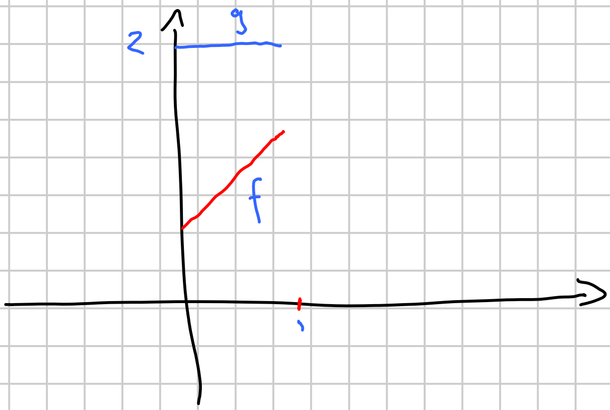
1. a)



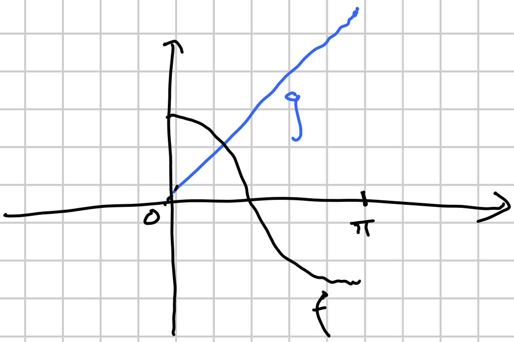
$$\langle f, g \rangle = \int_0^1 f(x)^* g(x)\, w(x)\, dx$$

$$= \int_0^1 (1+x) \times 2 \, dx$$

$$= 2 \int_0^1 1 + x \, dx = 2\left[x + \frac{x^2}{2} \right]_0^1$$

$$= 2\left(1 + \frac{1}{2}\right) = 2 \times \frac{3}{2} = 3$$

b)



$$\langle f, g \rangle = \int_0^\pi \cos x \cdot x \cdot \frac{1}{x} \, dx$$

$$= \int_0^\pi \cos x \, dx$$

$$= \left[\sin x \right]_0^\pi = 0$$

2. a) Orthogonal (by symmetry)

b) Not orthogonal (one is a multiple of the other)

c) Orthogonal (one is sin, the other is cos)

d) Orthogonal (by symmetry)

3. a) $\langle f, g \rangle = \int_{-\pi}^{\pi} \sin x \cdot \cos x \, dx$

$= \int_{-\pi}^{\pi} \frac{1}{2} \sin 2x \, dx$

$= \frac{1}{2} \left[\frac{-\cos(2x)}{2} \right]_{-\pi}^{\pi}$

$= \frac{1}{4} (1 - 1) = 0 \quad \Rightarrow$ orthogonal

b) $\langle f, g \rangle = \int_{0}^{\pi} \cos^2 x \sin x \, dx$

$= \left[\frac{\cos^3 x}{3} \right]_0^{\pi} = \frac{1}{3}(-1-1) = -\frac{2}{3} \quad \Rightarrow$ not orthogonal

c) $\langle f, g \rangle = \int_{-\pi}^{\pi} e^{ix} e^{2ix} \, dx$

$= \int_{-\pi}^{\pi} e^{3ix} \, dx = \frac{1}{3i} \left[e^{3ix} \right]_{-\pi}^{\pi}$

$= \frac{1}{3i} \left[e^{3i\pi} - e^{-3i\pi} \right] = \frac{1}{3i} \left(-1 - (-1) \right)$

$= 0 \quad \Rightarrow$ orthogonal.

4.

$$\mathcal{L} f = -\frac{d}{dx}\left(x \frac{df}{dx}\right) + \frac{2}{x} f$$

$$= -\frac{d}{dx}\left(x \frac{d}{dx}\left(3x^2\right)\right) + \frac{2}{x} 3x^2$$

$$= -\frac{d}{dx}\left(x . 6x\right) + 6x$$

$$= -\frac{d}{dx}\left(6x^2\right) + 6x$$

$$= -12x + 6x = -6x.$$

5. a) We have

$$\mathcal{L} \phi_1 = \lambda_1 \phi_1$$

So, taking the inner product with ϕ_1 on the left, we have

$$\langle \phi_1, \mathcal{L} \phi_1 \rangle = \lambda_1 \langle \phi_1, \phi_1 \rangle$$

Re-arranging,

$$\lambda_1 = \frac{\langle \phi_1, \mathcal{L} \phi_1 \rangle}{\langle \phi_1, \phi_1 \rangle} = \frac{\langle \phi_1, \mathcal{L} \phi_1 \rangle}{\| \phi_1 \|^2}.$$

b) Now $$\mathcal{L} \phi_1 = -\frac{d^2}{dx^2} \sin\left(\frac{\pi x}{L}\right)$$

$$= -\frac{d}{dx}\left(\frac{\pi}{L} \cos\left(\frac{\pi x}{L}\right)\right)$$

$$= + \frac{\pi^2}{L^2} \sin\left(\frac{\pi x}{L}\right)$$

So $$\langle \phi_1, \mathcal{L} \phi_1 \rangle = \int_0^L \sin\left(\frac{\pi x}{L}\right) \frac{\pi^2}{L^2} \sin\left(\frac{\pi x}{L}\right) dx$$

$$= \frac{\pi^2}{L^2} \int_0^L \sin^2\left(\frac{\pi x}{L}\right) dx$$

$$= \frac{\pi^2}{L^2} \int_0^L \frac{1}{2}\left(1 - \cos\left(\frac{2\pi x}{L}\right)\right) dx$$

$$= \frac{\pi^2}{L^2} \frac{1}{2} \left[x - \frac{\sin\left(\frac{2\pi x}{L}\right)}{2\pi/L} \right]_0^L$$

$$= \frac{\pi^2}{L^2} \frac{1}{2} \left(L - \frac{\sin(2\pi)}{\frac{2\pi}{L}} - 0 + \frac{\sin 0}{2\pi/L} \right)$$

$$= \frac{\pi^2}{2L}.$$

Also

$$\|\phi_1\|^2 = \int_0^L \sin^2\left(\frac{\pi x}{L}\right) dx$$

$$= \int_0^L \frac{1}{2}\left(1 - \cos\left(\frac{\pi x}{L}\right)\right) dx$$

$$= \frac{1}{2} \left[x - \frac{\sin \frac{\pi x}{L}}{\pi/L} \right]_0^L$$

$$= \frac{1}{2}\left(L - \frac{\sin \pi}{\pi/L}\right) - \frac{1}{2}\left(0 - \frac{\sin 0}{\pi/L}\right)$$

$$= \frac{L}{2}.$$

So

$$\lambda_1 = \frac{\langle \phi_1, \mathcal{L}\phi_1 \rangle}{\|\phi_1\|^2} = \frac{\frac{\pi^2}{2L}}{\frac{L}{2}} = \frac{\pi^2}{2L} \times \frac{2}{L}$$

$$= \frac{\pi^2}{L^2}.$$

6. First: the general solution to

$$-\frac{d^2\phi}{dx^2} = \lambda \phi .$$

You don't have to derive this if you can remember it correctly.

is

$$\phi(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$$

If we only want even solutions then we must have $B = 0$, since $\sin(x)$ is an odd function, so

$$\phi(x) = A\cos(\sqrt{\lambda}x)$$

Now we know $\phi(a) = 0$, so

$$\phi(a) = A\cos(\sqrt{\lambda}a) = 0$$

This is true when $\cos(\sqrt{\lambda}a) = 0$ i.e. $\sqrt{\lambda}a$ is an odd multiple of $\frac{\pi}{2}$:

$$\sqrt{\lambda}a = (2m+1)\frac{\pi}{2} \qquad \text{for } m = 0, 1, 2, \dots$$

We therefore have eigenvalues

$$\lambda_m = \left[\frac{1}{a}(2m+1)\frac{\pi}{2}\right]^2$$

with eigenfunctions

$$\phi_m = \cos(\sqrt{\lambda_m}x)$$

(We can leave out the A at the front, because eigenfunctions are unique only up to a multiplicative constant).

7. The expression in terms of eigenfunctions is

$$f(x) = \sum_{n=0}^{\infty} \frac{\langle f, \phi_n \rangle}{\|\phi_n\|^2} \phi_n(x)$$

Here we have

$$\phi_n = \cos(\sqrt{\lambda_n} x) \quad \text{with}$$

$$\lambda_n = \left(\frac{\pi}{2a}(2n+1)\right)^2$$

So

$$\langle f, \phi_n \rangle = \int_{-a}^{a} 2 \cos(\sqrt{\lambda_n} x)\, dx$$

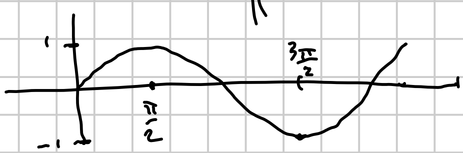
$$= 2 \left[\frac{\sin \sqrt{\lambda_n} x}{\sqrt{\lambda_n}} \right]_{-a}^{a}$$

$$= \frac{2}{\sqrt{\lambda_n}} \left[\sin(\sqrt{\lambda_n} a) - \sin(-\sqrt{\lambda_n} a) \right]$$

$$= \frac{4}{\sqrt{\lambda_n}} \sin(\sqrt{\lambda_n} a) = \frac{4}{\sqrt{\lambda_n}} \sin\left[\frac{\pi}{2\cancel{a}}(2n+1)\cancel{a}\right]$$

$$= \frac{4}{\sqrt{\lambda_n}} \sin\left(\frac{\pi}{2}(2n+1)\right)$$

odd integer! What happens to $\sin(x)$?



$$= \frac{4}{\sqrt{\lambda_n}} (-1)^n$$

$$= \frac{4}{\frac{\pi}{2a}(2n+1)} (-1)^n = \frac{8a}{\pi(2n+1)} (-1)^n$$

Also,

$$\|\phi_n\|^2 = \int_{-a}^{a} \sin^2(\sqrt{\lambda_n} x)\, dx$$

$$= \int_{-a}^{a} \frac{1}{2}\left(1 - \cos(2\sqrt{\lambda_n} x)\right) dx$$

$$= \frac{1}{2}\left[x - \frac{\sin(2\sqrt{\lambda_n}x)}{2\sqrt{\lambda_n}}\right]_{-a}^{a}$$

$$= \frac{1}{2}\left[a - \frac{\sin\left(2a\frac{\pi}{2a}(2n+1)\right)}{2\sqrt{\lambda_n}}\right] - \frac{1}{2}\left[-a + \frac{\sin\left(\frac{2a\pi}{2a}(2n+1)\right)}{2\sqrt{\lambda_n}}\right]$$

$$= \frac{1}{2}a + \frac{1}{2}a \qquad = 0 \quad \text{because} \quad \sin(2\pi(2n+1)) = 0$$

$$= a.$$

Finally,

$$c_n = \frac{\langle f, \phi_n \rangle}{\|\phi_n\|^2} = \frac{\frac{8a}{\pi}\frac{1}{(2n+1)}(-1)^n}{a} = \frac{8}{\pi}\frac{(-1)^n}{(2n+1)}$$

And so

$$f(x) = \sum_{n=0}^{\infty} \frac{8}{\pi}\frac{(-1)^n}{(2n+1)} \phi_n(x)$$

8. The general solution to

$$-\frac{d^2\phi}{dx^2} = \lambda\phi$$

is

$$\phi(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x).$$

We know that

$$\phi'(0) = 0, \quad \text{so}$$

$$\phi'(x) = -A\sin(\sqrt{\lambda}x) + B\cos(\sqrt{\lambda}x)$$

$$\Rightarrow \quad 0 = -A\times 0 + B\times 1$$

$$\Rightarrow \quad B = 0$$

We then have

$$\phi(x) = A\cos(\sqrt{\lambda}x)$$

We also know that

$$\phi'(2L) = 0$$

so

$$-A\sin(\sqrt{\lambda}.2L) = 0$$

and therefore

$$\sqrt{\lambda}.2L = m\pi \qquad \text{where } m = 1, 2, 3, \dots$$

(we exclude 0 because then we'd have $\lambda_m = 0$).

So the eigenfunctions are

$$\phi_m = \cos(\sqrt{\lambda_m}\, x)$$

with eigenfunctions

$$\lambda_m = \left(\frac{m\pi}{2L}\right)^2.$$

We now want to expand

$$f(x) = \begin{cases} x & 0 \leq x \leq L \\ 0 & L \leq x \leq 2L. \end{cases}$$

Now

$$\langle f, \phi_m \rangle = \int_0^{2L} f(x)\cos(\sqrt{\lambda_m}x)\,dx$$

$$= \int_0^L x\cos(\sqrt{\lambda_m}x)\,dx$$

$$= \left[\frac{x\sin(\sqrt{\lambda_m}x)}{\sqrt{\lambda_m}}\right]_0^L - \int_0^L \frac{\sin(\sqrt{\lambda_m}x)}{\sqrt{\lambda_m}}\,dx$$

$$= \frac{L}{\sqrt{\lambda_m}}\sin(\sqrt{\lambda_m}L) + \frac{1}{\lambda_m}\Big[\cos(\sqrt{\lambda_m}x)\Big]_0^L$$

$$= \frac{L}{\sqrt{\lambda_m}}\sin\left(\frac{m\pi}{2\cancel{L}}\cancel{L}\right) + \frac{1}{\lambda_m}\left[\cos\left(\frac{m\pi}{2L}L\right) - 1\right]$$

$$= \frac{2L^2}{\pi m}\sin\left(\frac{m\pi}{2}\right) + \left(\frac{2L}{\pi m}\right)^2\left(\cos\left(\frac{m\pi}{2}\right) - 1\right)$$

Then

$$\|\phi_m\|^2 = \int_0^{2L} \cos^2\left(\frac{m\pi}{2L}x\right) dx$$

$$= \frac{1}{2}\int_0^{2L}\left(1 + \cos\left(\frac{m\pi}{L}x\right)\right) dx$$

$$= \frac{1}{2}\left[x + \frac{\sin\left(\frac{m\pi}{L}x\right)}{m\pi/L}\right]_0^{2L}$$

$$= \frac{1}{2}\left[2L + \underbrace{\frac{\sin\left(\frac{m\pi}{2}\times 2L\right)}{\frac{m\pi}{L}}}_{=0} - 0 - \frac{\sin(0)}{\frac{m\pi}{L}}\right]$$

$$= L$$

So

$$f(x) = \sum_{m=1}^{\infty} c_m \phi_m(x)$$

where

$$c_m = \frac{2L}{\pi m}\sin\left(\frac{m\pi}{2}\right) + L\left(\frac{2}{\pi m}\right)^2\left(\cos\left(\frac{m\pi}{2}\right) - 1\right)$$