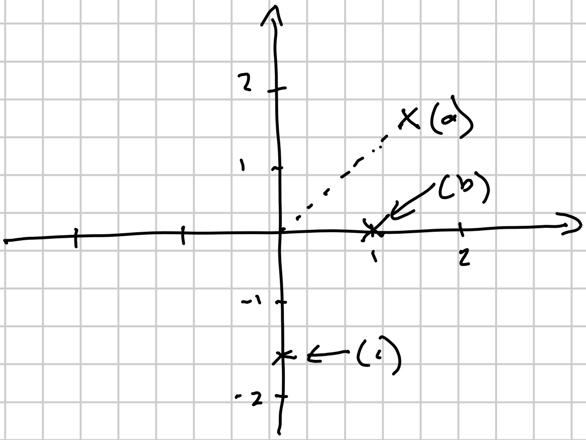
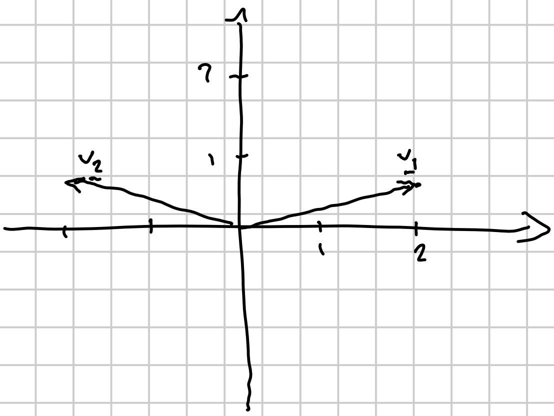
Problem Set 2 - Solutions

1.



2.



Both these vectors have the polar form

$$\underset{\sim}{v}_{1,2} = \sqrt{2^2 + \left(\tfrac{1}{2}\right)^2}\ \hat{\underset{\sim}{r}}$$

$$= \sqrt{4\tfrac{1}{4}}\ \hat{\underset{\sim}{r}}$$

The direction of these vectors is contained in the unit vector $\hat{\underset{\sim}{r}}$.

3. a) Starting from

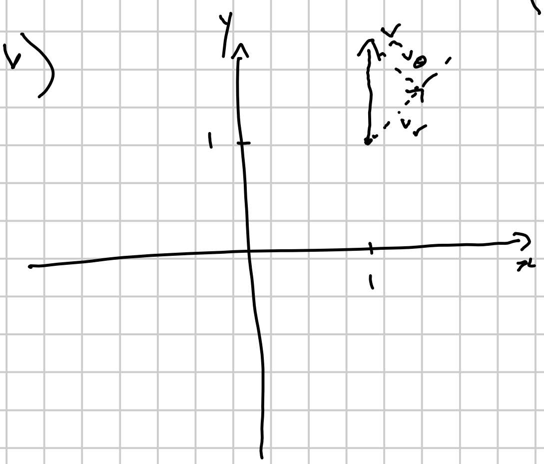
$$\begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} v_r \\ v_\theta \end{pmatrix}$$

we can invert the matrix:

$$\begin{pmatrix} v_r \\ v_\theta \end{pmatrix} = \frac{1}{\cos^2\theta + \sin^2\theta} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix}$$

b)



We know that

$$\begin{pmatrix} v_r \\ v_\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix}$$

$$= \begin{pmatrix} \cos\frac{\pi}{4} & \sin\frac{\pi}{4} \\ -\sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix}$$

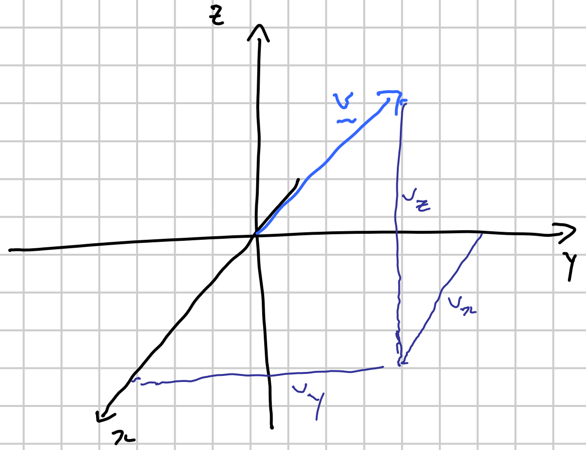
$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

So $v_r = \frac{1}{\sqrt{2}}$, $v_\theta = \frac{1}{\sqrt{2}}$

$$\Rightarrow \underset{\sim}{v} = \frac{1}{\sqrt{2}} \underset{\sim}{\hat{r}} + \frac{1}{\sqrt{2}} \underset{\sim}{\hat{\theta}}$$

4.



We know that

$$v_x = v_y = v_z$$

and also that

$$\sqrt{v_x^2 + v_y^2 + v_z^2} = 2$$

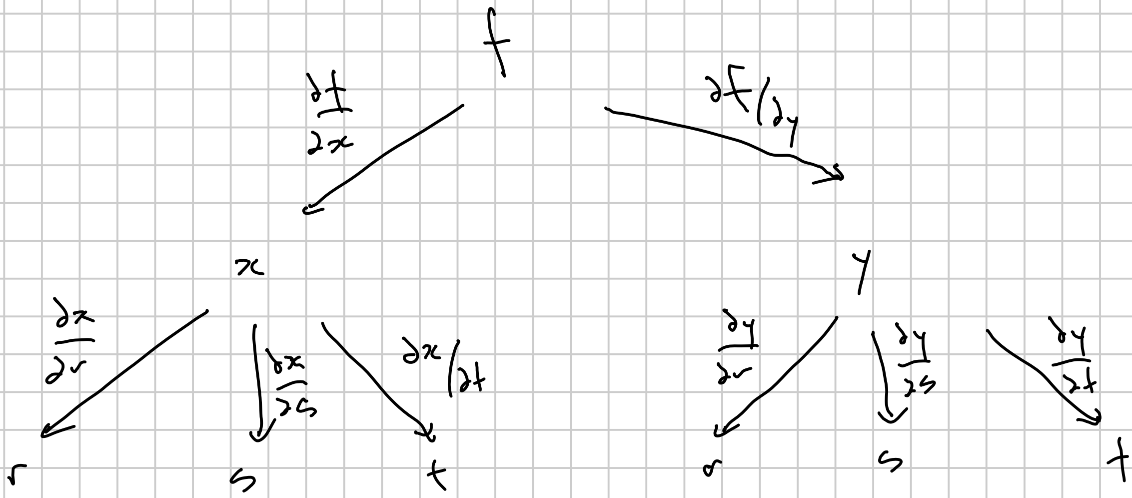
so $$\sqrt{3 v_x^2} = 2$$

$$\Rightarrow \quad v_x = \frac{2}{\sqrt{3}}$$

Therefore $$\underset{\sim}{v} = \frac{2}{\sqrt{3}} \langle 1, 1, 1 \rangle$$

in Cartesian coords.

5.



$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r}$$

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial x}\frac{\partial y}{\partial s}$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial x}\frac{\partial y}{\partial t}$$

6. We know that

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r}$$

Now $x = r\cos\theta$, $y = r\sin\theta$

So $\frac{\partial x}{\partial r} = \cos\theta$ $\frac{\partial y}{\partial r} = \sin\theta$

Also $\frac{\partial x}{\partial \theta} = -r\sin\theta$ $\frac{\partial y}{\partial \theta} = r\cos\theta$

Therefore
$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta$$

and
$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta}$$
$$= -r\sin\theta\frac{\partial f}{\partial x} + r\cos\theta\frac{\partial f}{\partial y}$$

So we have

$$\frac{\partial f}{\partial r} = \left(\cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}\right)f$$

$$\frac{\partial f}{\partial \theta} = r\left(-\sin\theta\frac{\partial}{\partial x} + \cos\theta\frac{\partial}{\partial y}\right)f$$

In matrix form

$$\begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{1}{r}\frac{\partial}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix}$$

7 We have in spherical coordinates

$$\underset{\sim}{v}(r, \theta, \varphi) = \frac{1}{r}\hat{\underset{\sim}{r}}$$

The divergence is given by

$$\nabla \cdot \underset{\sim}{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(v_\theta \sin\theta\right)$$
$$+ \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}\left(v_\varphi\right)$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right)$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{1}{r}\right)$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r\right) = \frac{1}{r^2}$$

8. We have

$$\underset{\sim}{B} = \frac{J_0 r^2}{3R}\hat{\underset{\sim}{\theta}} + B_0\hat{\underset{\sim}{z}}.$$

The curl in cylindrical coords is

$$\nabla \times \underset{\sim}{B} = \left(\frac{1}{r}\frac{\partial B_z}{\partial \theta} - \frac{\partial B_\theta}{\partial z}\right)\hat{\underset{\sim}{r}}$$
$$+ \left(\frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r}\right)\hat{\underset{\sim}{\theta}}$$
$$+ \frac{1}{r}\left(\frac{\partial}{\partial r}\left(r B_\theta\right) - \frac{\partial B_r}{\partial \theta}\right)\hat{\underset{\sim}{z}}$$

We have $B_r = 0$ and $B_z = \text{constant}$,

also B_θ only depends on r, so

$$\nabla \times B = \frac{1}{r}\left(\frac{\partial}{\partial r}\left(r B_\theta\right)\right)\hat{\underline{z}}$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(r \cdot \frac{J_0 r^2}{3R}\right)\hat{\underline{z}}$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{J_0}{3R} r^3\right)\hat{\underline{z}}$$

$$= \frac{1}{r}\frac{J_0}{3R} 3r^2 \hat{\underline{z}}$$

$$= \frac{J_0 r}{3R}\hat{\underline{z}}$$