

Problem Set 0 - solutions

$$\begin{aligned} \underline{1.} \quad a) \quad \underline{a} + \underline{b} &= (1, 2, 2) + (2, -3, 1) \\ &= (2+1, 2+(-3), 2+1) \\ &= (3, -1, 3) \quad \underline{\text{vector}} \end{aligned}$$

$$\begin{aligned} b) \quad 2\underline{a} - \underline{b} &= 2(1, 2, 2) - (2, -3, 1) \\ &= (2, 4, 4) - (2, -3, 1) \\ &= (0, 7, 3) \quad \underline{\text{vector}} \end{aligned}$$

$$c) \quad |\underline{a}| = \sqrt{1^2 + 2^2 + 2^2} = \sqrt{9} = 3 \quad \underline{\text{scalar}}$$

$$\begin{aligned} d) \quad \underline{a} \cdot \underline{b} &= a_1 b_1 + a_2 b_2 + a_3 b_3 \\ &= 2 - 6 + 2 \\ &= -2 \quad \underline{\text{scalar}} \end{aligned}$$

$$\underline{2.} \quad a) \quad \underline{a} \cdot \underline{b} = 2 + 0 + 0 = 2$$

b) We know

$$\underline{a} \cdot \underline{b} = |\underline{a}| |\underline{b}| \cos \theta$$

$$\begin{aligned} \text{so } \cos \theta &= \frac{\underline{a} \cdot \underline{b}}{|\underline{a}| |\underline{b}|} = \frac{2}{\sqrt{2^2 + 0 + 1} \sqrt{1^2 + (-3)^2 + 0}} = \frac{2}{\sqrt{5} \sqrt{10}} \\ &= \frac{2}{\sqrt{50}} \Rightarrow \theta = 1.29 \end{aligned}$$

$$c) \quad \hat{\underline{b}} = \frac{\underline{b}}{|\underline{b}|} = \frac{(1, -3, 0)}{\sqrt{1^2 + (-3)^2 + 0}} = \frac{1}{\sqrt{10}} (1, -3, 0)$$

d) The vector parallel to $\underline{b}$ and twice as long is

$$2\underline{b} = (2, -6, 0)$$

e) The scalar projection is

$$\begin{aligned}\vec{a} \cdot \hat{\vec{b}} &= (2, 0, 1) \cdot \hat{\vec{b}} \\ &= (2, 0, 1) \cdot \frac{(1, -3, 0)}{\sqrt{10}} \\ &= \frac{2 + 0 + 0}{\sqrt{10}} = \frac{2}{\sqrt{10}}\end{aligned}$$

3. a)

$$\begin{aligned}\vec{a} \times \vec{b} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 3 \\ 3 & 1 & -1 \end{vmatrix} = \hat{i} \begin{vmatrix} 1 & 3 \\ -1 & -1 \end{vmatrix} - \hat{j} \begin{vmatrix} 1 & 3 \\ 3 & -1 \end{vmatrix} + \hat{k} \begin{vmatrix} 1 & -1 \\ 3 & 1 \end{vmatrix} \\ &= -2\hat{i} + 10\hat{j} + 4\hat{k} \quad (= (-2, 10, 4))\end{aligned}$$

b)

$$\begin{aligned}\vec{a} \times \vec{b} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 4 & -3 \\ 2 & -3 & 4 \end{vmatrix} = \hat{i} \begin{vmatrix} 4 & -3 \\ -3 & 4 \end{vmatrix} - \hat{j} \begin{vmatrix} 2 & -3 \\ 2 & 4 \end{vmatrix} + \hat{k} \begin{vmatrix} 2 & 4 \\ 2 & -3 \end{vmatrix} \\ &= 7\hat{i} - 14\hat{j} - 14\hat{k}\end{aligned}$$

c)

$$\vec{a} \times \vec{b} =$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{vmatrix}$$

$$= \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(-1-1)$$

$$= -2\hat{k}$$

4. a)

$$\frac{\partial f}{\partial x} = 4x - 3y$$

$$\frac{\partial f}{\partial y} = -3x + 8y$$

b)

$$\frac{\partial g}{\partial x} = 2 \cos(2x + 3y)$$

$$\frac{\partial g}{\partial y} = 3 \cos(2x + 3y)$$

c)

$$\frac{\partial f}{\partial x} = yz \cos(xyz)$$

$$\frac{\partial f}{\partial y} = xz \cos(xyz)$$

$$\frac{\partial f}{\partial z} = 2z + xy \cos(xyz)$$

d)

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(y z^{-1}) = z^{-1} = \frac{1}{z}$$

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}(y z^{-1}) = y \left(-\frac{1}{z^2} \right) = -\frac{y}{z^2}$$

5. We have

$$f(x, y) = x^2 + 3xy + y^2$$

We want to find f_{xx} , f_{yy} , f_{xy} and f_{yx} .

Now

$$\frac{\partial f}{\partial x} = 2x + 3y$$

$$\frac{\partial f}{\partial y} = 3x + 2y$$

$$\Rightarrow \frac{\partial^2 f}{\partial x^2} = 2$$

$$\frac{\partial^2 f}{\partial y^2} = 2$$

$$\frac{\partial^2 f}{\partial x \partial y} = 3$$

$$\frac{\partial^2 f}{\partial y \partial x} = 3$$

So $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ as required.

$\nearrow$ f_{xy} $\nwarrow$ f_{yx}

(In this case we also have $f_{xx} = f_{yy}$
but this is a coincidence!)