**Surface Area of Implicitly Defined Surfaces**
Math 311

When $S$ is the graph of a differentiable function $f(x,y)$, we define the element of surface area

$$dS = \sqrt{f_x^2 + f_y^2 + 1}\,dA, \tag{1}$$

where $dA$ denotes an element of plane area in the domain of $f$. When $S$ is implicitly defined by an equation $g(x,y,z) = 0$, where $g$ is a diferentiable function, we compute the element of surface area $dS$ by thinking of $S$ (locally) as the graph of an implicit function of two variables and use implicit differentiation to find its partials. For example, when $z$ is an implicit function of $x$ and $y$, our formulas for implicit differentiation give

$$f_x^2 = \left(\frac{\partial z}{\partial x}\right)^2 = \left(-\frac{g_x}{g_z}\right)^2 = \frac{g_x^2}{g_z^2} \text{ and } f_y^2 = \left(\frac{\partial z}{\partial y}\right)^2 = \left(-\frac{g_y}{g_z}\right)^2 = \frac{g_y^2}{g_z^2},$$

and it follows from the formula in (1) that

$$dS = \sqrt{f_x^2 + f_y^2 + 1}\,dA = \sqrt{\frac{g_x^2 + g_y^2 + g_z^2}{g_z^2}}\,dA.$$

**Example:** The sphere $S$ of radius $a$ is defined implicitly by

$$g(x,y,z) = x^2 + y^2 + z^2 - a^2 = 0.$$

The upper and lower hemispheres are graphs of implicit functions of $x$ and $y$ given by $g$. Thus

$$\begin{aligned} dS &= \sqrt{\frac{4x^2+4y^2+4z^2}{4z^2}}\,dA = \sqrt{\frac{x^2+y^2+z^2}{z^2}}\,dA = \sqrt{\frac{a^2}{z^2}}\,dA \\ &= \frac{a}{|z|}\,dA = \pm\frac{a}{z}\,dA = \frac{\pm a}{\sqrt{a^2-x^2-y^2}}\,dA, \end{aligned}$$

where we use + sign when $z > 0$; this gives the upper hemisphere. By symmetry, the upper and lower hemispheres have the same surface area $SA$. Thus integrating over the upper hemisphere in polar coordinates and doubling our answer we have

$$\begin{aligned} SA &= \iint_S dS = 2\iint_R \frac{a}{\sqrt{a^2-x^2-y^2}}\,dA = 2a\int_0^{2\pi}\int_0^a \frac{r}{\sqrt{a^2-r^2}}\,dr\,d\theta \\ &= 4\pi a\int_0^a \frac{r}{\sqrt{a^2-r^2}}\,dr. \end{aligned}$$

Let $u = a^2 - r^2$; then $du = -2r\,dr$, $u(0) = a^2$ and $u(a) = 0$. Then

$$SA = 4\pi a\int_{a^2}^0 \frac{r}{\sqrt{u}}\frac{du}{-2r} = -2\pi a\left.\frac{u^{1/2}}{1/2}\right|_{a^2}^0 = 0 - \left(-4\pi a^2\right) = 4\pi a^2.$$