Vector Calculus and PDEs 37336
Problem Set 6: Integral theorems

1. (a) Evaluate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$ where
$$\mathbf{F} = -y^2\hat{\mathbf{i}} + xy\hat{\mathbf{j}} + z\hat{\mathbf{k}}$$
and S is the part of the paraboloid $x^2+y^2+z = 4$ that lies above xy plane, and that is upward-oriented.

   (b) State the divergence theorem.

   (c)Verify that the divergence theorem is true for $\mathbf{F}$ and for the region bound by the surface S.

2. Use the divergence theorem to calculate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where
$$\mathbf{F} = 3xy^2\hat{\mathbf{i}} + xe^z\hat{\mathbf{j}} + z^3\hat{\mathbf{k}}$$
and S is the outward-oriented surface of the solid bound by the cylinder $z^2+y^2 = 1$ and the planes $x = -1$ and $x = 2$.

3. Use Stokes theorem to evaluate
$$\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} \ ,$$
where $\mathbf{F} = x^2e^{yz}\hat{\mathbf{i}} + y^2e^{xz}\hat{\mathbf{j}} + z^2e^{xy}\hat{\mathbf{k}}$, and S is the hemisphere $x^2+y^2+z^2 = 4$, with $z \geq 0$, oriented upward. What would be the value of this integral if S is instead the the upward-oriented paraboloid $x^2 + y^2 + z = 4$, with $z \geq 0$?

4. Use the divergence theorem to evaluate the flux integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where
$$\mathbf{F}(x, y, z) = x\cos^2 z\hat{\mathbf{i}} + y\sin^2 z\hat{\mathbf{j}} + \sin x \cos x\hat{\mathbf{k}}$$
and S is the sphere centred at the origin with radius R.

5. Use Stokes' theorem to evaluate $\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S}$, where
$$\mathbf{F}(x, y, z) = -y\hat{\mathbf{i}} + x\hat{\mathbf{j}} + 2z\hat{\mathbf{k}}$$
and S is the outward-oriented part of the sphere $x^2 + y^2 + (z + 3)^2 = 25$ with $z \geq 0$.