1. a)

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

$A = 1$ $\quad C = 1$ $\quad B = 0$ $\Rightarrow$ $B^2 - 4AC = -4 > 0$

$\Rightarrow$ elliptic.

b)

$$\frac{\partial^2 \psi}{\partial x^2} + 2\frac{\partial^2 \psi}{\partial x \partial y} + \frac{\partial^2 \psi}{\partial y^2} = 3\frac{\partial \psi}{\partial x}$$

$A = 1$ $\quad 2B = 2$ $\Rightarrow B = 1$ $\quad C = 1$

$B^2 - 4AC = 1 - 4 = -3$ $\Rightarrow$ elliptic

c)

$$\frac{\partial^2 \psi}{\partial x^2} - \frac{1}{2}\frac{\partial^2 \psi}{\partial t^2} = \frac{\partial^2 \psi}{\partial x \partial t}$$

$A = 1$ $\quad C = -\frac{1}{2}$ $\quad B = -\frac{1}{2}$

$B^2 - 4AC = \frac{1}{4} - 4 \times 1 \times \left(-\frac{1}{2}\right) = \frac{1}{4} + 2 > 0$

$\Rightarrow$ hyperbolic

d)

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}\frac{\partial u}{\partial x}$$

is <u>nonlinear</u>

(so doesn't classify in this way).

2. a)

$$\frac{du}{dx} = \alpha u$$

General solution $u(x) = Ae^{\alpha x}$

b)

$$\frac{d^2 X}{dt^2} = k^2 X$$

has general solution $X(t) = A\cosh(kt) + B\sinh(kt)$

c) $$\frac{d^2 Y}{dy^2} = -\alpha^2 Y$$

has general solution

$$Y(y) = A\cos(\alpha y) + B\sin(\alpha y)$$

or equivalently,

$$Y(y) = Ae^{i\alpha y} + Be^{-i\alpha y}.$$

d) $$r\frac{R''}{R} + \frac{R'}{R} = 0$$

is a bit tricky: First multiply through by $R(r)$:

$$rR'' + R' = 0$$

or

$$r\frac{d}{dr}\left(\frac{dR}{dr}\right) + \frac{dR}{dr} = 0$$

which we can write as

$$\frac{d}{dr}\left(r\frac{dR}{dr}\right) = 0$$

Solving, $$r\frac{dR}{dr} = A$$ where A is a constant.

So $$\frac{dR}{dr} = \frac{A}{r}$$

Integrating again,

$$R = \int \frac{A}{r}\,dr$$

$$= A\ln|r| + B$$

which is the general solution.

3. We use the separation Ansatz

$$v(x,y) = X(x)\,Y(y).$$

Subbing in to the PDE, we find

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) X(x)\,Y(y) = 0$$

or

$$Y X'' + X Y'' = 0$$

Dividing by XY,

$$\frac{X''}{X} + \frac{Y''}{Y} = 0$$

And so the variables separate:

$$\frac{X''}{X} = \pm\lambda = -\frac{Y''}{Y}$$

We can satisfy $v(0,y) = v(\pi,y) = 0$ if $X(0) = X(\pi) = 0$. We therefore solve

$$\frac{X''}{X} = -\lambda$$

$$\Rightarrow \begin{cases} -X'' = \lambda X & \text{on} \quad 0 \leq x \leq \pi \\ X(0) = X(\pi) = 0 \end{cases}$$

which is a S-L problem. The general solution is

$$X(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$$

and $X(0) = 0 \Rightarrow A = 0$, so

$$X(x) = B\sin(\sqrt{\lambda}x)$$

Now $X(\pi) = 0$ implies

$$0 = \sin(\sqrt{\lambda}\pi)$$

so

$$\sqrt{\lambda} = n \quad \text{where} \quad n \in \mathbb{Z}.$$

The eigenfunctions are therefore

$$X_n = \sin(\sqrt{\lambda_n}x)$$

with

$$\lambda_n = n^2.$$

The problem for $Y(y)$ is then

$$-\frac{Y''}{Y} = -\lambda_n \quad \Rightarrow \quad Y'' = \lambda_n Y$$

The general solution is

$$Y_n''(y) = A\cosh(\sqrt{\lambda_n}y) + B\sinh(\sqrt{\lambda_n}y)$$

Now because $v(x,0) = 0$ we have $Y_n(0) = 0$, and so $A = 0$. Therefore

$$Y_n(y) = B\sinh(\sqrt{\lambda_n}y).$$

The general solution is then

$$v(x,y) = \sum_{n=1}^{\infty} b_n X_n(x)\, Y_n(y)$$

$$= \sum_{n=1}^{\infty} b_n \sin(\sqrt{\lambda_n}x) \cdot B\sinh(\sqrt{\lambda_n}y)$$

$$= \sum_{n=1}^{\infty} c_n \sin(nx)\sinh(ny).$$

We then have $v(x,a) = x(\pi - x) =: f(x)$

So

$$f(x) = \sum_{m=1}^{\infty} c_m X_m(x) Y_m(a).$$

Take the inner product with X_n:

$$\langle X_n, f \rangle = \sum_{m=1}^{\infty} c_m Y_m(a) \langle X_n, X_m \rangle$$

$$= c_n Y_n(a) \|X_n\|^2$$

So

$$c_n = \frac{\langle X_n, f \rangle}{Y_n(a) \|X_n\|^2}.$$

Now

$$\|X_n\|^2 = \int_0^{\pi} \sin^2(nx)\, dx$$

$$= \int_0^{\pi} \frac{1}{2}\left(1 - \cos(2nx)\right) dx$$

$$= \frac{1}{2}\left[x - \frac{\sin(2nx)}{2n}\right]_0^{\pi} = \left(\frac{\pi}{2} - 0\right) - (0 - 0)$$

$$= \frac{\pi}{2}.$$

Finally

$$\langle X_n, f \rangle = \int_0^{\pi} \sin(nx)(\pi x - x^2)\, dx$$

$$= \left[-\frac{\cos(nx)}{n}(\pi x - x^2)\right]_0^{\pi} - \int_0^{\pi}\left(-\frac{\cos(nx)}{n}\right)(\pi - 2x)\, dx$$

$$= 0 + \int_0^{\pi} \frac{\cos(nx)}{n}(\pi - 2x)\, dx$$

$$= \left[\frac{\sin(nx)}{n^2}(\pi - 2x)\right]_0^{\pi} - \int_0^{\pi} \frac{\sin(nx)}{n^2}(-2)\, dx$$

$$= 0 + \frac{2}{n^2}\int_0^{\pi} \sin(nx)\, dx$$

$$= \frac{2}{n^2}\left[\frac{-\cos(nx)}{n}\right]_0^{\pi} = \frac{-2}{n^3}\left(\cos(n\pi) - \cos(0)\right)$$

$$= \frac{-2}{n^3}\left((-1)^n - 1\right) = \frac{2}{n^3}\left(1-(-1)^n\right)$$

So

$$c_n = \frac{\langle X_n, f\rangle}{\|X_n\|^2}\frac{1}{Y_n(a)} = \frac{2}{n^3}\left(1-(-1)^n\right)\frac{2}{\pi}\frac{1}{\sinh(na)}$$

$$= \frac{4}{\pi n^3}\left(1-(-1)^n\right)\frac{1}{\sinh(na)}.$$

4. The general sol$\underline{^n}$ is

$$X(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x).$$

We have

$$X(0)=0 \implies A=0$$

$$X(\pi)=0 \implies 0 = B\sin(\sqrt{\lambda}\pi)$$

and so

$$\sqrt{\lambda} = m.$$

The eigenfunctions are

$$X_m(x) = \sin(mx)$$

with eigenvalues

$$\lambda_m = m^2.$$

Use the separation ansatz

$$u(x,t) = X(x)T(t)$$

Subbing in to the PDE, we find

$$XT' = kX''T$$

Dividing by kXT we have

$$\frac{1}{k}\frac{T'}{T} = \frac{X''}{X} = \pm\lambda$$

and so the variables separate.

Choosing the negative λ, we have the S-L problem

$$X'' = -\lambda X$$

$$X(0) = X(\pi) = 0$$

Which has the eigenfunctions given above.

The problem for T is then

$$\frac{1}{k}\frac{T'}{T} = -\lambda_n$$

Re-arranging:

$$T' = -\lambda_n k T$$

$$= -n^2 k T$$

This has the general solution

$$T_n(t) = A e^{-n^2 k t}.$$

We then expand the complete solution as

$$u(x,t) = \sum_{n=1}^{\infty} b_n X_n(x) T_n(t)$$

We would like $u(x,0) = f(x)$ at $t=0$, so

$$f(x) = \sum_{n=1}^{\infty} b_n X_n(x) T(0)$$

Taking the inner product with X_n,

$$\langle X_n, f \rangle = \sum_{m=1}^{\infty} b_m T(0) \langle X_n, X_m \rangle$$

$$= b_n T_n(0) \| X_n \|^2$$

so

$$b_n = \frac{\langle X_n, f \rangle}{T_n(0) \| X_n \|^2}$$

Now
$$\|X_n\|^2 = \int_0^\pi \sin^2(nx)\,dx$$
$$= \int_0^\pi \frac{1}{2}\left(1-\cos 2nx\right)dx = \frac{1}{2}\left[x - \frac{\sin(2nx)}{2n}\right]_0^\pi$$
$$= \frac{\pi}{2} - 0 - (0-0) = \frac{\pi}{2}.$$

Now we have to compute
$$\langle X_n, f\rangle = \int_0^\pi \sin(nx)\left\{\frac{1}{3}\sin(3x) + \sin(5x)\right\}dx.$$
and this is doable, but maybe a bit hard, unless we notice that
$$\sin(3x) = X_3(x)$$
$$\sin(5x) = X_5(x)$$
So
$$\frac{1}{3}\int_0^\pi \sin(nx)\sin(3x)\,dx$$
$$= \frac{1}{3}\langle X_n, X_3\rangle = \frac{1}{3}\delta_{n,3}\|X_3\|^2$$
"Kronecker delta"
$$= \frac{1}{3}\cdot\frac{\pi}{2}\delta_{n,3}$$
and
$$\int \sin(nx)\sin(5x)\,dx = \langle X_n, X_5\rangle$$
$$= \frac{\pi}{2}\delta_{n,5}.$$
Therefore
$$u(x,t) = \sum_{n=1}^{\infty}\left(\frac{1}{3}\frac{\pi}{2}\delta_{n,3} + \frac{\pi}{2}\delta_{n,5}\right)\frac{1}{\pi/2}X_n(x)T_n(t)$$
$$= \frac{1}{3}X_3(x)T_3(t) + X_5(x)T_5(t)$$
$$= \frac{1}{3}\sin(3x)e^{-k9t} + \sin(5x)e^{-k25t}$$ as required.

5. Use the separation ansatz

$$Y(x,t) = X(x)\,T(t).$$

Subbing into the PDE we find

$$X''T = \frac{1}{c^2} X T''$$

Dividing by XT we have

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} = \pm\lambda$$

and so the variable separate. Picking the negative sign for λ, we obtain a Sturm-Liouville problem for X:

$$-X'' = \lambda X$$
$$X(0) = X(a) = 0.$$

This has eigenfunctions

$$X_n = \sin\left(\frac{n\pi}{a}x\right)$$

with eigenvalues

$$\sqrt{\lambda_n} = \frac{n\pi}{a}.$$

The problem for $T_n(t)$ is then

$$\frac{1}{c^2}\frac{T_n''}{T_n} = -\lambda_n$$

$$\Rightarrow \quad T'' = -\lambda_n c^2 T$$

This has the general solution

$$T(t) = A_n\cos(\sqrt{\lambda_n}\,ct) + B_n\sin(\sqrt{\lambda_n}\,t)$$

Or
$$T_m(t) = A_m \cos\left(\frac{m\pi}{a}ct\right) + B_m \sin\left(\frac{m\pi}{a}ct\right).$$

The function $Y(x,t)$ can then be written as

$$Y(x,t) = \sum_{m=1}^{\infty} X_m(x)\,T_m(t)$$
$$= \sum_{m=1}^{\infty} \sin\left(\frac{m\pi}{a}x\right)\left[A_m \cos\left(\frac{m\pi}{a}ct\right) + B_m \sin\left(\frac{m\pi}{a}ct\right)\right].$$

as required.

Now, we are given that

$$\frac{\partial Y}{\partial t} = 0 \quad \text{when} \quad t=0$$

This will be satisfied if

$$T_m'(0) = 0$$

ie
$$-A_m \frac{m\pi}{a}c \sin 0 + B_m \frac{m\pi}{a}c \cos 0 = 0$$
$$\Rightarrow \quad B_m = 0$$

So we have

$$Y(x,t) = \sum_{m=1}^{\infty} X_m(x)\,A_m \cos\left(\frac{m\pi ct}{a}\right)$$

The other initial condition is

$$Y(x,0) = f(x) = x^2(a-x).$$

So
$$f(x) = \sum_{m=1}^{\infty} A_m X_m(x) \cos(0)$$
$$= \sum_{m=1}^{\infty} A_m X_m$$

We now take the inner product:

$$\langle X_n, f\rangle = \sum_{m=1}^{\infty} A_m \langle X_n, X_m\rangle$$

$$= A_n \|X_n\|^2$$

So

$$A_n = \frac{\langle X_n, f\rangle}{\|X_n\|^2}$$

Here, $$\|X_n\|^2 = \int_0^a \sin^2\left(\frac{n\pi x}{a}\right) dx$$

$$= \frac{1}{2}\int_0^a \left[1 - \cos\left(\frac{2n\pi x}{a}\right)\right] dx$$

$$= \frac{1}{2}\left[x - \frac{\sin\left(\frac{2n\pi x}{a}\right)}{\frac{2n\pi}{a}}\right]_0^a$$

$$= \frac{a}{2}$$

This is ugly, but doable:

Also $$\langle X_n, f\rangle = \int_0^a \sin\left(\frac{n\pi x}{a}\right) x^2(a-x)\, dx$$

$$= \int_0^a \sin\left(\frac{n\pi x}{a}\right)(ax^2 - x^3)\, dx$$

$$= \underbrace{\left[-\frac{\cos\left(\frac{n\pi x}{a}\right)}{\frac{n\pi x}{a}}(ax^2 - x^3)\right]_0^a}_{=0} + \int_0^a \frac{\cos\left(\frac{n\pi x}{a}\right)}{\frac{n\pi}{a}}(2ax - 3x^2)\, dx$$

$$= \frac{a}{n\pi}\underbrace{\left[\frac{\sin\left(\frac{n\pi x}{a}\right)}{n\pi/a}(2ax - 3x^2)\right]_0^a}_{=0} - \left(\frac{a}{n\pi}\right)^2 \int_0^a \sin\left(\frac{n\pi x}{a}\right)(2a - 6x)\, dx$$

$$= -\left(\frac{a}{n\pi}\right)^2\left[-\frac{\cos\left(\frac{n\pi x}{a}\right)}{n\pi/a}(2a - 6x)\right]_0^a + \left(\frac{a}{n\pi}\right)^2 \int_0^a \frac{\cos\left(\frac{n\pi x}{a}\right)}{\frac{n\pi}{a}}(-6)\, dx$$

$$= \left(\frac{a}{n\pi}\right)^3\left[\cos(n\pi)(-4a) - 2a\right] + \left(\frac{a}{n\pi}\right)^3(-6)\underbrace{\left[\frac{\sin\left(\frac{n\pi x}{a}\right)}{\frac{n\pi}{a}}\right]_0^a}_{=0.}$$

$$= -2a\left(\frac{a}{n\pi}\right)^3\left[2\cos(n\pi) + 1\right].$$

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$$A_n = \frac{\langle X_n, f\rangle}{\|X_n\|^2}$$

$$= \frac{-2a\left(\frac{a}{n\pi}\right)^3\left[2\cos(n\pi)+1\right]}{a/2}$$

$$= -4\left(\frac{a}{n\pi}\right)^3\left[2\cos(n\pi)+1\right]$$