1. A vector normal to the surface is

$$\underline{n} = \nabla\left(z - g(x,y)\right)$$

$$= \nabla\left(z - (x + y^2)\right)$$

$$= \langle -1, -2y, 1 \rangle$$

Alternatively, write the surface as

$$\underset{\sim}{r} = \langle x, y, x + y^2 \rangle$$

then

$$\frac{\partial \underline{r}}{\partial x} = \langle 1, 0, 1 \rangle, \quad \frac{\partial \underline{r}}{\partial y} = \langle 0, 1, 2y \rangle$$

and a normal to the surface is

$$\underset{\sim}{n} = \frac{\partial \underline{r}}{\partial x} \times \frac{\partial \underline{r}}{\partial y} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 1 & 0 & 1 \\ 0 & 1 & 2y \end{vmatrix}$$

$$= \underset{\sim}{i}(0 - 1) - \underset{\sim}{j}(2y - 0) + \underline{k}(1 - 0)$$

$$= \langle -1, -2y, 1 \rangle$$

$$= \langle -1, 4, 1 \rangle$$

at the point $\langle 1, 2, 5 \rangle$.

$\underline{2.}$ $$dS = \frac{\partial \underset{\sim}{r}}{\partial u} \times \frac{\partial \underset{\sim}{r}}{\partial v} \, du \, dv$$

Now

$$\underset{\sim}{r} = \langle x, y, z \rangle$$

$$= \langle uv, u+v, u-v \rangle$$

So $$\frac{\partial \underset{\sim}{r}}{\partial u} = \langle v, 1, 1 \rangle$$

$$\frac{\partial \underset{\sim}{r}}{\partial v} = \langle u, 1, -1 \rangle$$

$$\frac{\partial \underset{\sim}{r}}{\partial u} \times \frac{\partial \underset{\sim}{r}}{\partial v} = \begin{vmatrix} \hat{\underset{\sim}{i}} & \hat{\underset{\sim}{j}} & \hat{\underset{\sim}{k}} \\ v & 1 & 1 \\ u & 1 & -1 \end{vmatrix}$$

$$= \hat{\underset{\sim}{i}}(-1-1) - \hat{\underset{\sim}{j}}(-v-u) + \hat{\underset{\sim}{k}}(v-u)$$

$$= -2\hat{\underset{\sim}{i}} + (u+v)\hat{\underset{\sim}{j}} + (v-u)\hat{\underset{\sim}{k}}$$

$$= \langle -2, u+v, v-u \rangle$$

So

$$d\underset{\sim}{S} = \langle -2, u+v, v-u \rangle \, du \, dv$$

$$dS = |d\underset{\sim}{S}|$$

$$= \sqrt{4 + (u+v)^2 + (v-u)^2}\, du\,dv$$

$$= \sqrt{4 + u^2 + 2uv + v^2 + v^2 + u^2 - 2uv}\, du\,dv$$

$$= \sqrt{4 + 2u^2 + 2v^2}\, du\,dv$$

3. First write the surface parametrically: Let $u = x$, $v = y$.

Then

$$\underset{\sim}{r} = \langle x, y, z \rangle$$

$$= \langle u, v, 1 - u - 2v \rangle$$

The surface is

$$z = 1 - x - 2y$$

$$x \geq 0, \quad y \geq 0, \quad z \geq 0.$$

Now,

$$z \geq 0 \implies 1 - x - 2y \geq 0$$

$$\implies 1 - 2y \geq x$$

So

$$0 \leq x \leq 1 - 2y.$$

Also,

$$\underline{dS} = \frac{\partial \underline{r}}{\partial u} \times \frac{\partial \underline{r}}{\partial v}\, du\,dv$$

$$= \begin{vmatrix} \hat{\underline{i}} & \hat{\underline{j}} & \hat{\underline{k}} \\ 1 & 0 & -1 \\ 0 & 1 & -2 \end{vmatrix} du\,dv$$

$$= \left[\hat{\underline{i}}(0+1) - \hat{\underline{j}}(-2-0) + \hat{\underline{k}}(1-0) \right] du\,dv$$

$$= \langle 1, 2, 1 \rangle\, du\,dv$$

$$dS = |\underline{dS}| = \sqrt{1^2 + 2^2 + 1^2}\, du\,dv$$

$$= \sqrt{6}\, du\,dv.$$

The integral is then

$$\iint_S yz\, dS = \sqrt{6} \int_0^{\frac{1}{2}} \int_0^{1-2v} v(1-u-2v)\, du\,dv$$

$$= \sqrt{6} \int_0^{\frac{1}{2}} \int_0^{1-2v} \left[v(1-2v) - uv \right] du\,dv$$

$$= \sqrt{6} \int_0^{\frac{1}{2}} \left[uv(1-2v) - \frac{1}{2}u^2 v \right]_0^{1-2v} dv$$

$$= \sqrt{6} \int_0^{\frac{1}{2}} \left[v(1-2v)^2 - \frac{1}{2}v(1-2v)^2 \right] dv$$

$$= \sqrt{6} \int_0^{\frac{1}{2}} \frac{1}{2} v (1-2v)^2 \, dv = \frac{\sqrt{6}}{2} \int_0^{\frac{1}{2}} v(1+4v^2-4v) \, dv$$

$$= \frac{\sqrt{6}}{2} \int_0^{\frac{1}{2}} (v + 4v^3 - 4v^2) \, dv$$

$$= \frac{\sqrt{6}}{2} \left[\frac{1}{2}v^2 + v^4 - \frac{4}{3}v^3 \right]_0^{\frac{1}{2}}$$

$$= \frac{\sqrt{6}}{2} \left[\frac{1}{8} + \frac{1}{16} - \frac{1}{6} \right] = \frac{\sqrt{6}}{2} \left[\frac{6+3-8}{48} \right]$$

$$= \frac{\sqrt{6}}{96}$$

4. The area element is

$$dS = |d\underset{\sim}{S}|$$

$$= \left| \frac{\partial \underset{\sim}{r}}{\partial u} \times \frac{\partial \underset{\sim}{r}}{\partial v} \right| du\,dv$$

$$= \left| \begin{matrix} \hat{\underset{\sim}{i}} & \hat{\underset{\sim}{j}} & \hat{\underset{\sim}{k}} \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & 1 \end{matrix} \right| du\,dv$$

$$= \left| \hat{\underset{\sim}{i}} \sin v - \hat{\underset{\sim}{j}} \cos v + \hat{\underset{\sim}{k}} u \right| du\,dv$$

$$= \sqrt{\sin^2 v + \cos^2 v + u^2}\, du\,dv$$

$$= \sqrt{1+u^2}\, du\,dv$$

So

$$\text{Area} = \iint_S dS = \int_0^1 \int_0^\pi \sqrt{1+u^2}\, dv\,du$$

$$= \int_0^1 \left[v\sqrt{1+u^2} \right]_0^\pi du$$

$$= \pi \int_0^1 \sqrt{1+u^2}\, du \,.$$

This is a pretty hard integral.
We can do it by integrating by parts:

Let $$I = \int \sqrt{1+u^2}\, du .$$

$$= u\sqrt{1+u^2} - \int u \cdot \frac{\frac{1}{2} u \cdot 2}{\sqrt{1+u^2}}\, du$$

$$= u\sqrt{1+u^2} - \int \frac{u^2}{\sqrt{1+u^2}}\, du$$

$$= u\sqrt{1+u^2} - \int \left(\frac{1+u^2}{\sqrt{1+u^2}} - \frac{1}{\sqrt{1+u^2}} \right) du$$

$$= u\sqrt{1+u^2} - \int \sqrt{1+u^2}\, du + \int \frac{1}{\sqrt{1+u^2}}\, du$$

$$= u\sqrt{1+u^2} - I + \sin^{-1}(u)$$

So

$$2I = u\sqrt{1+u^2} + \sin^{-1}(u)$$

$$\Rightarrow \quad I = \frac{1}{2} u\sqrt{1+u^2} + \frac{1}{2}\sin^{-1}(u).$$

Then

$$A = \pi \int_0^1 \sqrt{1+u^2}\, du$$

$$= \pi \left[\frac{1}{2} u \sqrt{1+u^2} + \frac{1}{2} \sin^{-1}(u) \right]_0^1$$

$$= \pi\sqrt{2} + \frac{\pi}{2}.$$

5. The position vector on the surface is (using x, y as parameters)

$$\underset{\sim}{r}(x,y) = \langle x, y, 2-y \rangle$$

with $\quad 0 \le x \le 2, \quad 0 \le y \le 2.$

The infinitesimal area element is

$$d\underset{\sim}{S} = \frac{\partial \underset{\sim}{r}}{\partial x} \times \frac{\partial \underset{\sim}{r}}{\partial y}\, dx\,dy$$

$$= \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{vmatrix} dx\,dy$$

$$= \left[\hat{\underset{\sim}{i}}(0) - \hat{\underset{\sim}{j}}(-1-0) + \underset{\sim}{k}(1-0) \right] dx\,dy$$

$$= (\hat{\underset{\sim}{j}} + \underset{\sim}{k})\, dx\,dy = \langle 0, 1, 1 \rangle\, dx\,dy$$

The flux integral is then

$$\iint_S \underset{\sim}{F} \cdot d\underset{\sim}{S} = \int_0^2 \int_0^2 \langle x, 0, -(2-y)^2 \rangle \cdot \langle 0, 1, 1 \rangle\, dx\,dy$$

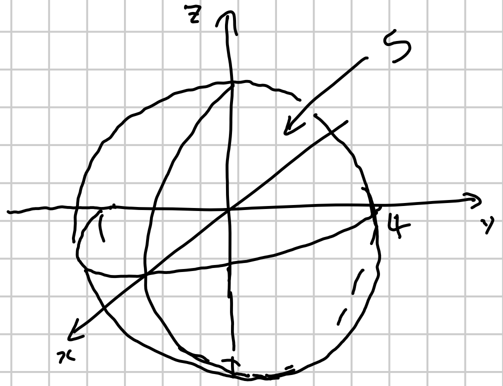
$$= \int_0^2 \int_0^2 -(2-y)^2 \, dx\,dy$$

$$= -2 \int_0^2 (2-y)^2 \, dy$$

$$= -2 \left[-\frac{1}{3}(2-y)^3 \right]_0^2$$

$$= -2 \times \left(0 + \frac{1}{3} \cdot 2^3 \right) = \frac{16}{3}.$$

6.



The surface area element in spherical coordinates is

$$\underset{\sim}{ds} = \rho^2 \sin\theta \, d\varphi \, d\theta \, \hat{\rho},$$

and S is defined by

$$\rho = 4 \qquad \text{radius}$$
$$0 \le \theta \le \pi \qquad \text{polar angle}$$
$$-\frac{\pi}{2} \le \varphi \le \frac{\pi}{2} \qquad \text{azimuthal angle}$$

So

$$\iint_S \underset{\sim}{F} \cdot \underset{\sim}{ds} = \int_0^{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{k}{\rho^2} \hat{\rho} \right) \cdot \left(\rho^2 \sin\theta \, d\varphi \, d\theta \, \hat{\rho} \right)$$

$$= \int_0^{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{k}{\cancel{\rho^2}} \cancel{\rho^2} \sin\theta \, d\varphi \, d\theta$$

$$= k \int_0^{\pi} \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin\theta \, d\varphi \right) d\theta$$

$$= k \int_0^{\pi} \sin\theta \left[\varphi \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta$$

$$= k \times \left(\frac{\pi}{2} + \frac{\pi}{2} \right) \times \int_0^{\pi} \sin\theta \, d\theta$$

$$= k\pi \times \left[-\cos\theta \right]_0^{\pi} = 2\pi k.$$