Second order differential equations (in 2D)

In two dimensions, the most general form of a linear 2nd-order PDE is

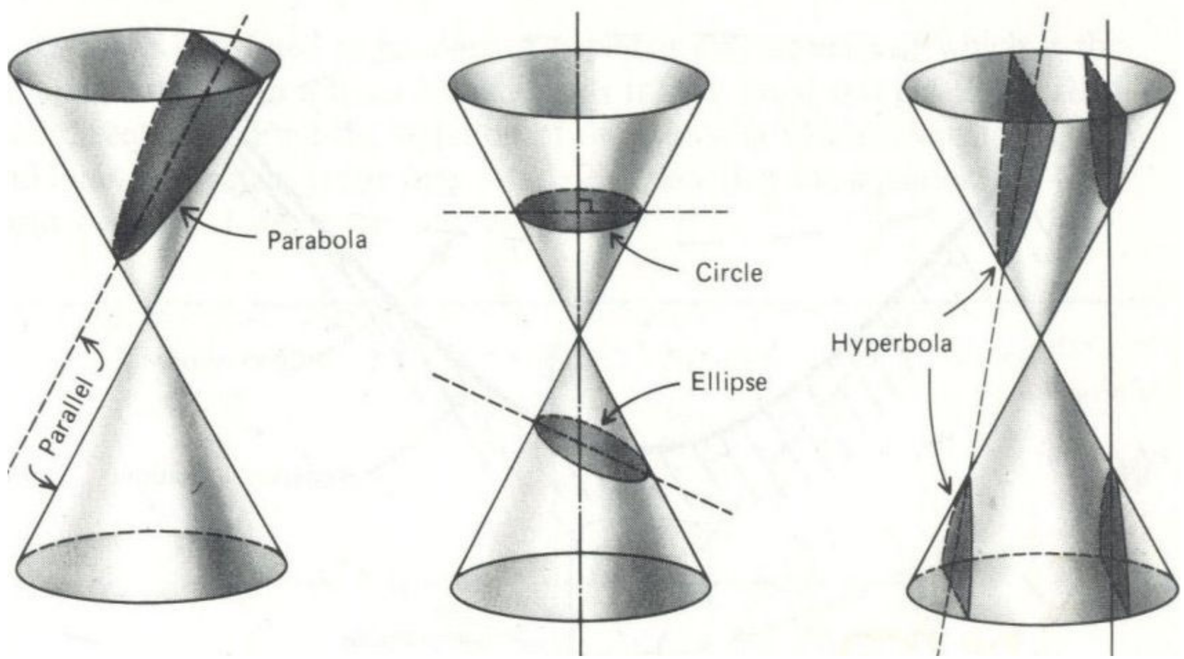
$$A(x,y)\frac{\partial^2 u}{\partial x^2} + 2B(x,y)\frac{\partial^2 u}{\partial x \partial y} + C(x,y)\frac{\partial^2 u}{\partial y^2} = F(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y})$$

We restrict ourselves for the moment to the case of *constant coefficients:*

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} = F(x,y,u,u_x,u_y)$$

Depending on A, B and C, the PDE falls into one of three categories:

1. $B^2 - 4\,AC > 0$: Hyperbolic
2. $B^2 - 4AC < 0$: Elliptic
3. $B^2 - 4AC = 0$: Parabolic



“Canonical” Examples:

The wave equation:

$$\frac{\partial^2 \psi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} = 0$$

hyperbolic

Laplace’s equation

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

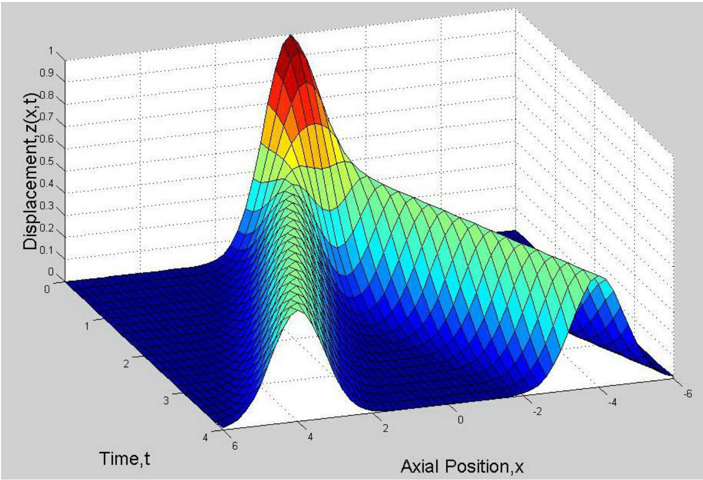
elliptic

The heat equation:

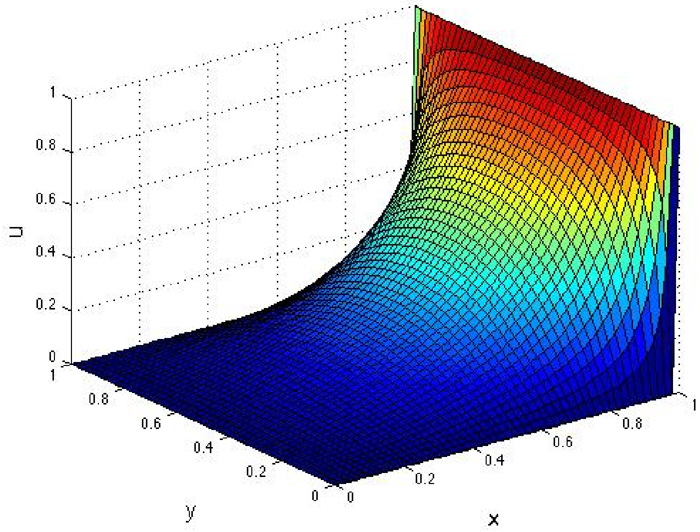
$$\frac{\partial^2 u}{\partial x^2} = a^2 \frac{\partial u}{\partial t}$$

parabolic

General characteristics of solutions to the different types of equations

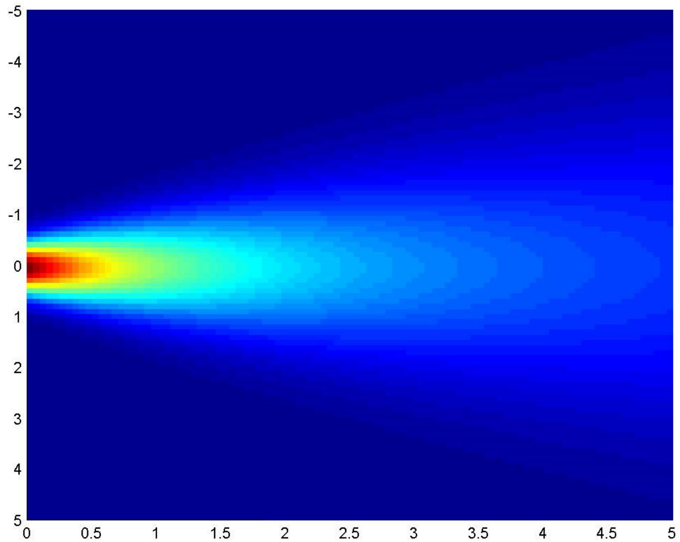


Hyperbolic:
Propagation of signals



Elliptic: as smooth as possible

Parabolic:
Spreading out



Boundary conditions for 2D PDEs

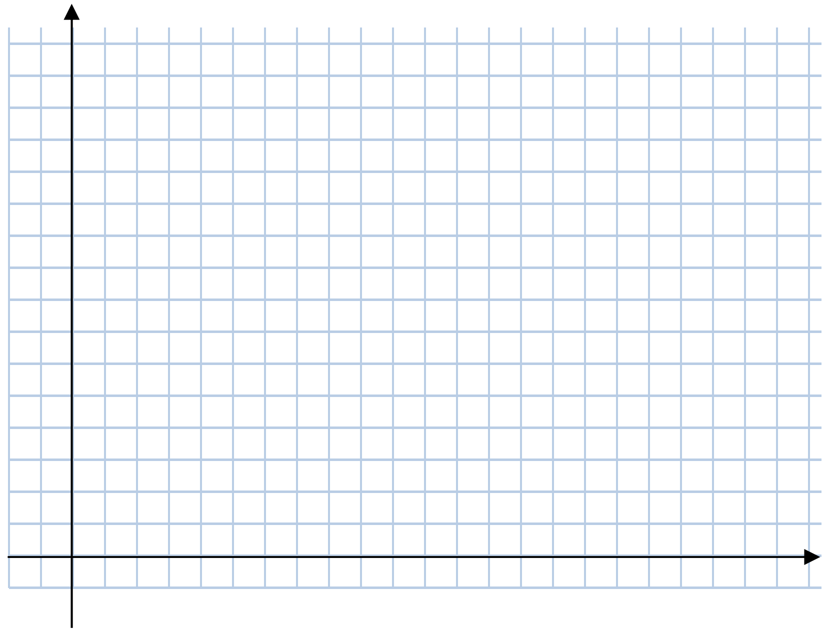
In two dimensions, a boundary line can be *parameterized*

$$\mathbf{r}(\mathbf{s}) = (x(s), y(s))$$

A boundary curve can be *open* or *closed*.

The unit normal vector to the boundary is

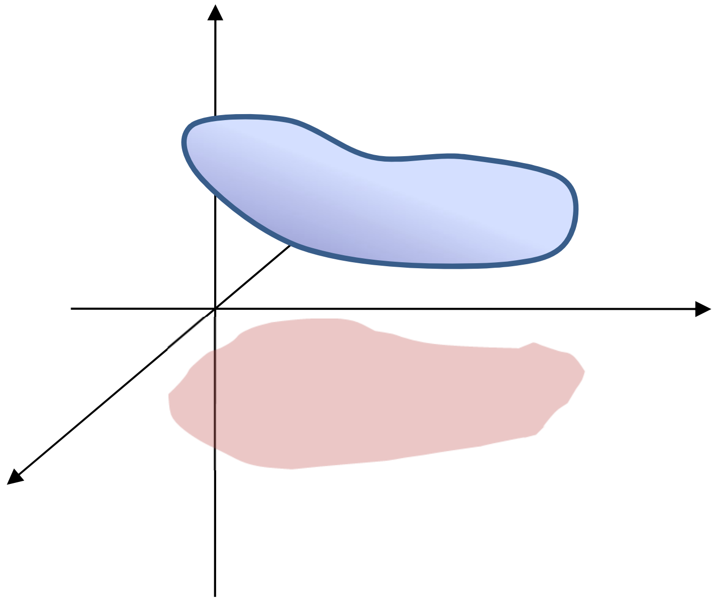
$$\hat{\mathbf{n}} = \left(\frac{dy}{ds}, -\frac{dx}{ds}\right) / \sqrt{x'(s)^2 + y'(s)^2}$$



The derivative of the solution $u(x,y)$ normal to the boundary is:

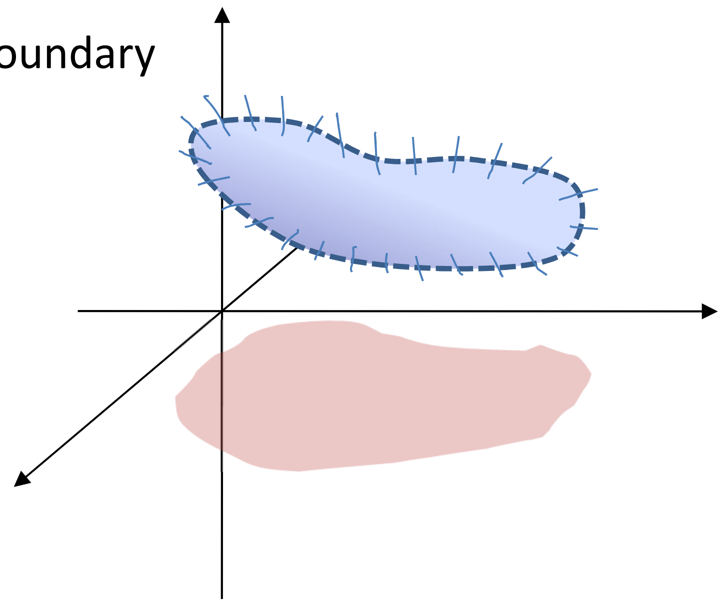
$$\frac{\partial u}{\partial n} :=$$

Types of boundary conditions:



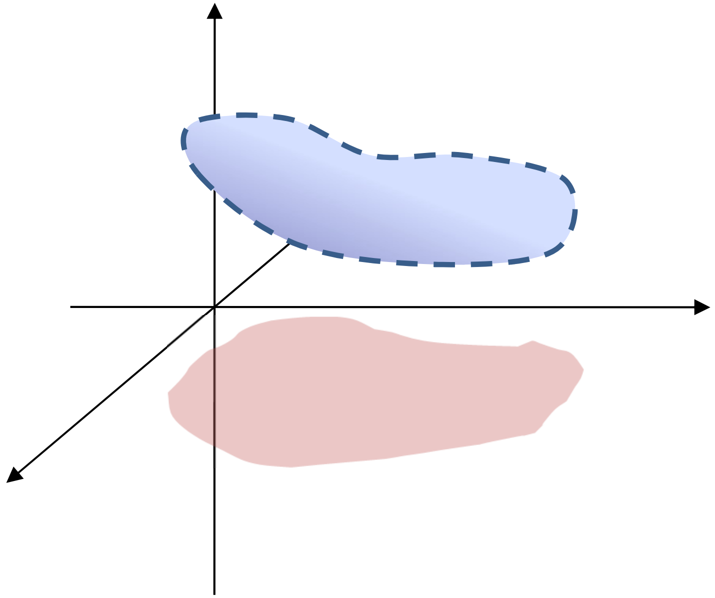
1. Dirichlet conditions:
   Specify the *value of the solution* on the boundary

2. Neumann conditions:
   Specify the *normal derivative* of the solution on the boundary

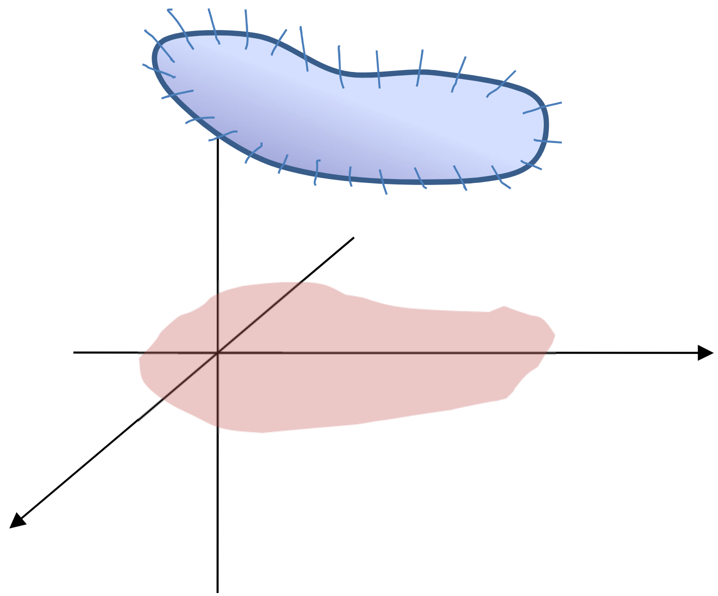


If the value or derivative is set to *zero,* these are known as *homogeneous* Dirichlet or Neumann conditions.

3. Mixed conditions:
   Specify some ratio of the value and the derivative on the boundary



4. Cauchy conditions:
   Specify *both* the normal derivative and the value on the boundary



Conditions	Boundary	Elliptic	Parabolic	Hyperbolic
Dirichlet/ Neumann/ mixed	Open	Insufficient	Sufficient; unique solution	Insufficient
	Closed	Sufficient, unique solution	Overspecified	Solution not unique
Cauchy	Open	Sufficient, unique (but unstable)	Overspecified	Sufficient, unique
	Closed	Overspecified	Overspecified	Overspecified

Methods of Solution of PDEs

1. Find a series of function that fit the PDE, then combine these to match the boundary conditions

2. Represent the solution as an integral over the boundary

We would like to find a general approach that we can use to apply to as wide a class of PDEs as possible.

In the next section, we will use the *Sturm-Liouville theory* from last week to construct such an approach.

Solving Partial Differential Equations using Separation of Variables

Canonical examples:

The heat equation:

$$\frac{\partial^2 u}{\partial x^2} = a^2 \frac{\partial u}{\partial t} \quad \leftarrow \text{parabolic}$$

Laplace's equation

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \quad \leftarrow \text{elliptic}$$

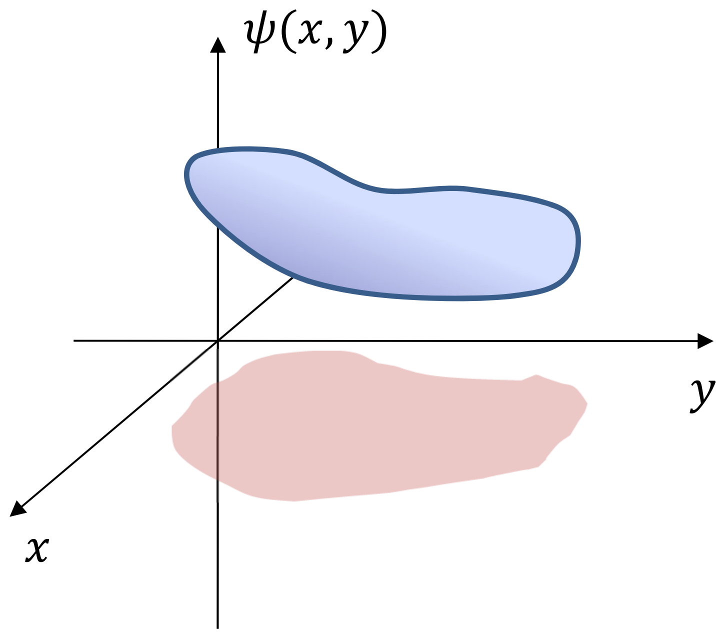
The wave equation:

$$\nabla^2 \psi - \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} = 0 \quad \leftarrow \text{hyperbolic.}$$

Consider Laplace's equation in 2D:

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

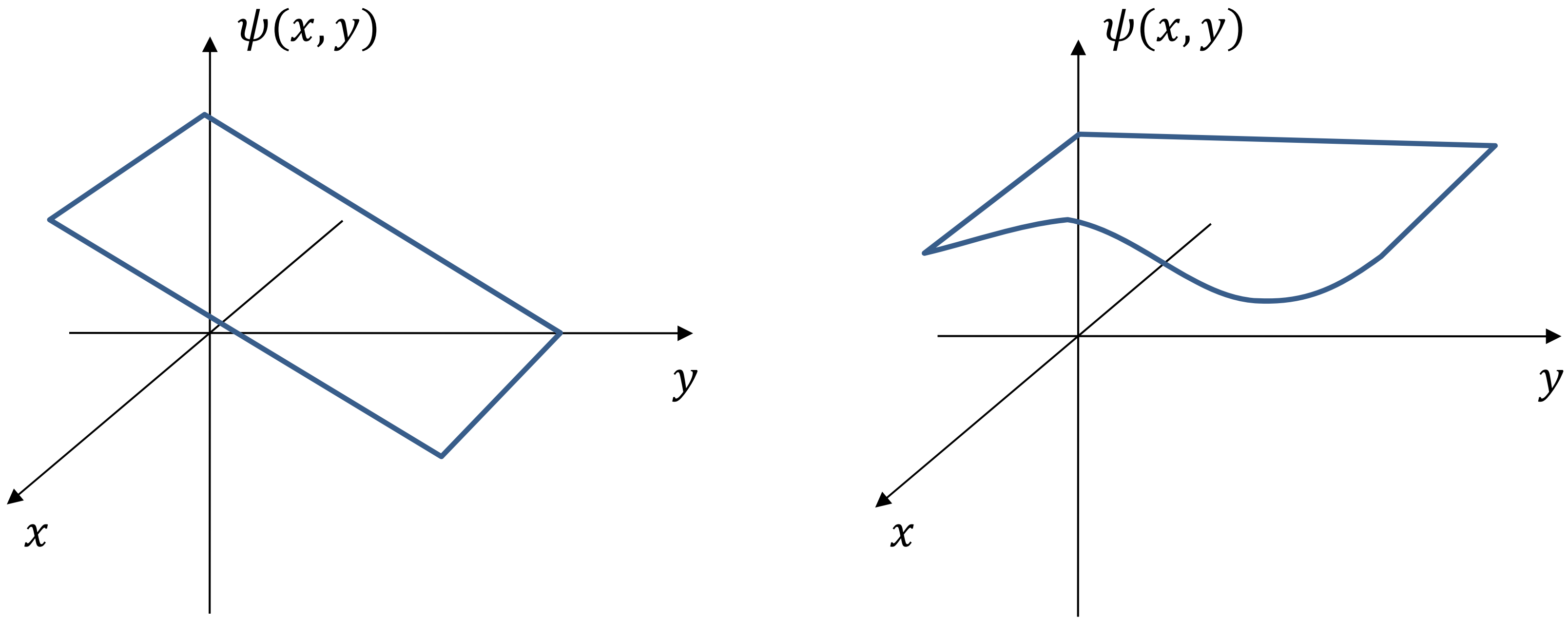
The solution is determined *uniquely* in some domain D
If we specify the value of ψ on the edge of the domain.



Solutions to Laplace's equation

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

have *no local maxima or minima.* The solutions are therefore "as smooth as possible, while still fitting the boundary conditions".

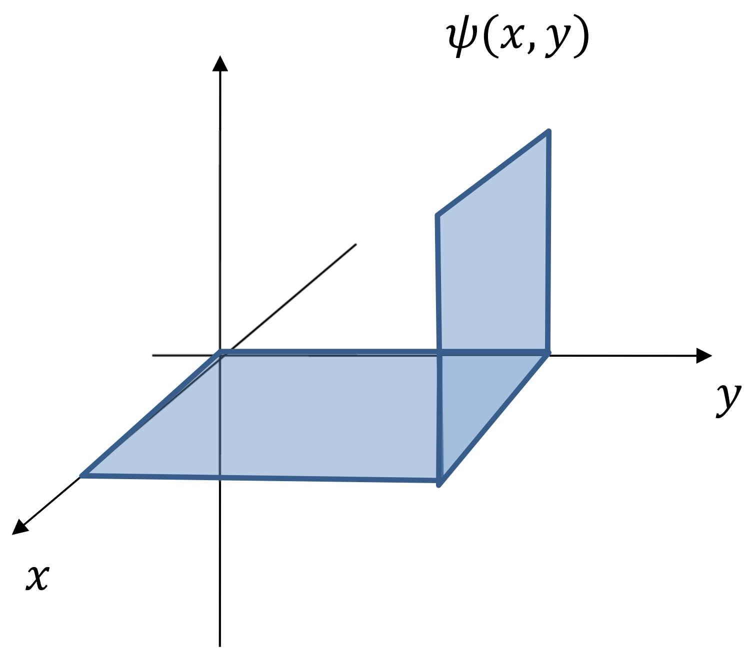


We will now find the solution to

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

On the domain $D = \{(x, y)|0 \leq x \leq 1, -L \leq y \leq L\}$
with the boundary conditions

$$\left.\begin{aligned} \psi(x,0) &= 0, \\ \psi(x,L) &= 1, \\ \psi(0,y) &= 0, \\ \psi(1,y) &= 0 \end{aligned}\right\}$$

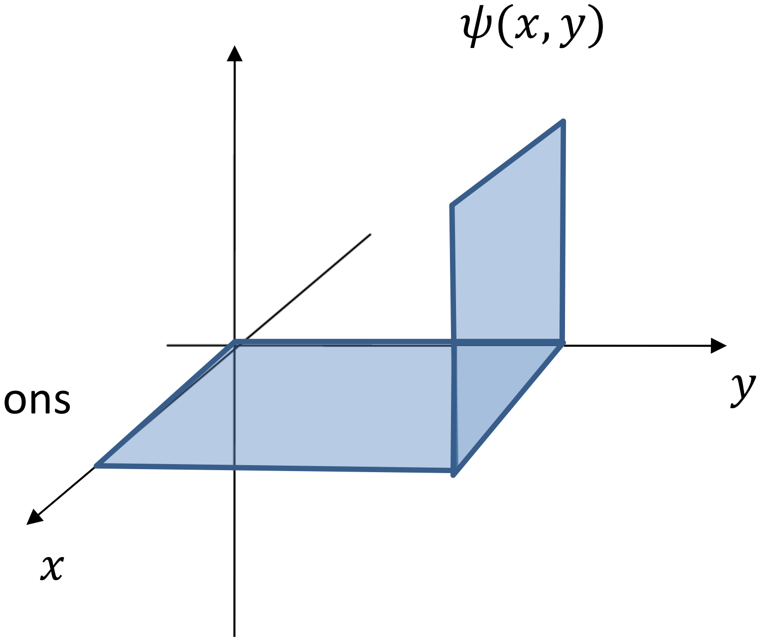


We first substitute the <u>Separation Ansatz:</u>

Let $\quad \psi(x, y) = X(x)Y(y)$

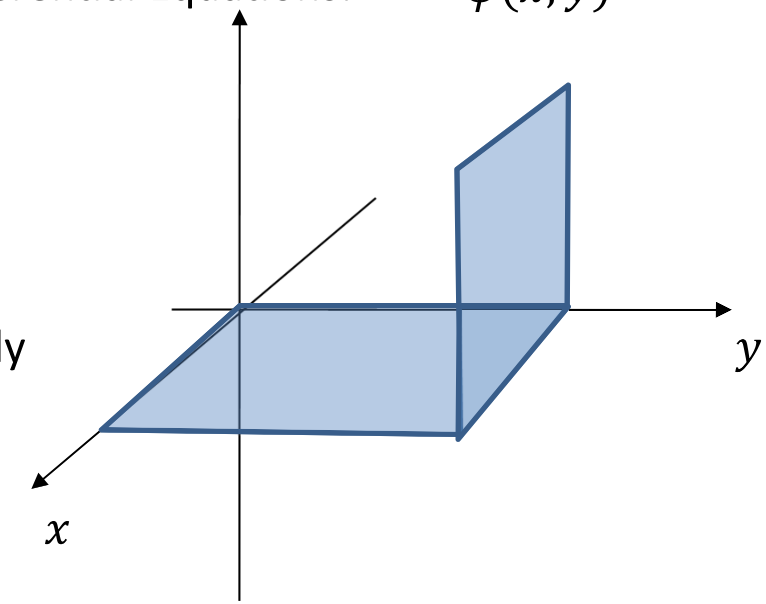
Now we have found

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = 0$$



The only way this can be true is if both of these fractions are constant. That is:

So we have converted the PDE into two Ordinary Differential Equations:



$$\frac{X''(x)}{X(x)} = \pm\lambda, \quad \frac{Y''(y)}{Y(y)} = - \pm \lambda$$

Two of the boundary conditions will be automatically satisfied if

$$X(0) = X(\pi) = 0$$

BCs
$$\begin{cases} \psi(x,0) = 0, \\ \psi(x,L) = 1, \\ \psi(0,y) = 0, \\ \psi(1,y) = 0 \end{cases}$$

Note that the problem

$$\frac{X''(x)}{X(x)} = -\lambda$$

$$X(0) = X(\pi) = 0$$

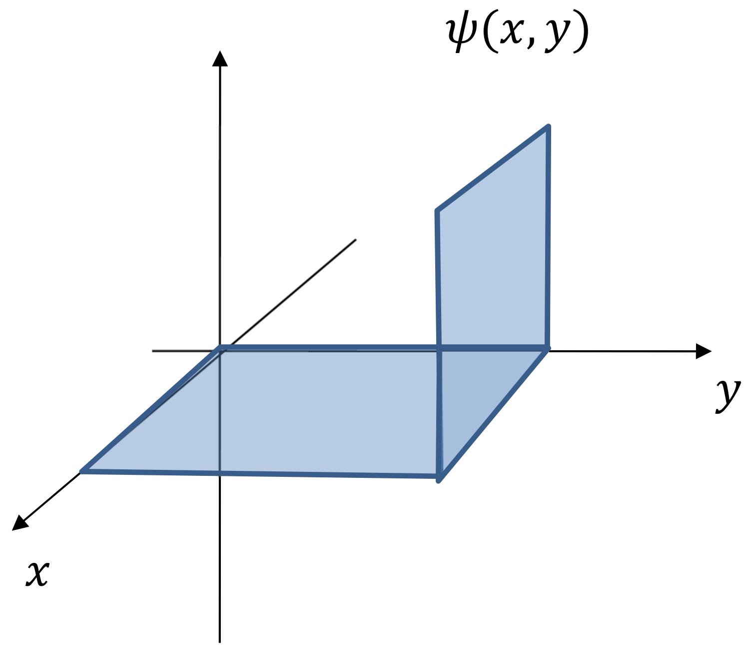
is a Sturm-Liouville problem.

So we have an infinite set of solutions for $X(x)$:

$$\begin{cases} X_n(x) \sin(\sqrt{\lambda_n} x) \\ \sqrt{\lambda_n} = \pi n \end{cases}$$

We can now solve for $Y(y)$:

$$\frac{Y''(y)}{Y(y)} = \lambda$$

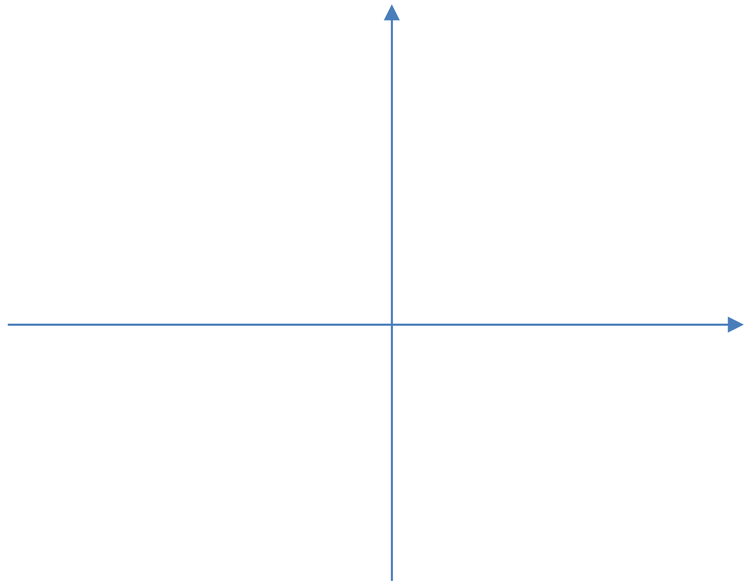


BCs $\begin{cases} \psi(x, 0) = 0, \\ \psi(x, L) = 1, \\ \psi(0, y) = 0, \\ \psi(1, y) = 0 \end{cases}$

(Aside: The *general solution* of

$$Y''(y) = k^2 Y(y)$$

is:

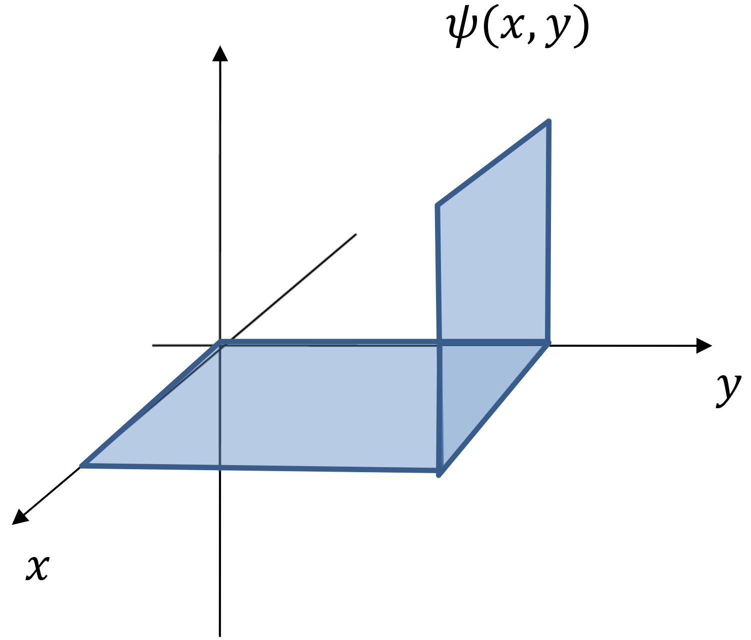


The total solution is therefore

$$\psi_n(x, y) = \sin(n\pi x)\sinh(n\pi y)$$

Does this fit all the boundary conditions?

BCs
$$\begin{cases} \psi(x,0) = 0, \\ \psi(x,L) = 1, \\ \psi(0,y) = 0, \\ \psi(1,y) = 0 \end{cases}$$

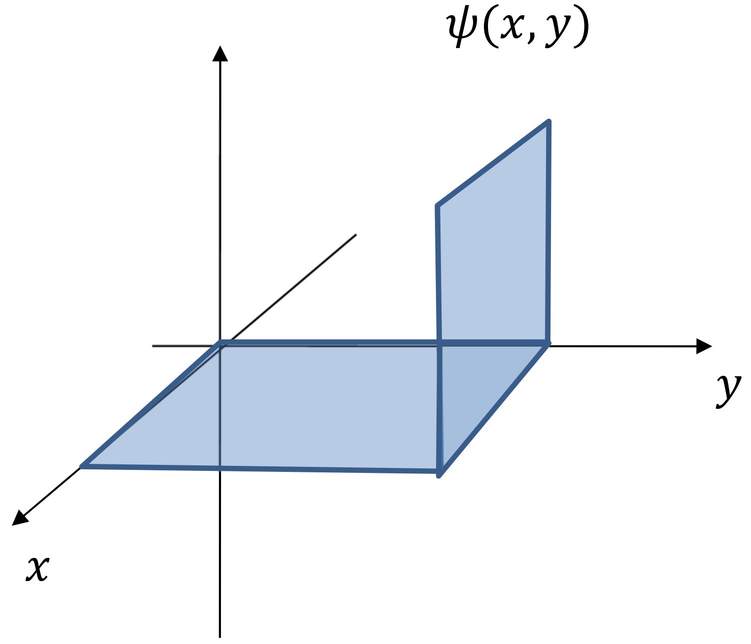


Solution: construct a *series of these functions to satisfy the remaining boundary condition*

We construct the total solution as a sum of the eigenfunctions X_n:

$$\psi(x,y) = \sum_{n=1}^{\infty} b_n X_n(x) Y_n(y)$$

We need $\psi(L) = 1$, so:



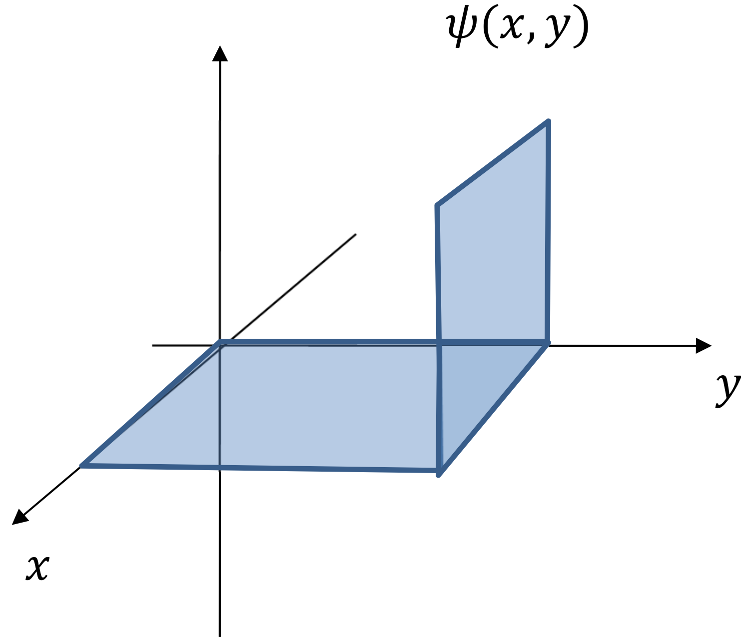
BCs
$$\begin{cases} \psi(x,0) = 0, \\ \psi(x,L) = 1, \\ \psi(0,y) = 0, \\ \psi(1,y) = 0 \end{cases}$$

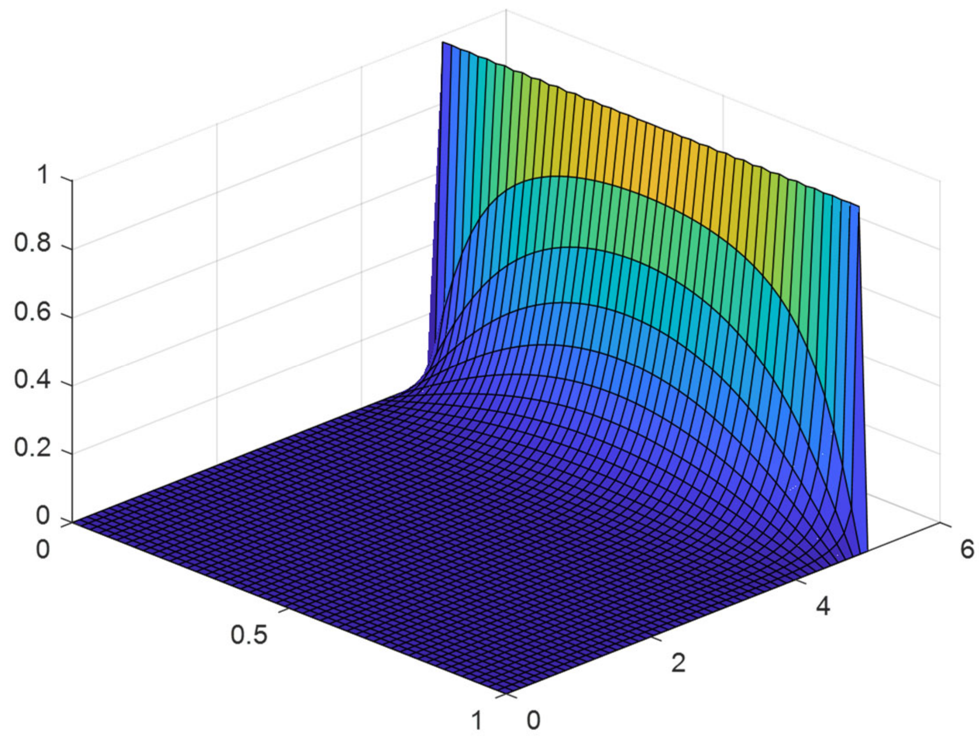
So the solution is

$$\psi(x,y) = \sum_{n=1}^{\infty} \frac{1}{Y_n(L)} \frac{\langle X_n, 1 \rangle}{||X_n||^2} X_n(x) Y_n(x)$$

with

$$X_n(x) = \sin(n\pi x)$$
$$Y_n(y) = \sinh(n\pi y)$$





1
0.8
0.6
0.4
0.2
0
0
0.5
1
0
2
4
6

The Diffusion and Wave equations

Canonical examples:

The heat equation:

$$\kappa \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$ ← parabolic

Laplace's equation

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$ ← elliptic

The wave equation:

$$\nabla^2 \psi - \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} = 0$$ ← hyperbolic.

The steps to solving the diffusion and wave equation are the same as for Laplace's equation, namely:

1. Separate variables

2. Identify the Sturm-Liouville problem and compute the eigenfunctions

3. Use an infinite series to match the boundary conditions.

Diffusion equation example:

The temperature of a metal bar of at position x and time t is given by

$$\kappa \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

With the boundary conditions

$$\begin{aligned} u(0,t) &= 0 \\ u(L,t) &= 0 \end{aligned}$$

$$u(x,0) = x(L-x)$$

Step 1: Separate the variables:

Step 2: identify the S-L problem and find the eigenfunctions

We have found:

$$\frac{X''(x)}{X(x)} = \frac{1}{\kappa}\frac{T'(t)}{T(t)}$$

3. Expand the solution in a series to match the boundary conditions

We have found

$$u(x,t) = \sum_{n=1}^{\infty} X_n(x)T(t)$$

with

$$X_n(x) = \sin(\frac{n\pi}{L}x)$$

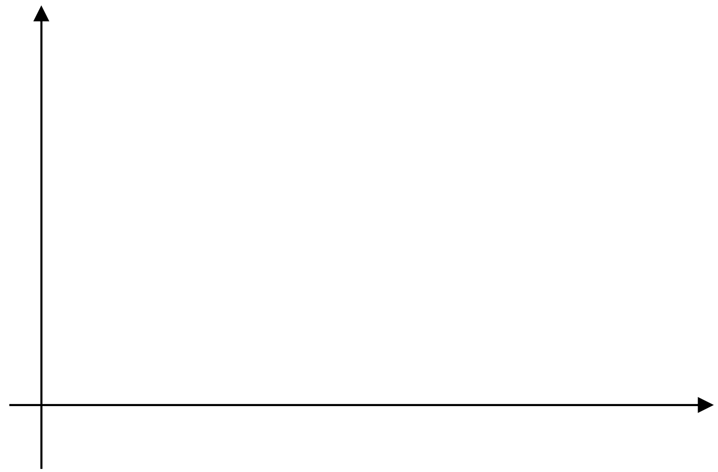
$$T_n(t) = e^{-(n\pi/\kappa L)t}$$

The remaining BC is

$$u(x,0) = x(L-x)$$

Wave equation example:
The z component of an electromagnetic wave travelling along a square metal pipe is of width a is given by the equation

$$\frac{d^2 E}{dx^2} - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = 0$$



With the boundary conditions

$$E(0,t) = 0$$
$$E(a,t) = 0$$

and the initial conditions

$$E(x,0) = f(x)$$

$$\frac{\partial E}{\partial x}(x,0) = 0$$

Step 1: Separate the variables:

Step 2: identify the S-L problem and find the eigenfunctions

We have found:

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{c^2T(t)}$$

3. Expand the solution in a series to match the boundary conditions

We have found

$$E(x,t) = \sum_{n=1}^{\infty} X_n(x) T_n(t)$$

with

$$X_n(x) = \sin(\frac{n\pi}{a} x)$$

$$T_n(t) = \cos\left(\frac{n\pi c t}{a}\right)$$

The remaining BC is

$$E(x,0) = f(x)$$