1. a) In cylindrical coordinates, Laplace's equation is

$$\left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \right) \psi = 0$$

Substituting

$$\psi(r, \theta, z) = R(r)\,\Theta(\theta)\,Z(z)$$

we have

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial R}{\partial r} \right) \Theta Z + \frac{1}{r^2} R Z \frac{\partial^2 \Theta}{\partial \theta^2} + R \Theta \frac{\partial^2 Z}{\partial z^2} = 0$$

Divide through by $R\Theta Z$:

$$\frac{1}{R} \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial R}{\partial r} \right) + \frac{1}{r^2} \frac{1}{\Theta} \frac{\partial^2 \Theta}{\partial \theta^2} + \frac{1}{Z} \frac{\partial^2 Z}{\partial z^2} = 0 .$$

The Z problem separates:

$$\frac{1}{Z} \frac{\partial^2 Z}{\partial z^2} = -\lambda$$

So

$$\frac{1}{R} \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial R}{\partial r} \right) + \frac{1}{r^2} \frac{1}{\Theta} \frac{\partial^2 \Theta}{\partial \theta^2} = \lambda$$

Multiply by r^2:

$$\frac{1}{R} r \frac{\partial}{\partial r} \left(r \frac{\partial R}{\partial r} \right) - \lambda r^2 + \frac{1}{\Theta} \frac{\partial^2 \Theta}{\partial \theta^2} = 0$$

The R and Θ problems separate:

$$\frac{1}{R} r \frac{\partial}{\partial r} \left(r \frac{\partial R}{\partial r} \right) - \lambda r^2 = \mu$$

$$\frac{1}{\Theta} \frac{\partial^2 \Theta}{\partial \theta^2} = -\mu .$$

b) The solution to

$$-\frac{Z''}{Z} = \lambda$$

with $Z(0) = Z(\pi) = 0$ is

$$Z_m(z) = A\sin(\sqrt{\lambda_m} z)$$

$$\lambda_m = m^2$$

c) The solution to

$$-\frac{\Theta''}{\Theta} = \mu$$

with periodic boundary conditions is

$$\Theta_l(\theta) = e^{il\theta}$$

$$\mu_l = l^2 .$$

d) The problem for $R(r)$ is then

$$\frac{1}{R} r \frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) - \lambda r^2 = \mu$$

or

$$\frac{1}{R} r \frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) - m^2 r^2 = l^2$$

Re-arranging:

$$r\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) - (m^2r^2 + l^2)R = 0$$

or

$$r^2\frac{\partial^2 R}{\partial r^2} + r\frac{\partial R}{\partial r} - (m^2r^2 + l^2)R = 0$$

Now let $z = mr$. Then $r^2\frac{\partial^2 R}{\partial r^2} = z^2\frac{\partial^2 R}{\partial z^2}$, $r\frac{\partial R}{\partial r} = z\frac{\partial R}{\partial z}$ and we have

$$z^2\frac{\partial^2 R}{\partial z^2} + z\frac{\partial R}{\partial z} - (z^2 + l^2)R = 0.$$

The solution to this is

$$R(z) = I_l(z)$$

$$= I_l(mr).$$

The general solution is then

$$\psi(r,\theta,z) = \sum_l \sum_m A_{lm} I_l(mr) e^{il\theta} \sin(mz).$$

e) The remaining boundary condition is

$$\psi(a,\theta,z) = \sin(z)$$

So

$$\sin z = A_0 + \sum_{l=-\infty}^{\infty} \sum_{m=1}^{\infty} A_{lm} I_l(ma) e^{il\theta} \sin(mz)$$

Taking the inner product with $e^{-in\theta}$, we find

$$\sin z \int_0^{2\pi} e^{-in\theta}\, d\theta = A_0 \int_0^{2\pi} e^{in\theta}\, d\theta$$

$$+ \sum_{l=-\infty}^{\infty} \sum_{m=1}^{\infty} A_{lm} I_l(ma) \sin(mz) \underbrace{\int_0^{2\pi} e^{i(l-n)\theta}\, d\theta}_{=2\pi\delta_{l,n}}$$

Or

$$\sin z\, \cancel{2\pi}\, \delta_{n,0} = A_0\, \delta_{n,0}\, \cancel{2\pi} + \sum_{m=1}^{\infty} A_{nm} I_n(ma) \sin(mz)\, \cancel{2\pi}$$

Taking the inner product with $\sin(kz) = Z_k(z)$,

$$\delta_{n,0} \langle Z_k, Z_1 \rangle = A_0 \delta_{n,0} \underbrace{\langle Z_k, 1 \rangle}_{=0} + \sum_{m=1}^{\infty} A_{nm} I_n(ma) \underbrace{\langle Z_k, Z_m \rangle}_{\delta_{k,m}}$$

$$\delta_{n,0}\, \delta_{k,1} \|Z_1\|^2 = A_{nk} I_n(ka) \|Z_k\|^2$$

$$\Rightarrow A_{nk} = \frac{1}{I_n(ka)} \delta_{n,0} \delta_{k,1}$$

So $$A_{nk} = \begin{cases} \dfrac{1}{I_0(a)} & n=0,\ k=1 \\ 0 & \text{otherwise.} \end{cases}$$

Finally

$$\psi(r,\theta,z) = A_{01} I_0(r) e^{i0\theta} \sin z$$

$$= \frac{I_0(r)}{I_0(a)} \sin z .$$

2. We want to solve

$$(\nabla^2 + k^2)\psi = 0$$

In spherical coordinates this is

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \psi}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \psi}{\partial \varphi^2}$$

$$+ k^2 \psi = 0.$$

Use the separation Ansatz

$$\psi(r,\theta,\varphi) = R(r)\Theta(\theta)\Phi(\varphi)$$

Then

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial R}{\partial r}\right)\Theta\Phi + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \Theta}{\partial \theta}\right) R\Phi$$

$$+ \frac{1}{r^2 \sin^2\theta} R\Theta \frac{\partial^2 \Phi}{\partial \varphi^2} + k^2 R\Theta\Phi = 0.$$

Divide by $R\Theta\Phi$:

$$\frac{1}{R}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial R}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{1}{\Theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \Theta}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{1}{\Phi}\frac{\partial^2 \Phi}{\partial \varphi^2} + k^2 = 0$$

Multiply by $r^2 \sin^2\theta$:

$$\sin^2\theta \frac{1}{R} \frac{\partial}{\partial r}\left(r^2 \frac{\partial R}{\partial r}\right) + \frac{\sin\theta}{\Theta} \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial\Theta}{\partial\theta}\right) + \frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\varphi^2}$$

$$+ k^2 r^2 \sin^2\theta = 0.$$

The φ problem separates:

$$\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\varphi^2} = -\lambda$$

Φ is also periodic, so

$$\Phi_m(\varphi) = e^{im\varphi}$$

$$\lambda_m = m^2 .$$

Therefore we have

$$\sin^2\theta \frac{1}{R} \frac{\partial}{\partial r}\left(r^2 \frac{\partial R}{\partial r}\right) + \frac{\sin\theta}{\Theta} \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial\Theta}{\partial\theta}\right) - m^2$$

$$+ k^2 r^2 \sin^2\theta = 0.$$

Divide by $\sin^2\theta$:

$$\frac{1}{R} \frac{\partial}{\partial r}\left(r^2 \frac{\partial R}{\partial r}\right) + k^2 r^2 + \frac{1}{\Theta\sin\theta} \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial\Theta}{\partial\theta}\right) - \frac{m^2}{\sin^2\theta} = 0 .$$

So the R, Θ problems separate:

$$\frac{1}{\Theta\sin\theta} \frac{\partial}{\partial\theta}\left(\sin \frac{\partial\Theta}{\partial\theta}\right) - \frac{m^2}{\sin^2\theta} = -\mu$$

$$\frac{1}{R} \frac{\partial}{\partial r}\left(r^2 \frac{\partial R}{\partial r}\right) + k^2 r^2 = \mu .$$

The solution to the Θ equation is

$$\Theta(\theta) = P_l^m(\cos\theta)$$

with

$$\mu = l(l+1) .$$

The problem for R is then

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + k^2 r^2 = l(l+1).$$

Pick $l=0$. Then we have to solve

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) = -k^2 r^2 R.$$

Try $R = \frac{e^{\alpha r}}{r}$.

Then

$$R' = -\frac{1}{r^2}e^{\alpha r} + \frac{\alpha e^{\alpha r}}{r}.$$

$$r^2 R' = -e^{\alpha r} + \alpha e^{\alpha r} r$$

$$\frac{\partial}{\partial r}(r^2 R') = \cancel{-\alpha e^{\alpha r}} + \cancel{\alpha e^{\alpha r}} + r\alpha^2 e^{\alpha r}$$

So we have, from

$$\frac{\partial}{\partial r}(r^2 R') = -k^2 R$$

$$\Longrightarrow \quad r\alpha^2 e^{\alpha r} = -k^2 r^2 \frac{e^{-\alpha r}}{r}$$
$$= -k^2 r e^{-\alpha r}$$

equating like powers of r,

$$\alpha^2 = -k^2$$

$$\Longrightarrow \quad \alpha = \pm ik.$$

Therefore

$$R(r) = A\frac{e^{\pm ikr}}{r}.$$

and so for $l=0$, $m=0$

$$\psi(r,\theta,\varphi) = R(r)\,\Theta_{0,0}(\theta)\,\Phi_0(\varphi)$$

$$= A\frac{e^{\pm ikr}}{r}\,P_0^0(\theta)\,e^{i0}$$

$$= A\,\frac{e^{\pm ikr}}{r}.$$

The general solution is therefore

$$\psi(r,\theta,\varphi) = A\,\frac{e^{+ikr}}{r} + B\frac{e^{-ikr}}{r}.$$