Vector Calculus and PDEs 37336
Problem Set 9: Separation of variables

1. Classify the following partial differential equations as elliptic, parabolic, hyperbolic, or neither:

   a)
$$\frac{\partial^2\psi}{\partial x^2} = -\frac{\partial^2\psi}{\partial y^2}$$

   b)
$$\frac{\partial^2\psi}{\partial x^2} + 2\frac{\partial^2\psi}{\partial x\partial y} + \frac{\partial^2\psi}{\partial y^2} = 3\frac{\partial\psi}{\partial x}$$

   c)
$$\frac{\partial^2\psi}{\partial x^2} - \frac{1}{2}\frac{\partial^2\psi}{\partial t^2} = \frac{\partial^2\psi}{\partial x\partial t}$$

   d)
$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}\frac{\partial u}{\partial x}$$

2. Write the general solutions to the following *ordinary* differential equations:

   a)
$$\frac{du}{dx} = \alpha u$$

   b)
$$\frac{d^2X}{dt^2} = k^2X$$

   c)
$$\frac{d^2Y}{dy^2} = -\alpha^2Y$$

   d)*
$$r\frac{R''(r)}{R(r)} + \frac{R'(r)}{R(r)} = 0$$

3. A scalar function $V(x,y)$ satisfies Laplace's equation
$$\frac{\partial^2V}{\partial x^2} + \frac{\partial^2V}{\partial y^2} = 0$$
in the region $(0 \le x \le \pi)$, $(0 \le y \le a)$, and is subject to the boundary conditions
$$\begin{aligned} V(0,y) &= 0, \quad 0 \le y \le a, \\ V(\pi,y) &= 0, \quad 0 \le y \le a, \\ V(x,0) &= 0, \quad 0 \le x \le \pi. \end{aligned}$$
Use the method of separation of variables to show that $V(x,y)$ can be written in the form
$$V(x,y) = \sum_{m=1}^{\infty} C_m \sinh(my)\sin(mx)\ .$$
Find the values of the constants B_m if the boundary condition at $y = a$ is
$$V(x,a) = x(\pi - x), \quad 0 \le x \le \pi\ .$$

4. Find the eigenvalues and eigenfunctions of the boundary value problem
$$\frac{d^2X}{dx^2} + \alpha^2X = 0\ , \quad X(0) = X(\pi) = 0\ .$$
The flow of heat in a thin bar of length π is governed by the heat equation
$$\frac{\partial u}{\partial t} = \kappa\frac{\partial^2u}{\partial x^2}$$
and is subject to the boundary conditions $u(0,t) = u(\pi,t) = 0$, together with the initial condition
$$u(x,0) = \frac{1}{3}\sin(3x) + \sin(5x)$$
Show, using separation of variables, that
$$u(x,t) = \frac{1}{3}\sin(3x)e^{-9\kappa t} + \sin(5x)e^{-25\kappa t}$$

5. The function $Y(x,t)$ satisfies the partial differential equation
$$\frac{\partial^2Y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2Y}{\partial t^2}\ , \quad 0 \le x \le a\ ,$$
and is subject to the boundary conditions that for all t, $Y = 0$ at $x = 0$ and $x = a$. Use the method of separation of variables to show that Y may be written in the form
$$Y(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{a}\right)\left[A_n\cos\left(\frac{n\pi ct}{a}\right) + B_n\sin\left(\frac{n\pi ct}{a}\right)\right]$$
where A_n and B_n are constants.

   Find the values of the constants A_n and B_n, if at $t = 0$ it is known that $Y(x,0) = x^2(a-x)$ and $\frac{\partial Y}{\partial t} = 0$ on the region $0 \le x \le a$.