Vector Calculus and PDEs 37336
Problem Set 3: Line and path integrals

1. A path $\mathcal{C}$ is defined as a line beginning at the point $\langle 1, 1, 0\rangle$ and ending at the point $\langle 0, 1, 1\rangle$.
   (a) Find a vector parallel to the path $\mathcal{C}$.
   (b) Find a *parametric representation* for the path $\mathcal{C}$ (i.e. find the position coordinate $\mathbf{r}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} + z(t)\hat{\mathbf{k}}$ in terms of some parameter t).

2. A path $\mathcal{C}$ in 3D space is parameterised by the equations

$$\begin{aligned} x &= t \\ y &= t^2 \\ z &= 1 \ , \end{aligned}$$

   where t starts at 0 and ends at 1. Evaluate the integral

$$\int_{\mathcal{C}} \sqrt{z + 4x^2} ds \ .$$

3. A path in 3D space consists of a circle parallel to the $x - z$ plane, centred at the point $\langle 0, 1, 0\rangle$ with radius $a = 2$. Sketch the path. Find a parametric representation for the path.

4. Evaluate the line integral

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$$

   where $\mathbf{F}(x, y, z) = yz\hat{\mathbf{i}} + xz\hat{\mathbf{j}} + xy\hat{\mathbf{k}}$ and $\mathcal{C}$ is the path parametrized by $\mathbf{r}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} + z(t)\hat{\mathbf{k}}$, where

$$\begin{aligned} x(t) &= 2 - t \\ y(t) &= t + 1 \\ z(t) &= 2t \end{aligned}$$

   with $0 \leq t \leq 2$.

5. Given $\mathbf{F} = \nabla\phi$, where $\phi = x^2 y + x \sin y$, evaluate

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

   where C is any path from $(1, 0)$ to $(2, \pi/2)$.

6. Evaluate $\int \mathbf{F} \cdot d\mathbf{r}$ when $\mathbf{F} = xy\hat{\mathbf{i}} + (x - y)\hat{\mathbf{j}}$ and $\mathcal{C}$ is the arc of the circle $x^2 + y^2 = 4$ lying in the plane $z = 1$, traversed counter-clockwise from $(2, 0, 1)$ to $(0, 2, 1)$.

7. Determine whether or not

$$\mathbf{F} = (x^3 + 4xy)\hat{\mathbf{i}} + (y^3 + 4xy)\hat{\mathbf{j}}$$

   is a conservative vector field; give reasons for your answer.

8. Show that

$$\mathbf{F} = \sin y\hat{\mathbf{i}} + x \cos y\hat{\mathbf{j}} - sinz\hat{\mathbf{k}}$$

   is a conservative vector field. Find the potential function for $\mathbf{F}$.