Vector Calculus and PDEs 37336

Problem Set 8: Inner Products, Operators and Sturm-Liouville eigenvalue problems

1. For each of the examples below, draw the functions f and g, and compute the inner product

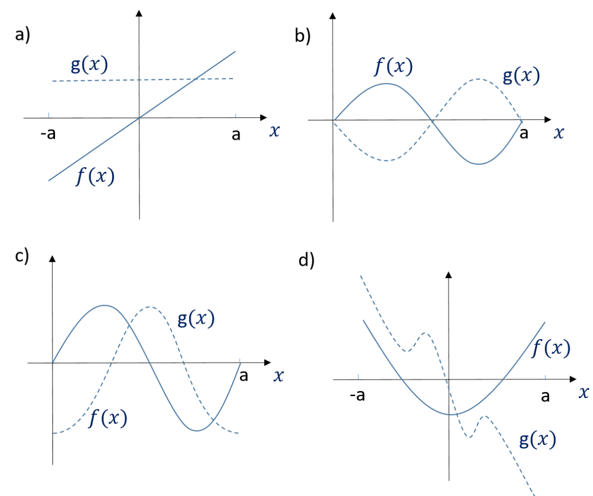
$$\langle f, g \rangle$$

over the given domains and with the given weights.

a) $f(x) = 1 + x$, $g(x) = 2$, on the domain $0 \le x \le 1$, with weight $w(x) = 1$.

b) $f(x) = \cos x$, $g(x) = x$, on the domain $0 \le x \le \pi$, with weight $w(x) = 1/x$.

2. For the functions shown, and without computing anything, state which are orthogonal. You can assume that the weight function $w(x) = 1$ in each case.



3. By computing the inner product determine whether or not f and g are orthogonal on the given domains. You can assume $w(x) = 1$.

a) $f(x) = \sin x$, $g(x) = \cos x$, on the domain $-\pi \le x \le \pi$.

d) $f(x) = \cos^2 x$, $g(x) = \sin x$, on the domain $0 \le x \le \pi$.

c) $f(x) = e^{ix}$, $g(x) = e^{2ix}$, on the domain $-\pi \le x \le \pi$.

4. Apply the operator

$$\mathcal{L} = -\left[\frac{d}{dx}\left(x\frac{d}{dx}\right) - \frac{2}{x}\right]$$

to the function $f(x) = 3x^2$.

5. * *NB: This is question is hard, but is important so try to work through it*
Consider a particular solution $\{\phi_1, \lambda_1\}$ of an eigenvalue problem with the operator $\mathcal{L}$, i.e.

$$\mathcal{L}\phi_1 = \lambda_1\phi_1 \ .$$

a) By taking the inner product of the above equation with ϕ_1, show that

$$\lambda_1 = \frac{\langle \phi_1, \mathcal{L}\phi_1 \rangle}{||\phi_1||^2}$$

b) Given the operator

$$\mathcal{L} = -\frac{d^2}{dx^2}$$

and the eigenfunction on the domain $0 \le x \le L$

$$\phi_1(x) = \sin(\pi x/L)$$

Use the formula in (a) to compute the eigenvalue λ_1.

6. Find all *even* solutions to the Sturm-Lioviulle eigenvalue problem

$$-\frac{d^2\phi}{dx^2} = \lambda\phi$$

on the domain $-a \le x \le a$, with the boundary condition $\phi(-a) = \phi(a) = 0$.

7. Expand the function

$$f(x) = 2$$

in the domain $-a \le x \le a$ as a series of the eigenfunctions ϕ_m appearing in the previous question.

8. Find all eigenfunctions of the Sturm-Liouville problem

$$-\frac{d^2\phi}{dx^2} = \lambda\phi$$

on the domain $0 \le x \le 2L$ with the Neumann conditions $\phi'(0) = \phi'(2L) = 0$. Expand the function

$$f(x) = \begin{cases} x & 0 \le x \le L \\ 0 & L < x < 2L \end{cases}$$

in terms of the eigenfunctions ϕ_n.