1. We want to find a general solution to

$$a u_{xx} + b u_x + c u_t = 0$$

Let

$$u(x,t) = X(x)T(t)$$

Then

$$a X'' T + b X' T + c X T' = 0$$

Divide by XT:

$$a \frac{X''}{X} + b \frac{X'}{X} + c \frac{T'}{T} = 0$$

So the variables separate:

$$a \frac{X''}{X} + b \frac{X'}{X} = \pm \lambda = -\frac{T'}{T}$$

Solving for T:

$$T' = -(\pm\lambda)T$$

The general sol$^{\underline{n}}$ is

$$T(t) = A e^{\mp \lambda t}$$

Solve for X:

$$a \frac{X''}{X} + b \frac{X'}{X} = \pm \lambda$$

So

$$a X'' + b X' \mp \lambda X = 0$$

The general solution is

$$X(x) = A e^{m_1 x} + B e^{m_2 x}$$

where

$$m = \frac{1}{2a}\left(-b \pm \sqrt{b^2 \pm 4a\lambda}\right)$$

So the general solution is

$$u(x,t) = \left(A e^{m_1 x} + B e^{m_2 x}\right) e^{\mp \lambda t}.$$

2. We want to solve

$$-\frac{d^2}{dx^2}\phi = \lambda\phi$$

On $-L \leq x \leq L$, with periodic boundary conditions.

The general solution is

$$\phi(x) = Ae^{i\sqrt{\lambda}x}$$

We want

$$\phi(x+2L) = \phi(x)$$

So

$$Ae^{i\sqrt{\lambda}(x+2L)} = Ae^{i\sqrt{\lambda}x}$$

$$\Rightarrow Ae^{i\sqrt{\lambda}x}e^{i\sqrt{\lambda}2L} = Ae^{i\sqrt{\lambda}x}$$

$$\Rightarrow e^{i\sqrt{\lambda}.2L} = 1 = e^{i2m\pi}$$

where $m \in \mathbb{Z}$. So we have

$$\sqrt{\lambda}\,2L = 2m\pi$$

or

$$\sqrt{\lambda} = \frac{m\pi}{L} \Rightarrow \lambda = \left(\frac{m\pi}{L}\right)^2$$

So the eigenfunctions are

$$\phi_m = e^{i\left(\frac{m\pi}{L}\right)x}$$

with eigenvalues $\lambda_m = \left(\frac{m\pi}{L}\right)^2$.

3. We have

$$u(r,\theta) = r^m e^{im\theta}.$$

Therefor

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(r\left[m r^{m-1} e^{im\theta}\right]\right)$$

$$+ \frac{1}{r^2} r^m (-m^2) e^{im\theta}$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(m r^m e^{im\theta}\right) - r^{m-2} m^2 e^{im\theta}$$

$$= \frac{1}{r}\left(m^2 r^{m-1} e^{im\theta}\right) - r^{m-2} m^2 e^{im\theta}$$

$$= m^2 r^{m-2} e^{im\theta} - r^{m-2} m^2 e^{im\theta}$$

$$= 0.$$

4. The general solution is

$$u(r,\theta) = A_0 + B_0 \ln r$$

$$+ \sum_{m=-\infty}^{\infty}\left(A_m r^{|m|} + B_m r^{-|m|}\right) e^{im\theta}.$$

Here we have $u(a,\theta) = 0$, so

$$0 = A_0 + B_0 \ln a$$

$$+ \sum_{m=-\infty}^{\infty}\left(A_m a^{|m|} + B_m a^{-|m|}\right) e^{im\theta}$$

which is satisfied if

$$A_0 = -B_0 \ln a \implies B_0 = \frac{-A_0}{\ln a}$$

$$A_m a^{|m|} = -B_m a^{-|m|} \implies B_m = -A_m a^{2|m|}.$$

So

$$u(r,\theta) = A_0 - A_0 \frac{\ln r}{\ln a} + \sum_{m=-\infty}^{\infty} \left(A_m r^{|m|} - A_m a^{2|m|} r^{-|m|}\right) e^{im\theta}.$$

We now have

$$u(b,\theta) = \cos 2\theta, \quad \text{so}$$

$$\cos 2\theta = A_0 - A_0 \frac{\ln b}{\ln a} + \sum_{m=-\infty}^{\infty} A_m \left(b^{|m|} - a^{2|m|} b^{-|m|}\right) e^{im\theta}.$$

Taking the inner product with $e^{in\theta}$:

$$\langle e^{in\theta}, \cos 2\theta \rangle = \langle e^{in\theta}, \underbrace{A_0 - A_0 \frac{\ln b}{\ln a} \rangle}_{=0} + \sum_{m=-\infty}^{\infty} A_m \left(b^{|m|} - a^{2|m|} b^{-|m|}\right) \langle e^{in\theta}, e^{im\theta} \rangle$$

$$= A_n \left(b^{|n|} - a^{2|n|} b^{-|n|}\right) . 2\pi .$$

Now

$$\langle e^{in\theta}, \cos 2\theta \rangle = \int_0^{2\pi} e^{-in\theta} \cos 2\theta \, d\theta$$

$$= \int_0^{2\pi} e^{-in\theta} \frac{1}{2}\left(e^{2i\theta} + e^{-2i\theta}\right) d\theta$$

$$= \frac{1}{2} \times 2\pi \times \delta_{n,2} + \frac{1}{2} \times 2\pi \times \delta_{n,-2}.$$

$$= \begin{cases} \pi & \text{if } n = 2, -2 \\ 0 & \text{otherwise.} \end{cases}$$

So

$$A_n = \frac{1}{2\pi} \frac{1}{b^{|n|} - a^{2|n|} b^{-|n|}} \langle e^{in\theta}, \cos 2\theta \rangle$$

$$= \begin{cases} \frac{1}{2} \frac{1}{b^2 - a^4 b^{-2}} & \text{if } n = 2, -2 \\ 0 & \text{otherwise} \end{cases}$$

Therefore

$$u(r,\theta) = \frac{1}{2} \frac{1}{b^2 - a^4 b^{-2}} r^2 e^{2i\theta}$$

$$\frac{1}{2} \frac{1}{b^2 - a^4 b^{-2}} r^2 e^{-2i\theta}$$

$$= \frac{1}{b^2 - a^4 b^{-2}} \frac{1}{2}\left(e^{2i\theta} + e^{-2i\theta}\right) r^2$$

$$= \frac{1}{b^2 - a^4 b^{-2}} \cos 2\theta \, r^2 .$$

$$= \frac{b^2}{b^4 - a^4} r^2 \cos 2\theta .$$

5. Now

$$\frac{d^2 y}{dt^2} + \frac{1}{t} \frac{dy}{dt} + \left(\frac{1}{a} - \frac{m^2}{t^2}\right) y = 0$$

Multiply by t^2:

$$t^2 \frac{d^2 y}{dt^2} + t \frac{dy}{dt} + \left(\frac{t^2}{a} - m^2\right) y = 0$$

Let $z^2 = \frac{t^2}{a} \Rightarrow z = \frac{t}{\sqrt{a}}$. Then

$$az^2 \frac{d^2 y}{d(az^2)} + az \frac{dy}{d(az)} + \left(z^2 - m^2\right) y = 0$$

or

$$z^2 y'' + z y' + (z^2 - m^2) y = 0$$

which has solution

$$y = Y_m(z) = Y_m\left(\frac{t}{\sqrt{a}}\right) .$$

6. The wave equation is

$$\nabla^2 \psi = \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2}$$

We try the separation Ansatz

$$\psi(r,\theta,t) = \Phi(r,\theta)\,T(t).$$

Then

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right]\Phi(r,\theta)T(t) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}T(t).\Phi(r,\theta)$$

So the variables separate:

$$\frac{\nabla^2\Phi}{\Phi} = \frac{1}{c^2}\frac{T''}{T} =: -\lambda =: -k^2$$

The solution for the problem in t is

$$T(t) = e^{\pm ikct}.$$

The problem for $\Phi(r,\theta)$ is

$$-\nabla^2\Phi = k^2\Phi$$

Together with the boundary condition

$$\frac{\partial\Phi}{\partial r} = 0\Big|_{r=a},$$

this is a Sturm-Liouville problem.

Again we use a separation Ansatz:

$$\Phi(r,\theta) = R(r)\Theta(\theta)$$

So

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + k^2\right] R\Theta = 0$$

Or

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' + k^2 R\Theta = 0$$

Again this separates:

$$r^2\frac{R''}{R} + r\frac{R'}{R} + k^2 r^2 = -\frac{\Theta''}{\Theta} = \lambda$$

The problem for Θ has solutions

$$\Theta(\theta) = e^{\pm i\sqrt{\lambda}\,\theta}$$

and because $\Theta(\theta+2\pi) = \Theta(\theta)$ we require $\sqrt{\lambda} = m$, where $m \in \mathbb{Z}$.

We therefore have

$$\Theta(\theta) = e^{\pm i m\theta}$$

The solutions to the $R(r)$ problem are Bessel functions:

$$R(r) = A J_m(kr) + B Y_m(kr)$$

Now, because u must remain non-singular at $r=0$ we require $B=0$. We therefore have

$$\Phi(r,\theta) = J_m(kr)\, e^{im\theta}$$

Putting in

$$\left.\frac{\partial \Phi}{\partial r}\right|_{r=a} = 0$$

we find solutions

$$\Phi_{m,n}(r,\theta) = J_m(k_{m,n} r)\, e^{im\theta}$$

where $k_{m,n}$ are the coefficients for which

$$J_m'(k_{m,n} a) = 0$$

ie

$$k_{m,n} = \frac{1}{a} d_{mn},$$

where d_{mn} is the n^{th} zero of the derivative of the m^{th}-order Bessel function.

7. The general solution to the PDE is therefore

$$u(r,\theta,t) = \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} \Big[a_{m,n} J_m(k_{m,n} r) e^{im\theta} e^{+ikct}$$

$$+ b_{m,n} J_m(k_{m,n} r) e^{im\theta} e^{-ikct} \Big]$$

We know that

$$u(r,\theta,0) = 0$$

So this means that $b_{m,n} = -a_{m,n}$.

Therefore

$$u(r,\theta,t) = \sum_{m,n} a_{m,n} . J_m(k_{m,n} r) e^{im\theta} \times 2i\sin(kct)$$

We also have

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = f(r,\theta)$$

So

$$\sum_{m,n} a_{m,n} \underbrace{J_m(k_{m,n} r) e^{im\theta}}_{= \Phi_{m,n}} 2ikc . \cos(0) = f(r,\theta)$$

Taking the inner product with $\Phi_{m',n'}$ we find

$$a_{m,n} = \frac{1}{2ikc} \langle \Phi_{m,n}, f \rangle$$

$$= \frac{1}{2ikc} \int_0^{2\pi} d\theta \int_0^a r\,dr \,.\, e^{im\theta} e^{-\frac{r^2}{b^2}} J_m(k_{m,n} r)$$

$$= \begin{cases} 0 & \text{if } m \neq 0 \\ \dfrac{\pi}{ikc} \displaystyle\int_0^a r e^{-\frac{r^2}{b^2}} J_0(k_{0,n} r)\, dr\,, & m = 0 \end{cases}$$

So

$$u(r,\theta,t) = \sum_{m=-\infty}^{\infty} \frac{2\pi}{kc} \int_0^a r' e^{-\frac{r'^2}{b^2}} J_0(k_{0,n} r')\, dr'$$

$$\times\ J_0(k_{0,n} r) \sin(kct)\ .$$