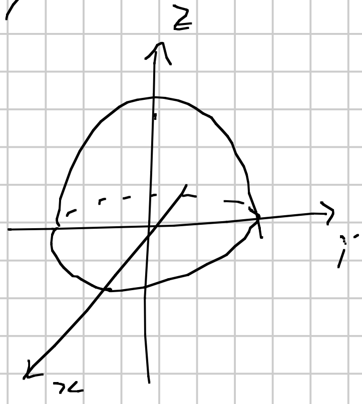
1. a) Parametize the curve:

$$\underset{\sim}{r} = \langle x, y, 4 - x^2 - y^2 \rangle$$



$$\frac{\partial \underset{\sim}{r}}{\partial x} = \langle 1, 0, -2x \rangle$$

$$\frac{\partial \underset{\sim}{r}}{\partial y} = \langle 0, 1, -2y \rangle$$

So

$$d\underset{\sim}{S} = \frac{\partial \underset{\sim}{r}}{\partial x} \times \frac{\partial \underset{\sim}{r}}{\partial y} \, dx\,dy$$

$$= \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 1 & 0 & -2x \\ 0 & 1 & -2y \end{vmatrix} dx\,dy$$

$$= \left[\hat{i}(2x) - \hat{j}(-2y) + \hat{k}(1) \right] dx\,dy$$

$$= \langle 2x, 2y, 1 \rangle \, dx\,dy.$$

Now

$$\underset{\sim}{F} = \langle -y^2, xy, z \rangle$$

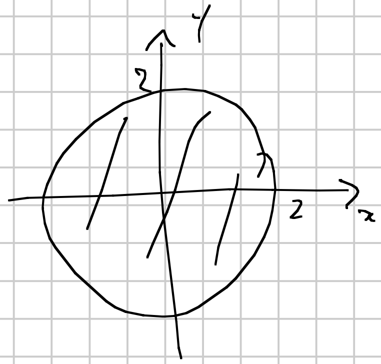
$$= \langle -y^2, xy, 4 - x^2 - y^2 \rangle$$

So

$$\underline{F} \cdot d\underline{S} = \langle -y^2, xy, 4 - x^2 - y^2 \rangle \cdot \langle 2x, 2y, 1 \rangle \, dx\,dy$$

$$= \left(-2xy^2 + 2xy^2 + 4 - x^2 - y^2 \right) dx\,dy$$

$$= \left(4 - x^2 - y^2 \right) dx\,dy$$



The region to be integrated over is a circle, so use polar coordinates:

$$\iint_S \underline{F} \cdot d\underline{S} = \int_0^2 \int_0^{2\pi} \left(4 - x^2 - y^2 \right) r \, d\theta \, dr$$

$$= \int_0^2 \int_0^{2\pi} \left(4 - r^2 \right) r \, d\theta \, dr$$

$$= 2\pi \int_0^2 \left(4r - r^3 \right) dr$$

$$= 2\pi \left[2r^2 - \frac{r^4}{4} \right]_0^2$$

$$= 2\pi \left(8 - \frac{16}{4} \right) = 8\pi$$

b) The divergence theorem states that, for a vector field $\underset{\sim}{F}$,

$$\iiint_V \nabla \cdot \underline{F} \, dV = \oiint_S \underset{\sim}{F} \cdot d\underset{\sim}{S}$$

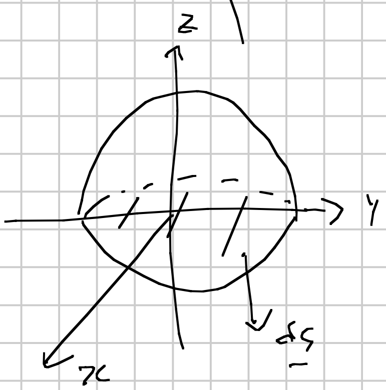
where S is the bounding surface of V.

c) We first have to calculate

$$\oiint_S \underline{F} \cdot d\underline{S} = \iint_{\text{upper}} \underline{F} \cdot d\underline{S} + \iint_{\text{lower}} \underline{F} \cdot d\underline{S}.$$

The integral over the upper surface is already calculated in part (a), we must find

$$\iint_{\text{lower}} \underline{F} \cdot d\underline{S}$$



The surface is perpendicular to the z-axis, so

$$d\underline{S} = \langle 0, 0, -1 \rangle \, dx\,dy$$

Then

$$\iint_{\text{lower}} \underline{F} \cdot d\underline{S} = \iint \langle -y^2, xy, z \rangle \cdot \langle 0, 0, -1 \rangle \, dx\,dy$$

$$= \iint z \, dx\,dy$$

but $z = 0$ on this surface, so

$$= \iint 0 \, dx\,dy = 0 .$$

Then

$$\oiint_S \underline{F} \cdot d\underline{S} = \frac{7\pi}{2} .$$

The volume integral is

$$\iiint_V \nabla \cdot F \, dx\,dy\,dz = \iiint 0 + x + 1 \quad dx\,dy\,dz$$

$$= \int_0^2 \int_0^{2\pi} \int_0^{4-x^2-y^2} \left(1 + r\cos\theta\right) r \, dz \, d\theta \, dr$$

$$= \int_0^2 \int_0^{2\pi} r\left(1 + r\cos\theta\right) \left[z \right]_0^{4-r^2} dr \, d\theta$$

$$= \int_0^2 \int_0^{2\pi} r\left(1 + r\cos\theta\right)\left(4 - r^2\right) dr \, d\theta$$

$$= \int_0^2 \int_0^{2\pi} \left[r\left(4 - r^2\right) + r^2\left(4 - r^2\right)\cos\theta \right] d\theta \, dr$$

$$= 2\pi \int_0^2 r\left(4 - r^3\right) dr + 0$$

$$= 2\pi \left[2r^2 - \frac{r^4}{4} \right]_0^2$$

$$= 2\pi \left(8 - 4 \right) = 8\pi$$

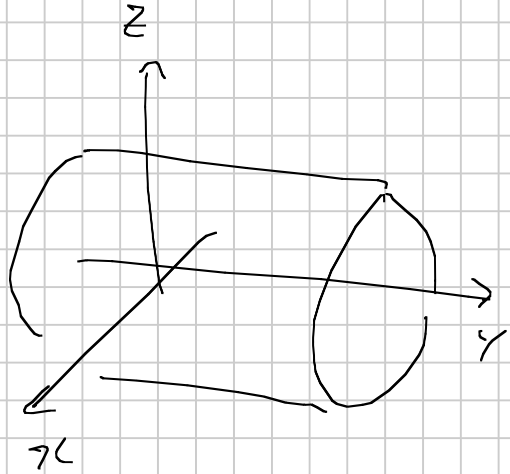
2\. The divergence theorem states that

$$\oiint_S \underline{F} \cdot d\underset{\sim}{S} = \iiint_V \nabla \cdot \underline{F} \, dV$$

where S is the bounding surface of the volume V.

Here $\nabla \cdot \underline{F} = 3y^2 + 0 + 3z^2$

Use cylindrical polar coordinates:



$$y = r\cos\theta$$

$$z = r\sin\theta$$

$$x = x$$

Then the volume has limits

$$0 \leq r \leq 1, \quad 0 \leq \theta \leq 2\pi$$

and

$$\iiint_V \nabla \cdot \underline{F} \, dV = \int_{-1}^{2} \int_{0}^{1} \int_{0}^{2\pi} (3y^2 + 3z^2)\, r \, d\theta \, dr \, dx$$

$$= \int_{-1}^{2}\int_{0}^{1}\int_{0}^{2\pi} 3r^3 \, d\theta \, dr \, dz$$

$$= 2\pi \int_{-1}^{2}\int_{0}^{1} 3r^3 \, dr \, dz$$

$$= 2\pi \int_{-1}^{2} \left[\frac{3}{4} r^4\right]_0^1 dz$$

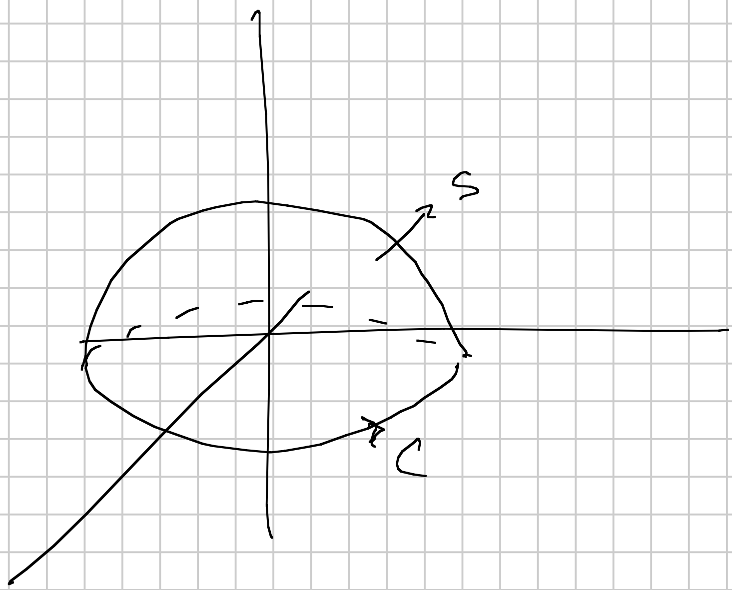
$$= 2\pi \cdot \frac{3}{4} \cdot \int_{-1}^{2} dz = \frac{2\pi}{4} \cdot 3 \cdot 3$$

$$= \frac{9\pi}{2} .$$

3. Stokes theorem states that for a vector field $\underset{\sim}{F}$

$$\iint_S \nabla \times \underset{\sim}{F} \cdot d\underset{\sim}{S} = \oint_C \underset{\sim}{F} \cdot d\underset{\sim}{r}$$

where C is the boundary of S, traversed in the positive sense.



Parametrize the curve: $z = 0$,

$x = 2\cos t$

$y = 2\sin t$

so $\underset{\sim}{r} = \langle 2\cos t, 2\sin t, 0 \rangle$

with $0 \leq t \leq 2\pi$.

Then

$$\frac{d\underset{\sim}{r}}{dt} = \langle -2\sin t, 2\cos t, 0 \rangle$$

and

$$\underset{\sim}{F} = \langle 4\cos^2 t, 4\sin^2 t, 0 \rangle$$

So

$$\int_C \underset{\sim}{F} \cdot d\underset{\sim}{r} = \int_0^{2\pi} \langle 4\cos^2 t, 4\sin^2 t, 0 \rangle \cdot \langle -2\sin t, 2\cos t, 0 \rangle \, dt$$

$$= \int_0^{2\pi} \left(-8\cos^2 t \sin t + 8\sin^2 t \cos t \right) dt$$

$$= \frac{-8}{3} \left[\cos^3 t \right]_0^{2\pi} + \frac{8}{3} \left[\sin^3 t \right]_0^{2\pi}$$

$$= 0$$

4. The divergence theorem states

$$\iint \underline{F} \cdot d\underline{S} = \iiint \nabla \cdot \underline{F} \, dV$$

Here $\underline{F} = \langle x\cos^2 z, \; y\sin^2 z, \; \cos x \sin x \rangle$

$$\nabla \cdot \underline{F} = \cos^2 z + \sin^2 z + 0$$

$$= 1 .$$

Then $\displaystyle\iint \underline{F} \cdot d\underline{S} = \iiint dV$

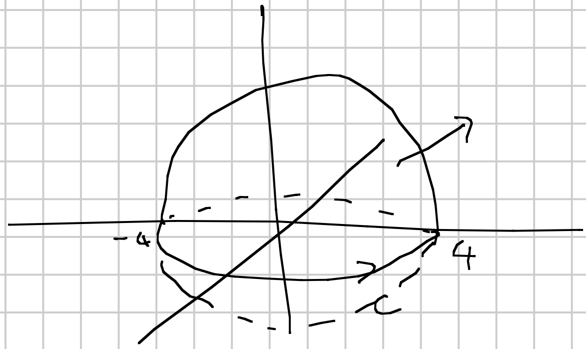
$= \text{Volume of region}$

$= \dfrac{4}{3}\pi R^3 .$

5. Stokes theorem states that

$$\iint_S \nabla \times \underline{F} \cdot d\underline{S} = \oint_C \underline{F} \cdot d\underline{r}$$

The curve is a circle in the $z=0$ plane, centred at $(0,0)$ with radius



$$R = \sqrt{25 - 3^2}$$

$$= 4.$$

Parametrize $x = 4\cos t$

$y = 4\sin t$ $\quad 0 \leq t \leq 2\pi$.

$z = 0$

$$\underline{r} = \langle 4\cos t, 4\sin t, 0 \rangle$$

$$\frac{d\underline{r}}{dt} = \langle -4\sin t, 4\cos t, 0 \rangle$$

$$\oint_C \underline{F} \cdot d\underline{r} = \int_0^{2\pi} \langle -4\sin t, 4\cos t, 0 \rangle \cdot \langle -4\sin t, 4\cos t, 0 \rangle \, dt$$

$$= \int_0^{2\pi} 16\left(\sin^2 t + \cos^2 t\right) dt$$

$$= 32\pi .$$