Vector Calculus and PDEs 37336
Problem Set 2: Changing coordinates

1. Draw a set of axes in 2D and on it mark the following points, which are expressed in 2D polar coordinates (r, θ). Express each point in Cartesian coordinates.
   (a)
   $$(2, \pi/4)$$
   (b)
   $$(1, 0)$$
   (c)
   $$(1.5, 3\pi/2)$$

2. Draw a set of 2D axes. Draw a position vector $\mathbf{v}_1 = 2\hat{\mathbf{i}} + 0.5\hat{\mathbf{j}}$. What is this position vector in 2D polar coordinates? Now draw the vector $\mathbf{v}_2 = -2\hat{\mathbf{i}} + 0.5\hat{\mathbf{j}}$, and write this in polar coordinates. These two vectors are pointing in different directions - in polar coordinates, where is the direction of the vector contained?

3. a) In lectures we gave the formula for transforming a vector from polar coordinates to Cartesian coordinates in 2D:
   $$\begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} v_r \\ v_\theta \end{bmatrix} \quad .$$
   Use this to obtain the *inverse* transformation, that is, to convert a vector from Cartesian coordinates to polar coordinates.

   b)A 2D vector field has a value of $\mathbf{v} = \hat{\mathbf{j}}$ at the point (1,1). Draw this in 2D, and draw the $\hat{\mathbf{r}}$ and $\hat{\theta}$ components of this vector. Convert this vector to 2D polar coordinates.

4. Draw a set of 3D axes. A vector field has the value of $\mathbf{v} = 2\hat{\mathbf{r}}$ in spherical polar coordinates, at the Cartesian point (1,1,1). Convert this vector to Cartesian coordinates.

5. Use a tree diagram to write out the chain rule for
   $$u = f(x, y) \ \ , \quad \text{where } x = x(r, s, t) \ , \ y = y(r, s, t) \ .$$

6. Starting from the definition of polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$, show that the derivatives with respect to r and θ can be written in the form
   $$\begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{1}{r}\frac{\partial}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} \quad .$$

7. Using the vector calculus identity given in lectures (or from the internet), calculate the divergence of the vector field in spherical coordinates
   $$\mathbf{v} = \frac{1}{r}\mathbf{r} \quad ,$$
   where $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$, and $r = |\mathbf{r}|$.

8. A magnetic field in the interior of a conducting cylinder is given by
   $$\mathbf{B}(r, \theta, z) = \frac{J_0 r^2}{3R}\hat{\theta} + B_0\hat{\mathbf{z}}$$
   where J_0, R, and B_0 are all constants. Compute the curl of $\mathbf{B}$.