Vector Calculus and PDEs 37336
Problem Set 1: Div, Grad and Curl

1. State clearly what the difference is between a *vector field* and a *scalar field*. Classify the following as either vector fields, scalar fields, or something else (state what this is).
   (a) $f(x,y,z) = x^2 + 2y$
   (b) $\mathbf{f}(x,y,z) = 2x\hat{\mathbf{i}} + y^2\hat{\mathbf{j}} + (x+y)\hat{\mathbf{k}}$
   (c) $f(x,y,z) = 3\mathbf{r}$, where $\mathbf{r} =< x,y,z >$
   (d) $g(x,y,z) = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$
   (e) $f(x,y,z) = 2 + x^2 + 3\hat{\mathbf{i}} + 4y\hat{\mathbf{j}}$
   (f) $\mathbf{f} \cdot \mathbf{g}$ where $\mathbf{f} = 2x\hat{\mathbf{i}} + y^2\hat{\mathbf{j}} + (x+y)\hat{\mathbf{k}}$ and $\mathbf{g} = 6x\hat{\mathbf{i}} - y^3\hat{\mathbf{j}} + (x+y)\hat{\mathbf{k}}$
   (g) $\mathbf{f} \times \mathbf{g}$ where $\mathbf{f} = 2x\hat{\mathbf{i}} + y^2\hat{\mathbf{j}} + (x+y)\hat{\mathbf{k}}$ and $\mathbf{g} = 6x\hat{\mathbf{i}} - y^3\hat{\mathbf{j}} + (x+y)\hat{\mathbf{k}}$

2. Calculate the vector quantity

$$\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle f(x,y,z) \quad ,$$

   where $f(x,y,z) = 3x + 2y + z^2$.

3. Find the gradients of the following functions:
   (a) $g(x,y,z) = x^2y^3z^4$
   (b) $f(x,y,z) = e^z \sin x \ln(y)$

4. Find the gradient ∇f of the function

$$f(x,y,z) = \frac{1}{x^2+y^2+z^2} \ .$$

   In which direction does the vector ∇f point?

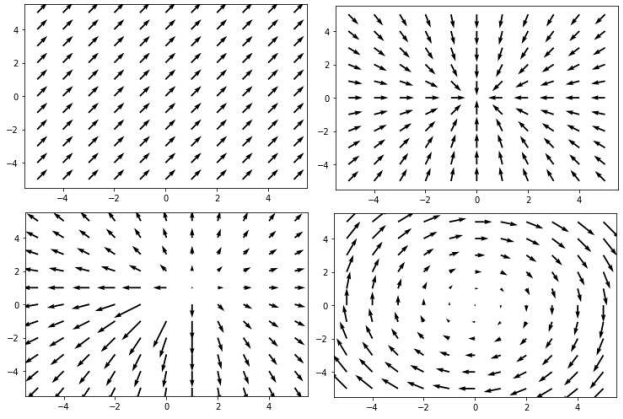
5. Sketch the 2D vector field corresponding to the *gradient* of
   (a)

$$f(x,y) = x^2$$

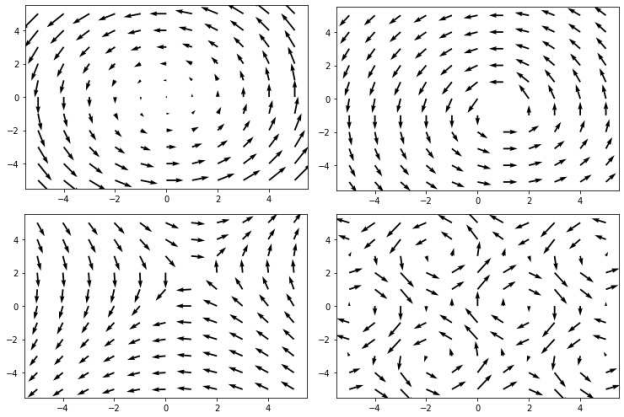
   (b)

$$f(x,y) = e^{-(x^2+y^2)}$$

6. For the following vector fields, indicate where the divergence is zero, where it is positive (i.e. a source), and where it is negative (i.e. a sink):



7. For the following vector fields, indicate where the curl is zero, and where it points into or out of the page:



8. Calculate the divergence and curl of the vector field

$$\mathbf{v}(x,y,z) =< 2xy, -yz, 3z > \quad .$$

9. Find (i) the divergence, and (ii) the curl of the following vector functions:
   (a) $\mathbf{v} = x^2\hat{\mathbf{i}} + 2xy\hat{\mathbf{j}} + 3z^2\hat{\mathbf{k}}$
   (b) $\mathbf{v} = -y\hat{\mathbf{i}} + x\hat{\mathbf{j}}$

10. Calculate the Laplacian ($\nabla^2\phi$) of the following functions:
    (a) $\phi = x^2 + (y+1)^2 + xz^2$
    (b) $\phi = \cos x \cos y \cos z$
    For (b) above, where is the Laplacian equal to zero?