1. A vector field is a function whose value is a vector at each position, whereas a scalar field is a scalar-valued function of position.

a) scalar field

b) vector field

c) vector field

d) (constant) vector field

e) Not defined (doesn't make sense to mix vectors and scalars in this way)

f) scalar field

g) vector field

2.

$$\left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle f$$

$$= \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

$$= \langle 3, 2, 2z \rangle$$

3. a)

$$\nabla g = \left\langle \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z} \right\rangle$$

$$= \langle 2xy^3z^4, 3x^2y^2z^4, 4x^2y^3z^3 \rangle$$

b)

$$\nabla f = \left\langle e^z \cos x \ln y, e^z \sin x \frac{1}{y}, e^z \sin x \ln y \right\rangle$$

4.

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

Now

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[\left(x^2+y^2+z^2\right)^{-1}\right]$$

$$= -\left(x^2+y^2+z^2\right)^{-2} \cdot \frac{\partial}{\partial x}\left(x^2+y^2+z^2\right)$$

$$= -\frac{1}{\left(x^2+y^2+z^2\right)^2} \times 2x = \frac{-2x}{\left(x^2+y^2+z^2\right)^2}$$

Similarly

This word is very useful here

$$\frac{\partial f}{\partial y} = \frac{-2y}{(x^2+y^2+z^2)^2}$$

$$\frac{\partial f}{\partial z} = \frac{-2x}{(x^2+y^2+z^2)^2}$$

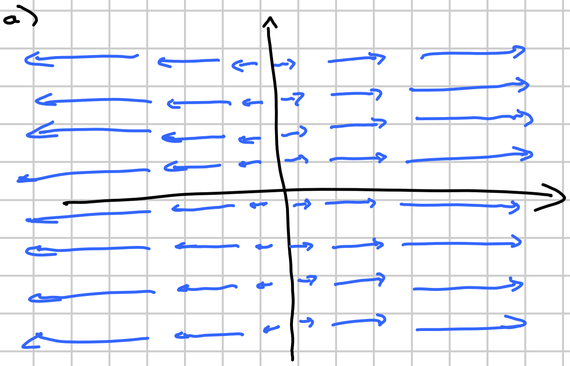
So

$$\nabla f = \left\langle \frac{-2x}{(x^2+y^2+z^2)^2}, \frac{-2y}{(x^2+y^2+z^2)^2}, \frac{-2z}{(x^2+y^2+z^2)^2} \right\rangle$$

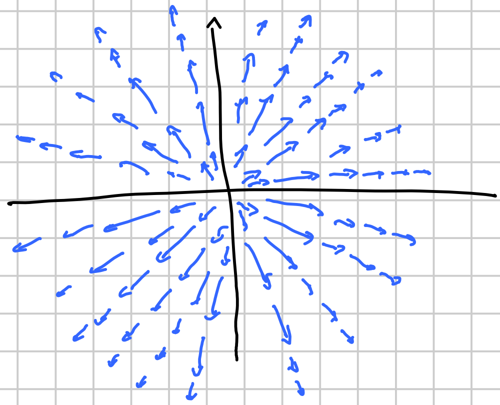
$$= \frac{2}{(x^2+y^2+z^2)^2} \langle -x, -y, -z \rangle$$

The first term is just a positive scalar. The second term always points inwards towards the origin. ∇f therefore points towards the origin.

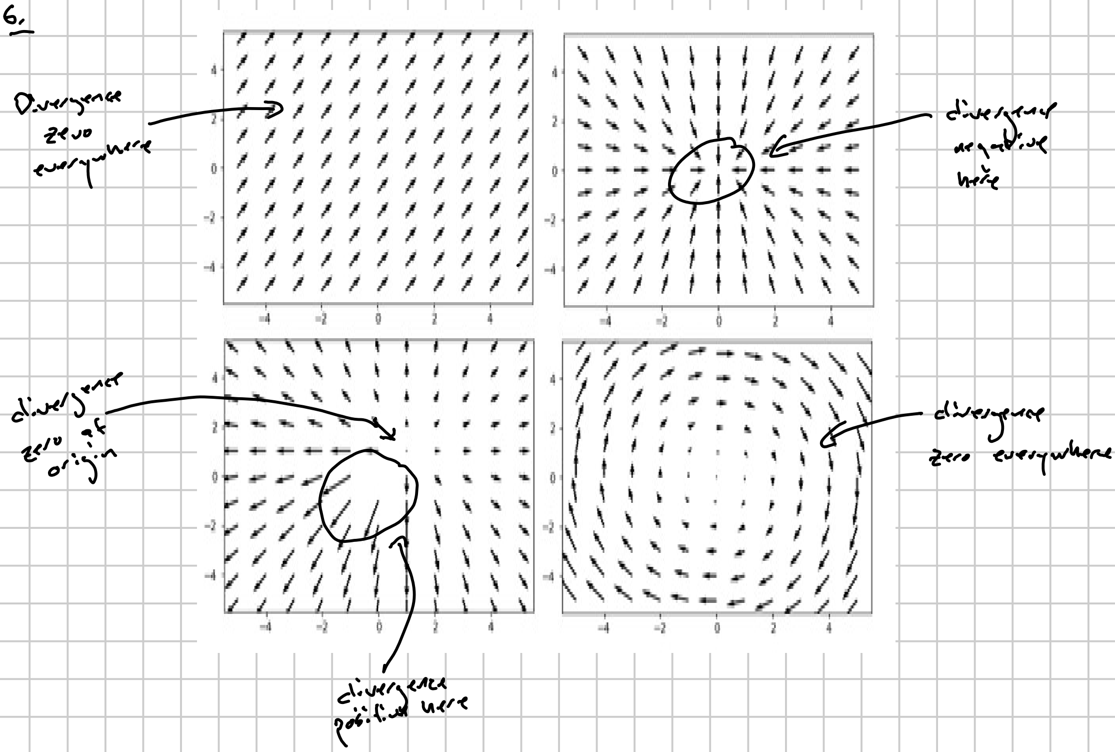
5. a)



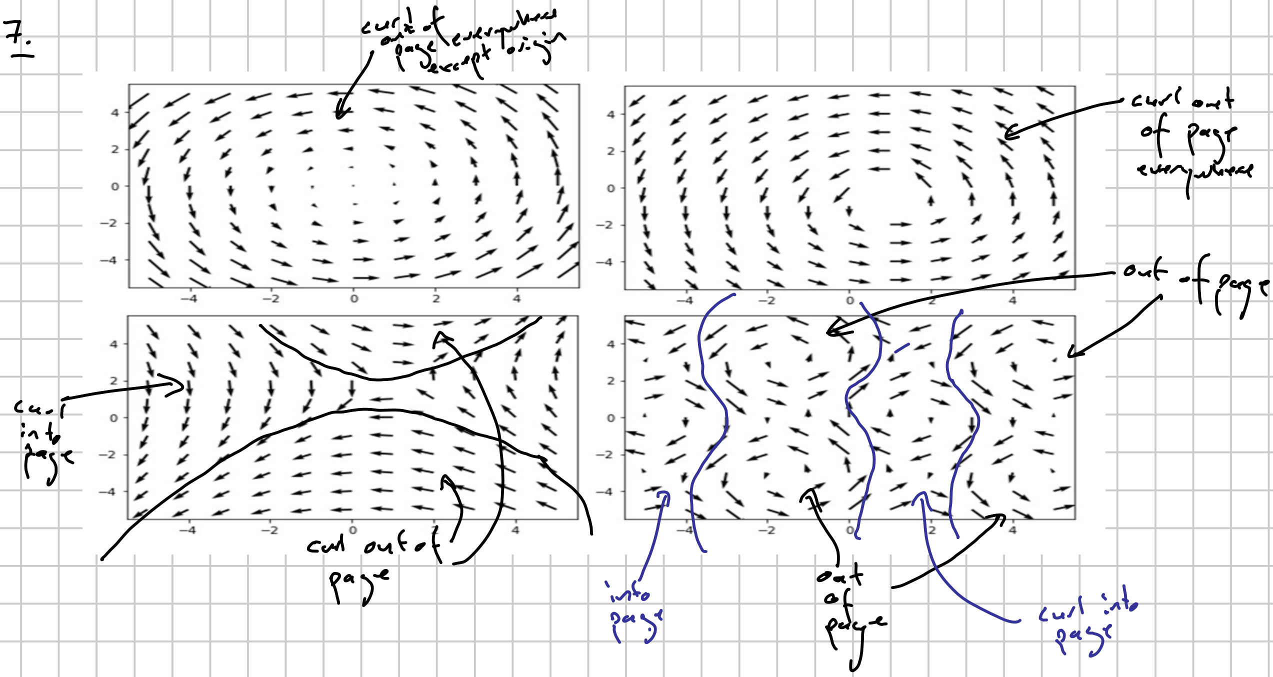
b)



6.



7.



8. We have

$$\underset{\sim}{v}(x,y,z) = \langle 2xy, -yz, 3z \rangle$$

So

$$\nabla \cdot \underset{\sim}{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

$$= 2y - z + 3$$

and

$$\nabla \times \underset{\sim}{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy & -yz & 3z \end{vmatrix} = \hat{i}(0+y) - \hat{j}(0-0) + \hat{k}(0-2x)$$

$$= y\hat{i} - 2x\hat{k}$$

$$= \langle y, 0, -2x \rangle$$

9. a) (i) $\nabla \cdot \underset{\sim}{v} = 2x + 2x + 6z$

$$= 4x + 6x$$

(ii)

$$\nabla \times \underset{\sim}{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & 2xy & 3z^2 \end{vmatrix}$$

$$= \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(2y-0)$$

$$= 2y\hat{k} = \langle 0, 0, 2y \rangle$$

b) (i) $\underset{\sim}{v} = \langle -y, x, 0 \rangle$

$$\nabla \cdot \underline{v} = \frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial z}(0)$$

$$= 0 + 0 + 0 = 0$$

(ii)
$$\nabla \times \underline{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix}$$

$$= \underset{\sim}{\hat{i}}(0-0) - \hat{j}(0-0) + \underset{\sim}{\hat{k}}(1+1)$$

$$= 2\underset{\sim}{\hat{k}}.$$

10. a)
$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

$$= \frac{\partial}{\partial x}\left(2x + z^2\right) + \frac{\partial}{\partial y}\left(2(y+1)\right) + \frac{\partial}{\partial z}\left(2xz\right)$$

$$= 2 + 2 + 2x$$

$$= 2x + 4$$

b)
$$\nabla^2 \phi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) \cos x \cos y \cos z$$

$$= \cos y \cos z \frac{\partial^2}{\partial x^2}(\cos x) + \cos x \cos z \frac{\partial^2}{\partial y^2} \cos y$$

$$+ \cos x \cos y \frac{\partial^2}{\partial z^2} \cos z$$

$$= -\cos x \cos y \cos z - \cos x \cos y \cos z - \cos x \cos y \cos z$$

$$= -3 \cos x \cos y \cos z.$$

This is zero whenever $\cos x$, $\cos y$ or $\cos z$ is zero, i.e. along the planes in 3D given by $x = (2l+1)\frac{\pi}{2}$, $y = (2m+1)\frac{\pi}{2}$, $z = (2n+1)\frac{\pi}{2}$ for l, m, n, being any integers.