Vector Calculus and PDEs 37336
Problem Set 7: Vector calculus in electromagnetic theory

1. In lectures, we found that the electric field of a point charge Q is in spherical coordinates by the vector field

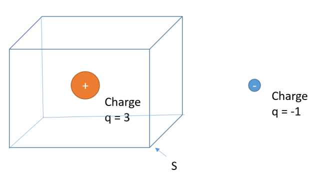
$$\mathbf{E}(r,\theta,\varphi) = \frac{Q}{4\pi\varepsilon_0}\frac{1}{r^2}\hat{\mathbf{r}}$$

Consider two point charges separated by a distance d, as shown below. By drawing the electric field from each of the charges separately, sketch the total electric field.

a) Two equal positive charges:

b) Two equal but opposite charges:

2. Consider the charge distribution shown (units are in Coulombs):



Use Gauss's law to compute the flux integral of the electric field $\mathbf{E}$ through the surface S.

3. Suppose the electric field in some region is found to depend on the distance from the origin r via the equation

$$\mathbf{E} = kr^3\hat{\mathbf{r}}$$

where k is some constant and $\hat{\mathbf{r}}$ is the radial unit vector in spherical polar coordinates. Use Gauss's law to compute the charge density $\rho(\mathbf{r})$.

4. Find a potential V for the uniformly-increasing electric field

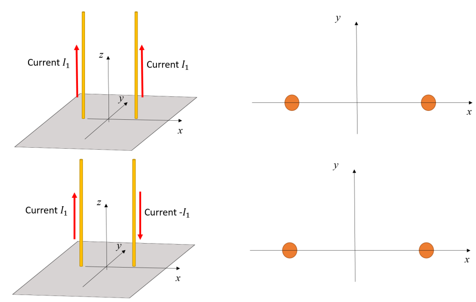
$$\mathbf{E}(x,y,z) = Cx + D\hat{\mathbf{i}}$$

where C and D are constants.

5. In lectures, we found the magnetic field of a line current I flowing in the z direction is given by

$$\mathbf{B} = \frac{\mu_0}{2\pi}\frac{I}{r}\hat{\boldsymbol{\theta}}$$

Two wires carrying currents are directed perpendicular to the z-axis, as shown. By drawing the magnetic fields of the wires separately, plot the magnetic fields in the x, y plane.



6. We showed in lectures that the electric field obeys the partial differential equation (PDE)

$$\nabla^2\mathbf{E} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{E}\ .$$

Show that the following field, which represents a plane wave travelling in the x direction, is a solution to this PDE.

$$\mathbf{E} = \hat{\mathbf{j}}A\cos\left(\frac{w}{c}x - \omega t\right)$$

where A and ω are constants.

7. For an electromagnetic signal with a particular frequency ω, the electric field and magnetic field are related by the curl equation

$$\nabla\times\mathbf{E} = i\omega\mathbf{B}\ .$$

Use this to compute $\mathbf{B}$ for the plane wave in the previous question. Show that $\mathbf{E}$ and $\mathbf{B}$ are perpendicular.