MULTIVARIABLE CALCULUS

OLIVER KNILL, MATH 21A

Lecture 26: Flux integrals I

Definition

If $\vec{F}(x,y,z)$ is a vector field and $\vec{r}(u,v)$ parametrizes a surface S on a parameter domain R, then

$$\int_S \vec{F} \cdot d\vec{S} = \int_R \vec{F}(\vec{r}(u,v)) \cdot \vec{r}_u \times \vec{r}_v \, dudv$$

is the **flux integral** of $\vec{F}$ through S. It measures how much field passes through S in unit time. It is common to abbreviate $d\vec{S} = \vec{r}_u \times \vec{r}_v \; dudv$. Compare this with $dS = |\vec{r}_u \times \vec{r}_v| \; dudv$, which appears in scalar integrals. Note that the flux integral depends on the orientation of the surface.

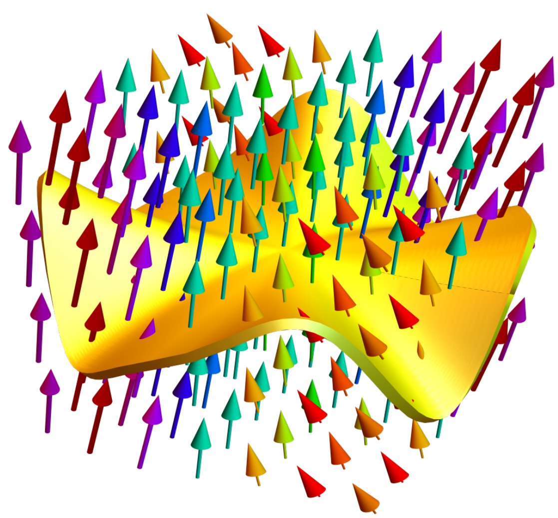

FIGURE 1. The flux of a vector field $\vec{F}$ through a surface measures how much field passes through S in unit time.

Example

Let $\vec{F}(x,y,z) = \langle 0,0,z\rangle$ and S be the unit sphere $x^2+y^2+z^2=1$ oriented outwards. In order to compute the flux, we have to parametrize the surface first

$$\vec{r}(\phi,\theta) = \langle \sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi)\rangle \ .$$

We have $\vec{F}(\vec{r}(u,v)) = \langle 0,0,\cos(\phi)\ \rangle$. We also have

$$\vec{r}_\phi(\phi,\theta) \times \vec{r}_\theta(\phi,\theta) = \sin(\phi)\langle \sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi)\rangle \ .$$

Now we can compute the integral

$$\int_0^{2\pi}\int_0^{\pi} \langle 0,0,\cos(\phi)\rangle \cdot \langle \sin(\phi)\langle \sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi)\rangle d\phi d\theta \ .$$

This simplifies to $\int_0^{2\pi}\int_0^{\pi} \cos^2(\phi)\sin(\phi)\ d\phi d\theta = 4\pi/3$. The answer is positive. We could have seen that also by noticing that at every point $\vec{r}_\phi \times \vec{r}_v$.

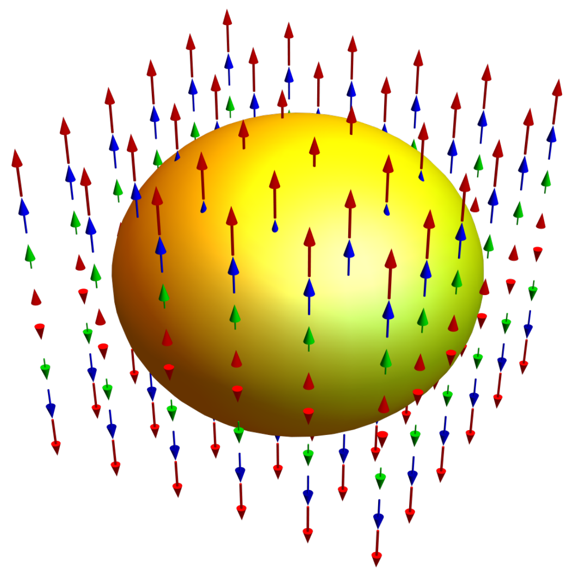

Figure 2. The vector field $\vec{F}(x,y,z) = \langle 0,0,z\rangle$ will be important later on. It has constant divergence 1. We will see later that the computation just done was actually just the computation of the volume of the sphere.

About flux and surface integrals

Line integrals and flux integrals are orientation sensitive. Arc length and surface area or more generally surface integrals orientation oblivious. When we write $\int_a^b f'(t)\ dt = f(b)-f(a)$, the integration depends on the orientation. If you compute volume, area or length you do not care how you parametrize. If you define the function $f(x,y,z) = \vec{F}(x,y,z)\cdot\vec{n}(x,y,z)$, the flux integral $\int\int_S \vec{F}\cdot d\vec{S}$ becomes the surface integral $\int\int_S f d\vec{S}$. The reason is that $\vec{n}(x,y,z) = \vec{r}_u \times \vec{r}_v/|\vec{r}_u \times \vec{r}_v|$ and so $d\vec{S} = \vec{n}dS$. It is good advise to avoid the $\vec{n}$ notation however, especially when computing things. It is not only more complicated, there can also be places, where it is not defined.