Vector Calculus and PDEs 37336
Problem Set 4: Volume and double integrals

1. Use polar coordinates to evaluate the integral

$$\int\int_R \sqrt{x^2+y^2}dA$$

where R is the region in the first quadrant enclosed by the circle $x^2+y^2=9$.

2. For the change of coordinates $x = u\cos v$, $y = u\sin v$, calculate the determinant of the Jacobian matrix $\left|\frac{\partial(u,v)}{\partial(x,y)}\right|$

3. Use the transformation $u = x+2y$, $v = x-y$, to evaluate

$$\iint_R \frac{\cos(x-y)}{x+2y}dA$$

where R is the region bound by the lines $y=x$, $y=x-\pi/2$, $x+2y=1$, and $x+2y=2$.

4. Use a geometric argument (or otherwise) to evaluate

$$\iint xdA$$

where R is the region bounded by the circle $(x-3)^2+(y+1)^2=9$.

5. a) Transform the expression $(x-a)^2+y^2=a^2$ into polar coordinates centred at $(a,0)$ and sketch the region bound by the curve.

b) Use a double integral in polar coordinates to find the area of the region.

6. A sphere has radius R m and has an interior density

$$g(x,y,z) = a(x^2+y^2+z^2)+b \ \ \text{kg/m}^3$$

The mass M of the sphere is the integral of the density over the sphere's volume. Compute M.

7. The density of air in a hot air balloon changes linearly from 1.0 kg/m^3 at the base to 1.2 kg/m^3 at the top. For a spherical balloon of total radius 10m, compute the total mass of air in the balloon.

8. * The fuel section for a solid booster engine consists of a solid lying within a cylinder of radius 1 m, lying below the plane z = 20 and above the paraboloid $z = 1-x^2-y^2$. The density $g(x,y,z)$ is proportional to the distance from the axis of the cylinder. Find the centre of mass, which is given by the formula

$$\overline{z} = \frac{1}{M}\iiint_E zg(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

where M is the total mass of the engine and E is the region in space that it occupies.