Vector Calculus and PDEs 37336
Problem Set 11: PDEs in 3D and spherical coordinates

1. Consider Laplace's equation in 3D *cylindrical* coordinates (r, θ, z), defined on the domain $D = \{(r, \theta, z) : r \leq a, 0 \leq \theta \leq 2\pi, 0 \leq z \leq \pi\}$:
$$\nabla^2 \psi(r, \theta, z) = 0 \ ,$$
with the boundary conditions on the curved surface of the cylinder:
$$\psi(a, \theta, z) = \sin z \ ,$$
as well as boundary conditions at the two ends of the cylinder:
$$\psi(r, \theta, 0) = \psi(r, \theta, \pi) = 0 \ .$$

   a) Using the separation ansatz
$$\psi(r, \theta, z) = R(r)\Theta(\theta)Z(z) \ ,$$
separate the variables to find three ODEs for R,Θ and Z, with two separation constants.

   b) Use the boundary conditions at $z = 0$ and $z = \pi$ to solve the Sturm-Liouville problem for $Z(z)$.

   c) Use the fact that Θ is a periodic function to solve the Sturm-Liouville problem for Θ.

   d) The DE for R should be of the general form
$$z^2 \frac{d^2 y}{dz^2} + z \frac{dy}{dz} - (z^2 + m^2) y = 0 \ .$$
The non-singular solutions to this equation are *modified Bessel functions of the first kind*
$$y(z) = I_m(z) \ .$$
Use these special functions to write down the solution for R, and hence write the general solution to the PDE.

   e) Apply the boundary condition at $r = a$ to solve the the PDE in terms of the modified Bessel functions.

2. a) Separate the following PDE in spherical polar coordinates (r, θ, φ):
$$\nabla^2 \psi + k^2 \psi = 0 \ .$$

   b) In the particular case when the solution does not depend on θ and ϕ (i.e. when we have polar order $\ell = 0$ and azimuthal order $m = 0$), use the Ansatz $R(r) = e^{\alpha r}/r$ to find the general solution to the PDE.