel{Y} X} + \frac{\cancel{X} Y''}{\cancel{X} Y} = 0$$

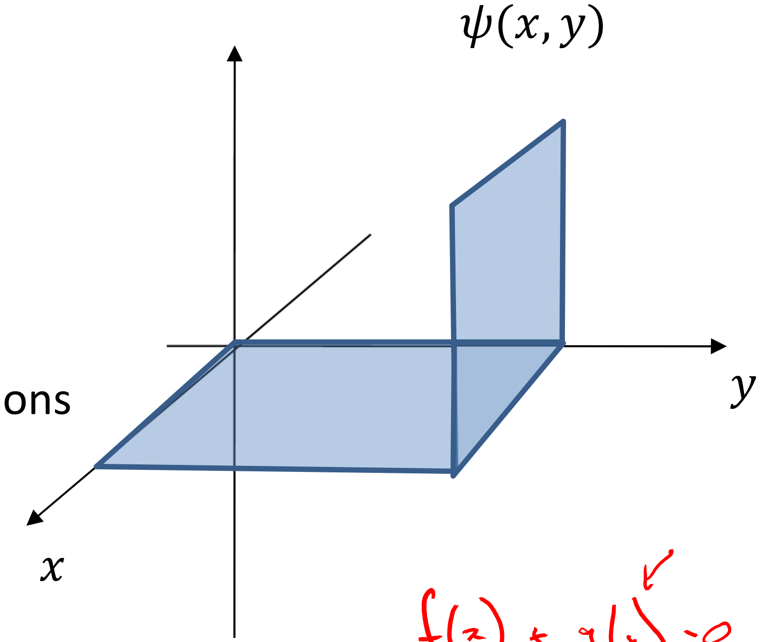
Now we have found

$f(x)$ $\quad\quad$ $g(y)$

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = 0$$

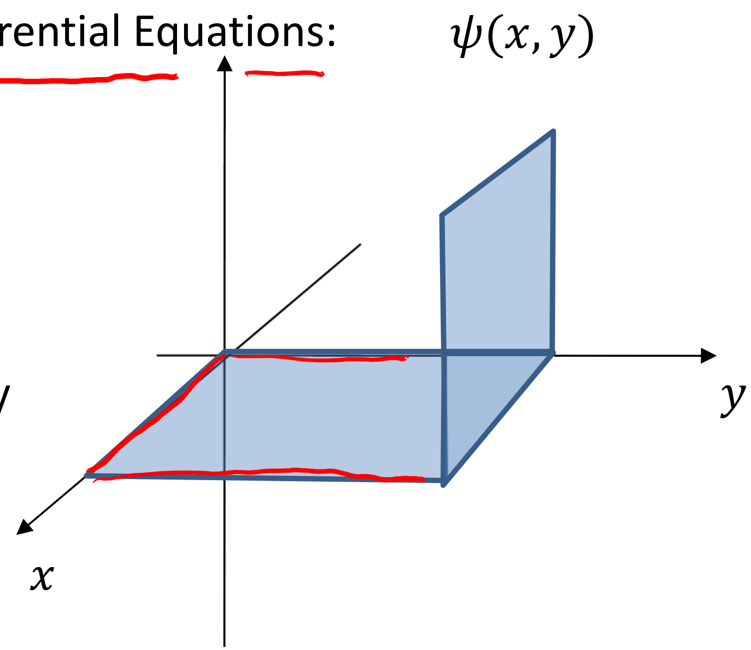
The only way this can be true is if both of these fractions are constant. That is:

$$\frac{X''(x)}{X(x)} = \pm\lambda \qquad \frac{Y''(y)}{Y(y)} = \mp\lambda$$



$f(x) + g(y) = 0$

So we have converted the PDE into two Ordinary Differential Equations:



$$\frac{X''(x)}{X(x)} = \pm\lambda, \qquad \frac{Y''(y)}{Y(y)} = -(\pm\lambda)$$

Two of the boundary conditions will be automatically satisfied if

$$X(0) = X(\cancel{\pi}\,1) = 0$$

BCs $\begin{cases} \psi(x,0) = 0, \\ \psi(x,L) = 1, \\ \psi(0,y) = 0, \\ \psi(1,y) = 0 \end{cases}$

$\psi(0,y) = Y(y)X(0)$

Note that the problem

$$\frac{X''(x)}{X(x)} = -\lambda$$

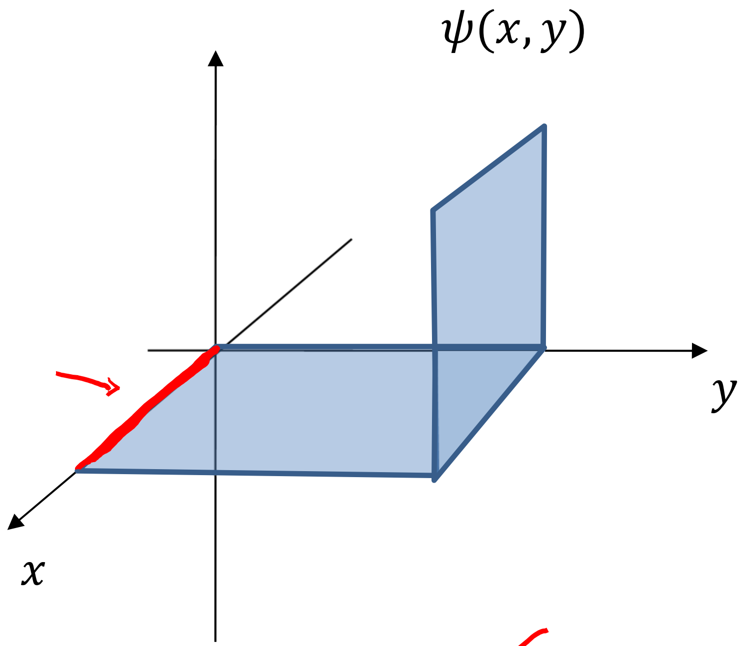
$$X(0) = X(1) = 0$$

is a Sturm-Liouville problem.

→ eigenfunctions $\begin{cases} X_n(x) = \sin(n\pi x) \\ \sqrt{\lambda_n} = n\pi . \end{cases}$

So we have an infinite set of solutions for $X(x)$:

$$\begin{cases} X_n(x)\,\sin(\sqrt{\lambda_n}x) \\ \sqrt{\lambda_n} = \pi n \end{cases}$$



We can now solve for $Y(y)$:

$$\frac{Y''(y)}{Y(y)} = \lambda$$

BCs
$$\begin{cases} \psi(x,0) = 0, \\ \psi(x,L) = 1, \\ \psi(0,y) = 0, \\ \psi(1,y) = 0 \end{cases}$$

Re-arrange:

$$Y'' = \lambda Y$$

General solution

$$Y(y) = \left[A\cosh(\sqrt{\lambda}y) + B\sinh(\sqrt{\lambda}y)\right].$$

$$= B\sinh(\sqrt{\lambda}y).$$

$$Y(0) = 0$$

$$\Rightarrow A\cosh(\sqrt{\lambda}0) + \cancel{B\sinh(\sqrt{\lambda}0)} = 0$$

$$\Rightarrow A = 0$$

(Aside: The *general solution* of

$$Y''(y) = k^2 Y(y)$$

is:

$$Y'' - k^2 Y = 0$$

Characteristic equation

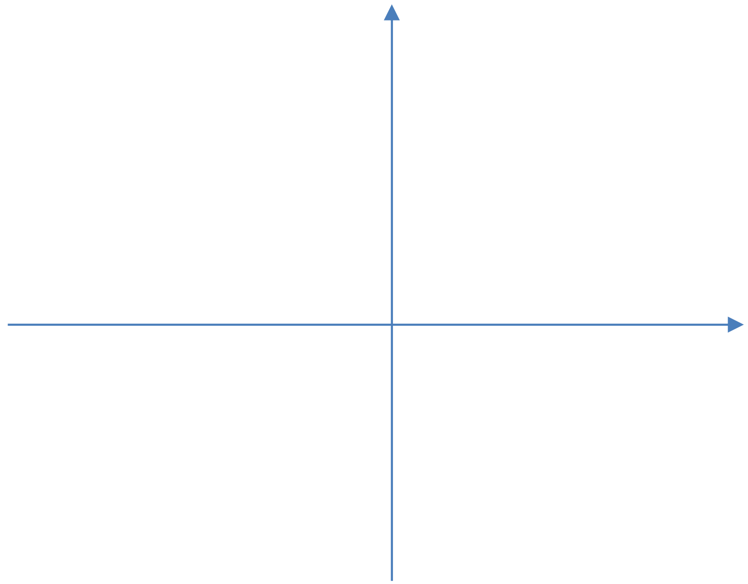
$$m^2 - k^2 = 0 \implies m = \pm k$$

General solution e^{+ky}, e^{-ky}

$$Y(y) = Ae^{ky} + Be^{-ky}$$

$$= C\cosh(ky) + D\sinh(ky)$$

).



The total solution is therefore

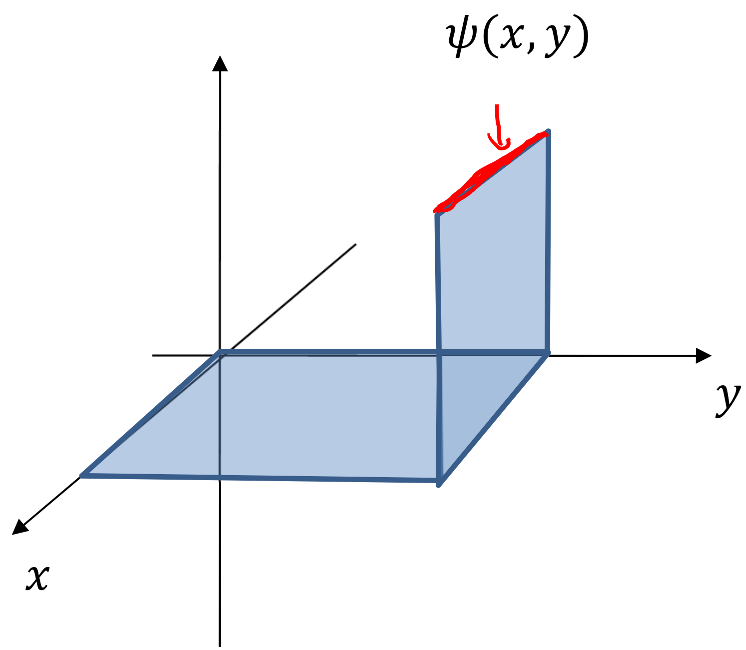
$X_n(x)$ $\quad$ $Y_n(y)$

$$\psi_n(x,y) = \underbrace{\sin(n\pi x)\sinh(n\pi y)}$$

Does this fit all the boundary conditions?

$y=0$

$$\text{BCs}\begin{cases}\psi(x,0) = 0, & \checkmark\\ \psi(x,L) = 1, & \times\\ \psi(0,y) = 0, & \checkmark\\ \psi(1,y) = 0 & \checkmark\end{cases}$$



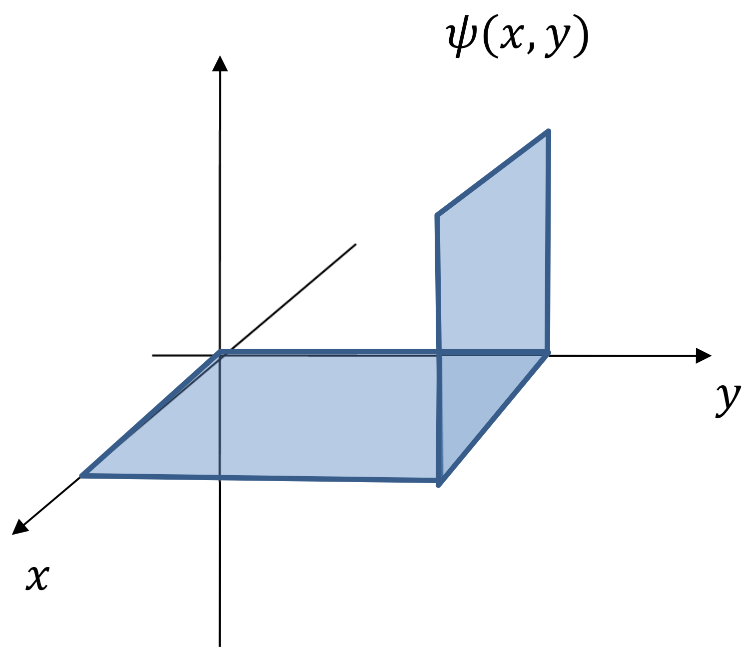
Solution: construct a *series of these functions to satisfy the remaining boundary condition* !

$$\psi(x,y) = \sum_{n=1}^{\infty} b_n \psi_n(x,y)$$

We construct the total solution as a sum of the eigenfunctions X_n:

$$\psi(x,y) = \sum_{n=1}^{\infty} b_n X_n(x) Y_n(y)$$

We need $\psi(L) = 1$, so:



$$\text{BCs} \begin{cases} \psi(x,0) = 0, \\ \psi(x,L) = 1, \\ \psi(0,y) = 0, \\ \psi(1,y) = 0 \end{cases}$$

$$1 = \sum_{n=1}^{\infty} b_n X_n(x) Y_n(1)$$

Take the inner product with $X_m(x)$:

$$\langle X_m, 1 \rangle = \langle X_m, \sum_{n=1}^{\infty} b_n X_n Y_n(1) \rangle$$

$$= \sum_{n=1}^{\infty} b_n Y_n(1) \underbrace{\langle X_m, X_n \rangle}_{=0 \text{ unless } n=m}$$

$$= b_m Y_m(1) \langle X_m, X_m \rangle .$$

So

$$b_m = \langle X_m, 1 \rangle / Y_m(1) \|X_m\|^2 .$$

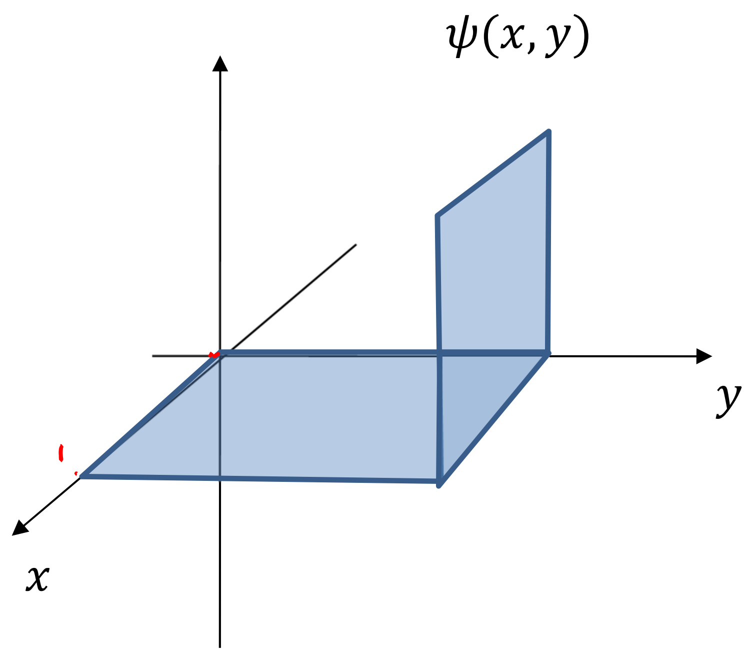
So the solution is

$$\psi(x,y) = \sum_{n=1}^{\infty} \underbrace{\frac{1}{Y_n(L)}\frac{\langle X_n, 1\rangle}{||X_n||^2}}_{b_n} X_n(x) Y_n(\cancel{x}\, y)$$

with

$$X_n(x) = \sin(n\pi x)$$

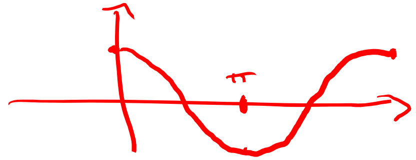
$$Y_n(y) = \sinh(n\pi y)$$



$$\langle X_n, 1\rangle = \int_0^1 \sin(n\pi x)\cdot 1 \, dx$$

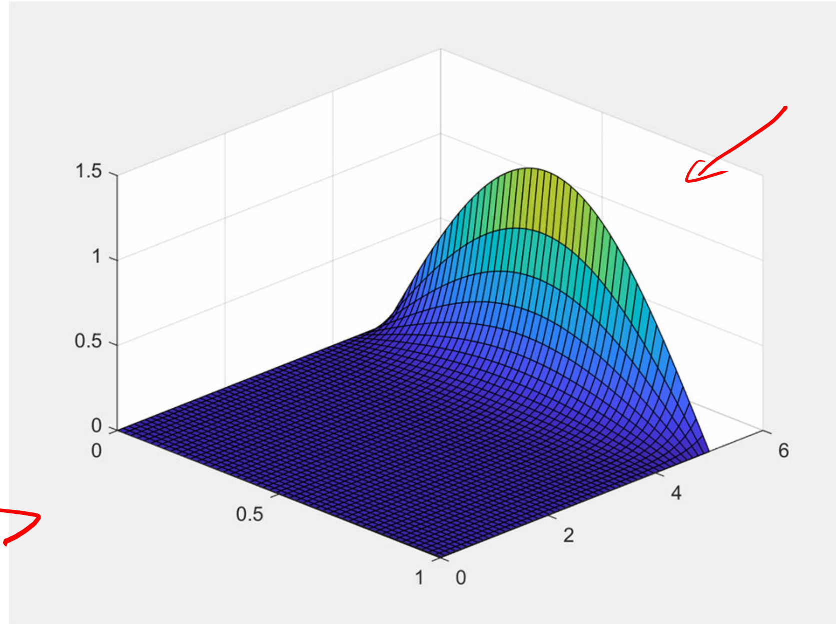
$$= \left[-\frac{1}{n\pi}\cos(n\pi x)\right]_0^1 = \left(-\frac{1}{n\pi}\cos(n\pi) + \frac{1}{n\pi}\cos(0)\right)$$

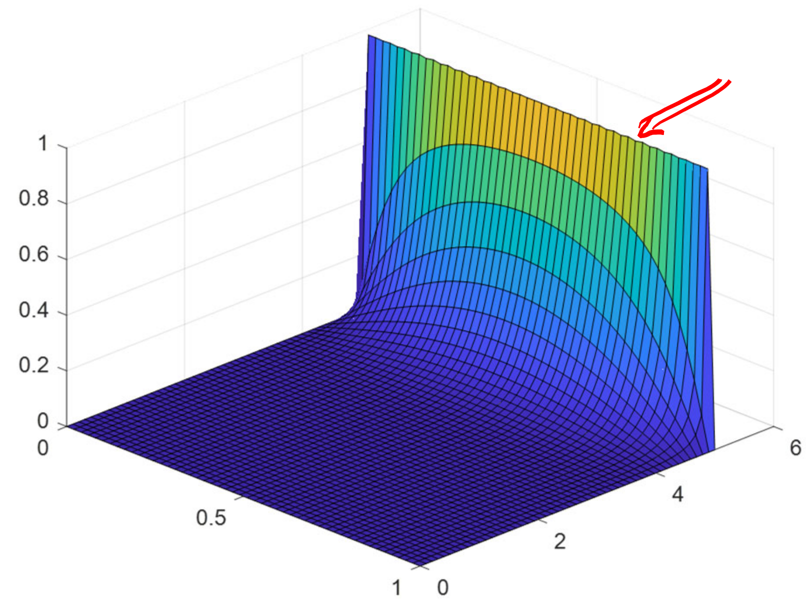
$$= \frac{1}{n\pi}\left(1 - \underbrace{\cos(n\pi)}_{(-1)^n}\right)$$



$$||X_n||^2 = \int_0^1 \sin^2(n\pi x)\, dx = \int_0^1 \frac{1}{2}\left(1 - \cos(2n\pi x)\right) dx = \frac{1}{2}.$$

$$\Psi(x,y) = \sum_{n=1}^{\infty} \frac{1}{\sinh(nL)} \frac{1}{2}\left(1-(-1)^n\right) \times \sin(n\pi x)\sinh(ny).$$





The Diffusion and Wave equations

Canonical examples:

The heat equation:

$$\kappa \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$ ← parabolic

Laplace's equation

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$ ← elliptic

The wave equation:

$$\nabla^2 \psi - \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} = 0$$ ← hyperbolic.

The steps to solving the diffusion and wave equation are the same as for Laplace's equation, namely:

1. Separate variables
2. Identify the Sturm-Liouville problem and compute the eigenfunctions
3. Use an infinite series to match the boundary conditions.

Diffusion equation example:

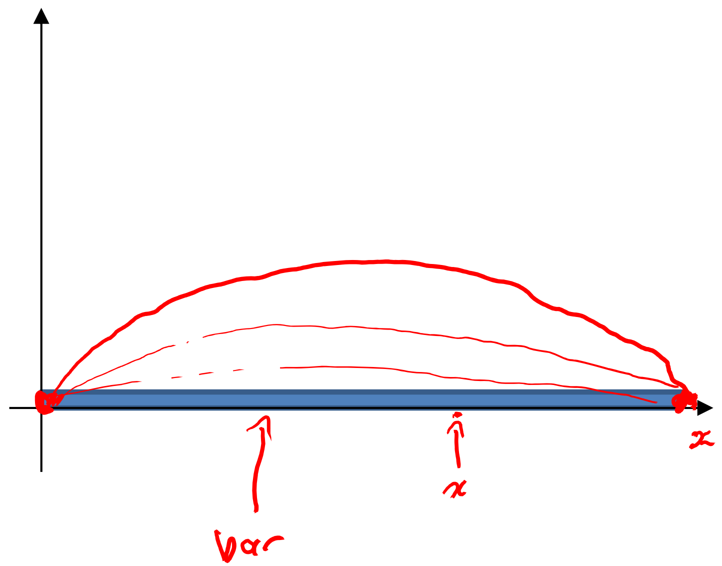
The temperature of a metal bar of at position x and time t is given by

$$\kappa \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

With the boundary conditions

$$\begin{aligned} u(0,t) &= 0 \\ u(L,t) &= 0 \end{aligned}$$

$$u(x,0) = x(L-x)$$



Step 1: Separate the variables:

$$\underbrace{\kappa\frac{\partial^2 u}{\partial x^2}} = \frac{\partial u}{\partial t}$$

$$u(x,t) = X(x)\,T(t)$$

Subbing this in to the PDE:

$$\kappa\frac{\partial^2}{\partial x^2}(XT) = \frac{\partial}{\partial t}(XT)$$

$$\kappa T X'' = X T'$$

Rearrange, divide by XT:

$$\frac{\kappa \cancel{T} X''}{X\cancel{T}} = \frac{\cancel{X} T'}{\cancel{X} T} \Rightarrow \underbrace{\frac{X''}{X}}_{f(x)} = \frac{T'}{\kappa T}$$

$g(t)$

So the variables separate.

Step 2: identify the S-L problem and find the eigenfunctions

We have found:

$$\frac{X''(x)}{X(x)} = \frac{1}{\kappa}\frac{T'(t)}{T(t)} = \pm\lambda$$

We want $u(0,t) = u(L,t) = 0$

This is satisfied if $X(0)=0,\ X(L)=0$.

So the problem is

$$\frac{X''}{X} = -\lambda, \quad X(0) = X(L) = 0$$

is a S-L problem. The solutions are

$$\to X_n(x) = \sin\left(\frac{n\pi}{L}x\right)$$

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2.$$

The problem for T is

$$\frac{T'(t)}{\kappa T(t)} = -\lambda_n$$

$$\frac{dT}{dt} = -(\kappa\lambda_n)T$$

The solution is

$$T(t) = Ae^{-\kappa\lambda_n t}$$

$$= Ae^{-\kappa\left(\frac{n\pi}{L}\right)^2 t}.$$

3. Expand the solution in a series to match the boundary conditions

We have found

$$u(x,t) = \sum_{n=1}^{\infty} \overset{b_n}{X_n(x)T_n(t)}$$

with

$$X_n(x) = \sin(\frac{n\pi}{L}x)$$

$$T_n(t) = e^{-(n\pi/L)^2 \kappa t}$$

The remaining BC is

$$u(x,0) = x(L-x) = f(x)$$

So

$$f(x) = \sum_{n=1}^{\infty} b_n X_n(x) T_n(0)$$

Taking the inner product with $X_m(x)$:

$$\langle X_m, f \rangle = \langle X_m, \sum_{n=1}^{\infty} b_n X_n T_n(0) \rangle$$

$$\langle X_m, f \rangle = \sum_{n=1}^{\infty} b_n T_n(0) \langle X_m, X_n \rangle$$

$$= b_m \underbrace{T_m(0)}_{= e^{-0} = 1} \| X_m \|^2$$

So

$$b_m = \frac{\langle X_m, f \rangle}{\| X_m \|^2}$$

Now

$$\| X_m \|^2 = \int_0^L \sin^2\left(\frac{m\pi}{L} x\right) dx$$

$$= \int_0^L \frac{1}{2}\left(1 - \cos\left(\frac{2m\pi}{L} x\right)\right) dx$$

$$= \frac{L}{2} - \underbrace{\left[\frac{L}{2m\pi} \sin\left(\frac{2m\pi x}{L}\right)\right]_0^L}_{=0}$$

$$= \frac{L}{2}.$$

$$\langle X_n, f \rangle = \int_0^L \sin\left(\frac{n\pi x}{L}\right) x(L-x)\, dx$$

$$= \cdots$$

$$= \frac{2L}{n^3\pi^3}\left(1 - \cos(n\pi)\right).$$

The full solution is

$$u(x,t) = \sum_{n=1}^{\infty} \frac{\langle X_n, f \rangle}{L/2} \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 t}.$$

Wave equation example:

The z component of an electromagnetic wave travelling along a square metal pipe is of width a is given by the equation

$$\frac{\partial^2 E}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = 0$$

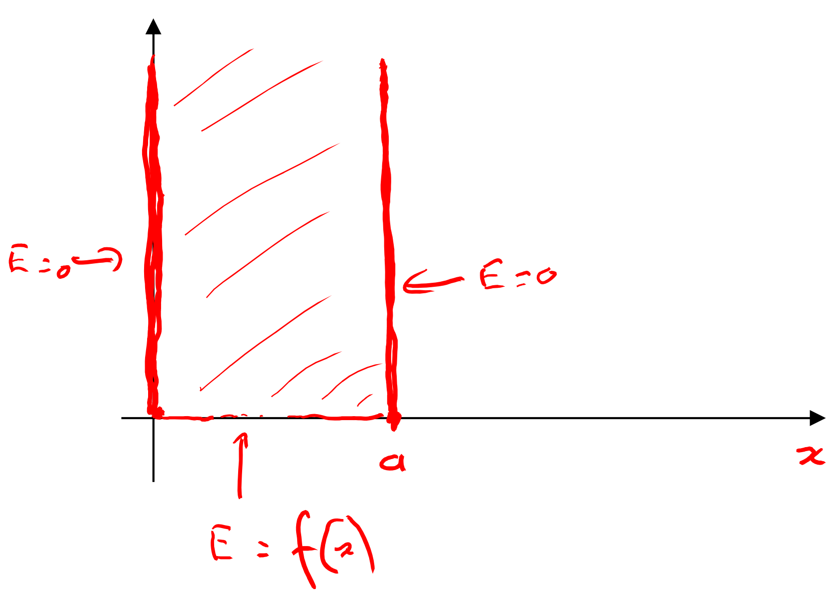
With the boundary conditions

$$E(0,t) = 0$$
$$E(a,t) = 0$$

and the initial conditions

$$E(x,0) = f(x)$$

$$\frac{\partial E}{\partial \cancel{x}\, t}(x,0) = 0$$



Step 1: Separate the variables:

$$\frac{\partial^2 E}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = 0$$

Let $E(x,t) = X(x)T(t)$

Subbing in:

$$\frac{\partial^2}{\partial x^2}(XT) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}(XT) = 0$$

$$TX'' - \frac{1}{c^2}XT'' = 0$$

Re-arranging:

$$\frac{X''}{X} - \frac{1}{c^2}\frac{T''}{T} = 0$$

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} \Rightarrow \text{variable separate.}$$

Step 2: identify the S-L problem and find the eigenfunctions

We have found:

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{c^2 T(t)} = \pm\lambda$$

$X(0)=0$ $\rightarrow$ $E(0,t)=0$ ($x=0$)

$X(a)=0$ $\rightarrow$ $E(a,t)=0$ ($x=a$)

$$E(x,0) = f(x)$$

$$\frac{\partial E}{\partial t}(x,0) = 0$$

We want $X(0) = X(a) = 0$.

The problem for $X(x)$ is

$$X'' = -\lambda X$$

with $X(0) = X(a) = 0$, and λ is a S-L problem. The solutions are

$$X_n(x) = \sin\left(\frac{n\pi x}{a}\right)$$

$$\sqrt{\lambda_n} = \frac{n\pi}{a}$$

The problem for $T(t)$ is

$$\frac{T''}{c^2 T} = -\lambda$$

or

$$T'' = -\lambda c^2 T$$

The general solution is

$$T_n(t) = A\cos\left(\sqrt{\lambda_n c^2}\, t\right) + B\sin\left(\sqrt{\lambda_n c^2}\, t\right)$$

$$T_n(t) = A\cos(\omega_n t) + B\sin(\omega_n t)$$

where $\omega_n = \sqrt{\lambda_n c^2} = c\sqrt{\lambda_n}$

$= c\frac{n\pi}{a}$.

But $\left.\frac{\partial E}{\partial t}(x,0) = 0 = \frac{\partial}{\partial t}\big(X(x)T(t)\big)\right|_{t=0}$.

$t=0$

This is satisfied if $\underline{T'(0)=0}$

So $0 = -A\omega_n \cancel{\sin(\omega_n 0)} + B\omega_n\cos(0)$.

$\Rightarrow B = 0$

So $T_n(t) = A\cos(\omega_n t)$.

3. Expand the solution in a series to match the boundary conditions

We have found

$$E(x,t) = \sum_{n=1}^{\infty} X_n(x) T_n(t)$$

b_n

with

$$X_n(x) = \sin\left(\frac{n\pi}{a} x\right)$$

$$T_n(t) = \cos\left(\frac{n\pi c t}{a}\right)$$

The remaining BC is

$$E(x,0) = f(x)$$

$$f(x) = \sum_{n=1}^{\infty} b_n X_n(x) T_n(0)$$

Take the inner product with $X_m(x)$:

$$\langle X_m, f \rangle = \langle X_m, \sum_{n=1}^{\infty} b_n X_n T_n(0) \rangle$$

$$= \sum_{n=1}^{\infty} b_n T_n(0) \langle X_m, X_n \rangle$$

$$= b_m T_m(0) \| X_m \|^2$$

So

$$b_m = \frac{\langle X_m, f \rangle}{T_m(0) \| X_m \|^2}$$

$=1$

$$\| X_m \|^2 = \int_0^a \sin^2\left(\frac{m\pi}{a} x\right) dx = \frac{a}{2}.$$

So

$$E(x,t) = \sum_{n=1}^{\infty} \frac{2}{a} \langle X_n, f \rangle \sin\left(\frac{n\pi}{a} x\right) \times \cos\left(\frac{n\pi c}{a} t\right).$$