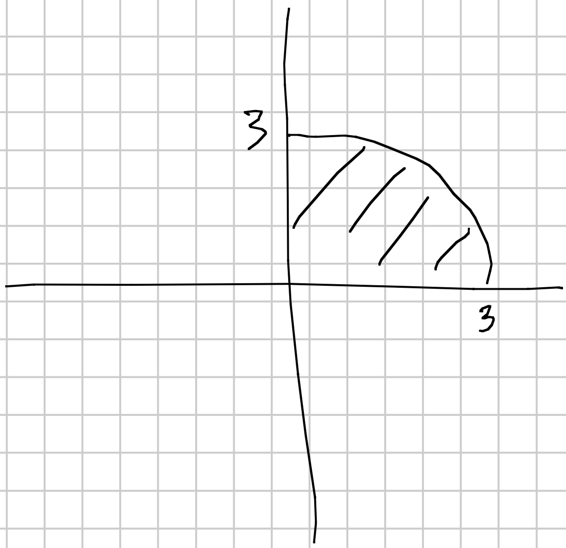
1. Let $x = r\cos\theta$

$y = r\sin\theta$



with $0 \le r \le 3$

$0 \le \theta \le \frac{\pi}{2}$.

then $dA = r\,dr\,d\theta$

and

$$\iint_S \sqrt{x^2+y^2}\,dA = \int_0^3 \int_0^{\frac{\pi}{2}} \sqrt{r^2\cos^2\theta + r^2\sin^2\theta}\; r\,d\theta\,dr$$

$$= \int_0^3 \int_0^{\frac{\pi}{2}} r \cdot r\,d\theta\,dr$$

$$= \frac{\pi}{2}\left[\frac{r^3}{3}\right]_0^3 = \frac{\pi}{2}\cdot\frac{27}{3}$$

$$= \frac{9\pi}{2}$$

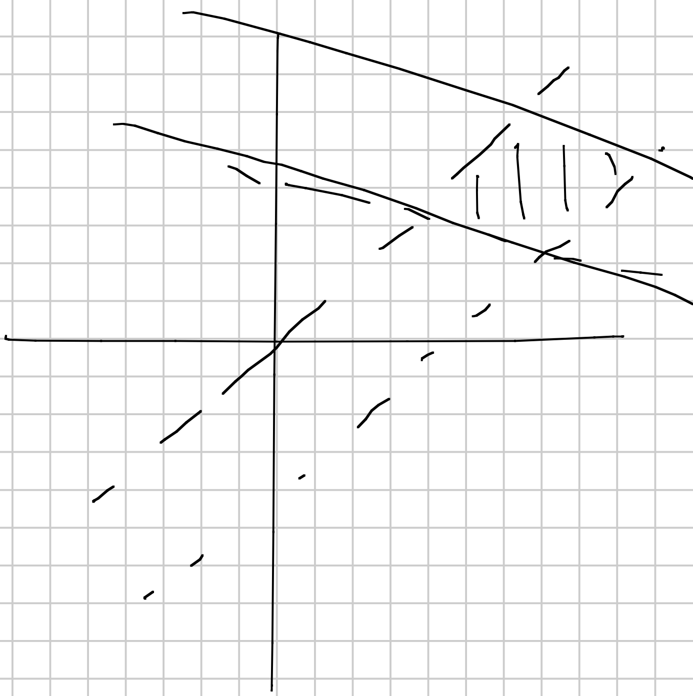
2.

$$\left|\frac{\partial(u,v)}{\partial(x,y)}\right| = \frac{1}{\left|\frac{\partial(x,y)}{\partial(u,v)}\right|}$$

$$= \frac{1}{\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}} = \frac{1}{\begin{vmatrix} \cos v & -u\sin v \\ \sin v & u\cos v \end{vmatrix}}$$

$$= \frac{1}{u\cos^2 v + u\sin^2 v} = \frac{1}{u}$$

3. The limits of the domain are:



$y = x$ $\qquad\qquad$ $v = 0$

$y = x - \frac{\pi}{2}$ $\Rightarrow$ $v = \frac{\pi}{2}$

$x + 2y = 1$ $\qquad\qquad$ $u = 1$

$x + 2y = 2$ $\qquad\qquad$ $u = 2$

Using the Jacobian,

$$dA = dx\,dy = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} du\,dv$$

$$= \frac{1}{\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}} du\,dv$$

$$= \frac{1}{\begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix}} du\,dv = \frac{1}{-3} du\,dv$$

So

$$\iint_R \frac{\cos(x-y)}{x+2y}\,dx\,dy = \int_0^{\frac{\pi}{2}}\int_1^2 \frac{\cos v}{u}\left(-\frac{1}{3}\right)du\,dv$$

$$= -\frac{1}{3}\int_0^{\frac{\pi}{2}} \cos v \left[\ln u\right]_1^2 dv$$

$$= -\frac{1}{3}\left(\ln 2 - \ln 1\right)\int_0^{\frac{\pi}{2}} \cos v\,dv$$

$$= -\frac{1}{3}\ln 2\left[\sin v\right]_0^{\frac{\pi}{2}} = -\frac{1}{3}\ln 2.$$

4. The centre of the circle is located at $(3,-1)$, so $\bar{x} = 3$.

Now,

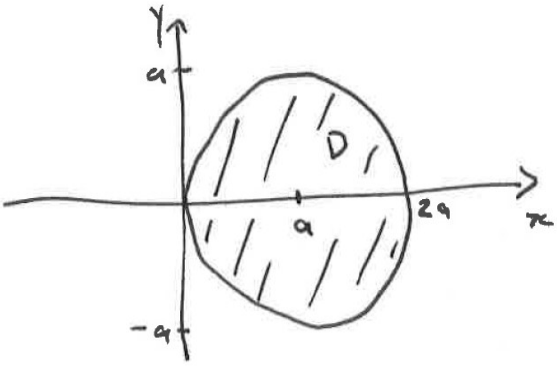
$$\bar{x} = \frac{1}{A}\iint_R x\,dx\,dy$$

So

$$\iint_R x\,dx\,dy = A\bar{x} = 3 \times \text{Area}$$
$$= 3 \times \pi \times 9 = 27\pi$$

5.

a)




$x = a + r\cos\theta$

$y = r\sin\theta$

b) The area is

$$\text{Area} = \iint_D 1 \, dx\,dy$$

$$= \int_0^{2\pi}\int_0^a 1 \cdot r \, dr\,d\theta$$

$$= \int_0^{2\pi} \left[\frac{r^2}{2}\right]_0^a d\theta$$

$$= \int_0^{2\pi} \frac{a^2}{2} \, d\theta = \frac{2\pi}{2} a^2$$

$$= \pi a^2.$$

6. The mass is


$$M = \iiint_S g(x,y,z)\, dV$$

$$= \iiint_S \left[a(x^2+y^2+z^2)+b\right] dx\,dy\,dz$$

In spherical coordinates

$$x = \rho \cos\theta \sin\phi$$

$$y = \rho \sin\theta \sin\phi$$

$$z = \rho \cos\phi$$

with volume element $dV = \rho^2 \sin\phi \, d\rho\, d\theta\, d\phi$.

Therefore

$$M = \int_0^R \int_0^\pi \int_0^{2\pi} (a\rho^2+b)\rho^2 \, d\theta \sin\phi \, d\phi \, d\rho$$

because $\rho^2 = x^2+y^2+z^2$.

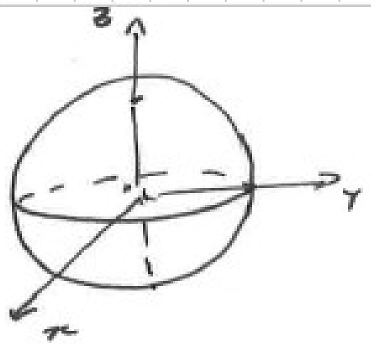
$$= \int_0^R \int_0^\pi \rho^2(a\rho^2+b)\cdot 2\pi \sin\phi \, d\phi \, d\rho$$

$$= \int_0^R 2\pi\, \rho^2(a\rho^2+b)\, d\rho \times \int_0^\pi \sin\phi \, d\phi$$

$$= 2\pi \int_0^R a\rho^4 + \rho^2 b \, d\rho \times \left[-\cos\phi\right]_0^\pi$$

$$= 4\pi \left[\frac{a}{5}\rho^5 + \frac{b}{3}\rho^3\right]_0^R = 4\pi\left(\frac{a}{5}R^5 + \frac{b}{3}R^3\right)$$

7.




We can write the density as

$$g(x,y,z) = kz + b \quad \leftarrow \text{general linear relationship.}$$

when $z = -R$, $g = 1.0$, and
when $z = +R$, $g = 1.2$, so

$$1.0 = -kR + b$$
$$1.2 = +kR + b$$
$$2.2 = 2b \implies b = 1.1$$
$$0.2 = 2kR \implies k = \frac{0.2}{2R} = \frac{0.2}{20} = 0.0$$

(It's less "fiddly" to leave $k$ and $b$ as they are for the moment - we can substitute in later.)

The total mass is

$$M = \iiint_S g(x,y,z)\, dx\,dy\,dz$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{R} (kz+b)\,\rho^2 \sin\phi \; d\rho\, d\phi\, d\theta .$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{R} (k\rho\cos\phi + b)\,\rho^2 \sin\phi \; d\rho\, d\phi\, d\theta .$$

$$= \int_0^{2\pi} \int_0^{\pi} \left[ \frac{k\rho^4}{4} \cos\phi \sin\phi + b\frac{\rho^3}{3} \sin\phi \right]_0^R d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi} \left[ \frac{kR^4}{4} \cos\phi \sin\phi + b\frac{R^3}{3} \sin\phi \right] d\phi \, d\theta$$

$$= \int_0^{2\pi} \left[ \frac{kR^4}{4} \frac{1}{2} \sin^2\phi + b\frac{R^3}{3} (-\cos\phi) \right]_0^{\pi} d\theta$$

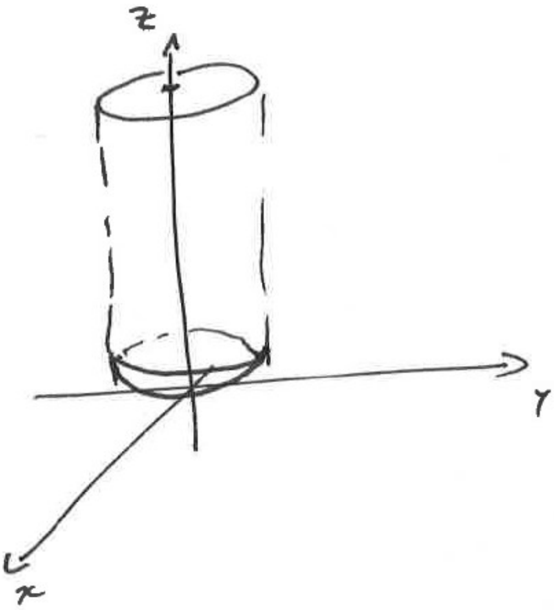
$$= 2\pi \times \left[ \cancel{\frac{kR^4}{4} \frac{1}{2} \sin^2 \pi} + \frac{bR^3}{3} (-\cos\pi) - \cancel{\frac{kR^4}{4} \frac{1}{2} \sin^2 0} - \frac{bR^3}{3} (-\cos 0) \right]$$

$$= 2\pi \times \frac{bR^3}{3} (1 + 1) = \frac{4\pi b}{3} R^3$$

$$= \frac{4\pi}{3} R^3 \times 1.1$$

$$= \frac{4\pi}{3} \times 10^3 \times 1.1 \approx 4610 \text{ kg}.$$

8.




Use cylindrical coordinates:

$$x = r\cos\theta$$

$$y = r\sin\theta \qquad\qquad dV = r\,dr\,d\theta\,dz.$$

$$z = z.$$

The total mass is

$$M = \iiint_V \rho(x,y,z)\,dx\,dy\,dz$$

$$= \iiint_V kr \cdot r\,dr\,d\theta\,dz.$$

$$= \int_0^1 \int_0^{2\pi} \int_{1-r^2}^{20} kr^2 \;dz\,d\theta\,dr$$

$$= \int_0^1 \int_0^{2\pi} \Big[kr^2 z\Big]_{1-r^2}^{20} d\theta\,dr$$

$$= \int_0^1 \int_0^{2\pi} kr^2\left(20 - (1-r^2)\right) d\theta\, dr$$

$$= \int_0^1 \int_0^{2\pi} 20kr^2 - kr^2(1-r^2)\ d\theta\, dr$$

$$= 2\pi \int_0^1 20kr^2 - kr^2 + kr^4\ dr$$

$$= 2\pi \int_0^1 \left(19kr^2 + kr^4\right) dr$$

$$= 2\pi \left[\frac{19}{3}kr^3 + \frac{k}{5}r^5\right]_0^1$$

$$= 2\pi\left(\frac{19}{3}k + \frac{k}{5}\right) = 2\pi k\left(\frac{19}{3} + \frac{1}{5}\right).$$

Then the centre of mass is

$$\bar{z} = \frac{1}{M}\iiint_E zkr\ \ r\,dr\,d\theta\,dz.$$

$$= \frac{1}{M}\int_0^{2\pi}\int_0^1\int_{1-r^2}^{20} kr^2 z\ dz\,dr\,d\theta$$

$$= \frac{2\pi}{M}\int_0^1\int_{1-r^2}^{20} kr^2 z\ dz\,dr$$

$$= \frac{2\pi}{M}\int_0^1 \left[kr^2\frac{z^2}{2}\right]_{1-r^2}^{20} dr$$

$$= \frac{2\pi}{M} \int_0^1 kr^2 \left( 200 - \frac{1}{2}(1-r^2)^2 \right) dr$$

$$= \frac{2\pi k}{M} \int_0^1 200r^2 - r^2(1 + r^4 - 2r^2)\, dr$$

$$= \frac{2\pi k}{M} \int_0^1 200r^2 - r^2 - r^6 + 2r^4 \; dr$$

$$= \frac{2\pi k}{M} \left[ \frac{200}{3} r^3 - \frac{r^3}{3} - \frac{r^7}{7} + \frac{2}{5} r^5 \right]_0^1$$

$$= \frac{2\pi k}{M} \left( \frac{199}{3} - \frac{1}{7} + \frac{2}{5} \right)$$

so

$$\bar{z} = \frac{\frac{199}{3} - \frac{1}{7} + \frac{2}{5}}{\frac{19}{3} + \frac{1}{5}} \approx 10.2\,m.$$