**Vector Calculus and PDEs 37336**
**Problem Set 10: Separation of variables with polar coordinates**

1. Find a separation ansatz and hence the general solution to the equation

$$au_{xx} + bu_x + u_t = 0$$

where $a$ and $b$ are constants.

2. Find all eigenvalues and eigenvectors of the periodic Sturm-Liouville problem

$$-\frac{d^2}{dx^2}\phi = \lambda\phi$$

defined on the domain $-L \leq x \leq L$, with boundary conditions

$$\phi(-L) = \phi(L) \quad , \phi'(-L) = \phi'(L) \ .$$

3. Show that the function

$$u(r,\theta) = r^m e^{im\theta}$$

is a solution to the Laplace equation in polar coordinates.

4. Starting with the general solution for the Laplace equation in 2D polar coordinates, solve

$$\nabla^2 u = 0$$

on the domain $a \leq r \leq b$, with Dirichlet boundary conditions $u = 0$ at $r = a$ and $u = \cos(2\theta)$ at $r = b$.

5. The Bessel function

$$f(z) = Y_m(z)$$

satisfies the equation

$$z^2 f'' + zf' + (z^2 - m^2)f = 0 \ .$$

Write the solution to the differential equation

$$\frac{d^2y}{dt^2} + \frac{1}{t}\frac{dy}{dt} + \left(\frac{1}{a} - \frac{m^2}{t^2}\right)y = 0$$

in terms of the $Y_m$ Bessel functions.

6. Find all eigenfunctions for the problem

$$\nabla\phi = \lambda\phi$$

defined on the domain in polar coordinates $r \leq a$, and with Neumann boundary conditions

$$\left.\frac{\partial\phi}{\partial r}\right|_{r=a} = 0$$

defined on the edge of the domain.

7. A large circular plate of radius $a$ is struck by a hammer, causing vibrations to be set up in the material of the plate. The displacement $\psi(r,\theta)$ of the plate from equilibrium is governed by the wave equation

$$\nabla^2\psi = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}$$

and Neumann boundary conditions $\frac{\partial\psi}{\partial r} = 0$ hold on the edge of the plate. The effect of the hammer is embodied in the initial condition

$$\psi(0,r,\theta) = 0 \quad , \frac{\partial\psi}{\partial t}(0,r,\theta) = e^{-r^2/b^2} \ .$$

Use your eigenfunctions from the previous question, solve this problem to find $\psi$ for all values of $t$. You can leave your answer as an infinite series of integrals.