COMPLEX ANALYSIS LECTURE NOTES

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Contents

1. A review of Elementary Calculus

1.1. **Limits and Continuity.** Calculus is primarily concerned with the behaviour of continuous functions. So we begin by revising some important ideas.

Definition 1.1. We define limit points and limits of functions as follows.

(1) A point x is a limit point of a set $X \subseteq \mathbb{R}$ if there is a sequence $\{x_n\}_{n=1}^{\infty} \subset X$ such that $x_n \to x$. If there is no such sequence, then x is an isolated point.

(2) Let $X \subseteq \mathbb{R}$, $f : X \to \mathbb{R}$ and x_0 a limit point of X. Then L is the limit of f as $x \to x_0$ if and only if, given $\epsilon > 0$, there exists $\delta > 0$ such that $x \in X$, $|x - x_0| < \delta$ implies $|f(x) - L| < \epsilon$.

Limits of functions satisfy the usual arithmetic properties.

Theorem 1.2. *Let $f, g : X \to \mathbb{R}$ be functions and c a constant. If x_0 is a limit point of X and $\lim_{x\to x_0} f(x) = L$ and $\lim_{x\to x_0} g(x) = M$, then*

$$\lim_{x\to x_0} cf(x) = cL \tag{1.1}$$

$$\lim_{x\to x_0} (f(x) + g(x)) = L + M \tag{1.2}$$

$$\lim_{x\to x_0} f(x)g(x) = LM \tag{1.3}$$

$$\lim_{x\to x_0} f(x)/g(x) = L/M, \tag{1.4}$$

provided $M \neq 0$ and g is nonzero.

Proofs of these results are exercises with the triangle inequality and are left to the reader. We can define right and left limits for functions.

Definition 1.3. Let $f : X \to \mathbb{R}$, where $X \subseteq \mathbb{R}$. We say that

$$\lim_{x\to a^+} f(x) = L,$$

if for every $\epsilon > 0$, there exists $\delta > 0$ such that $a < x < a + \delta$ implies $|f(x) - L| < \epsilon$. Similarly we say

$$\lim_{x\to a^-} f(x) = L,$$

if for every $\epsilon > 0$, there exists $\delta > 0$ such that $a - \delta < x < a$ implies $|f(x) - L| < \epsilon$.

An easy result follows.

Proposition 1.4. *Let $f : X \to \mathbb{R}$, where $X \subseteq \mathbb{R}$. Then $\lim_{x\to a} f(x) = L$ if and only if $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L$.*

The proof is an exercise. Finally we define the limit at infinity.

Definition 1.5. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Then $\lim_{x\to\infty} f(x) = L$ if for every $\epsilon > 0$ there exists an $M > 0$ such that $x \geq M$ implies $|f(x) - L| < \epsilon$. Similarly we say that $\lim_{x\to-\infty} f(x) = L$ if for every $\epsilon > 0$ there exists $M < 0$ such that $x \leq M$ implies $|f(x) - L| < \epsilon$.

Having established the essentials about limits of functions, we introduce the crucial idea of continuity.

Definition 1.6. A function $f : X \to \mathbb{R}$ is said to be continuous at x if for any sequence $\{x_n\}_{n=1}^{\infty} \subset X$ which converges to x, we have

$$\lim_{n\to\infty} f(x_n) = f(x).$$

This can be recast in the following form.

Definition 1.7. A function $f : X \to \mathbb{R}$ is continuous at $x \in X$ if for any $\epsilon > 0$, we can find a $\delta_x > 0$ such that $|x - y| < \delta_x$ implies $|f(x) - f(y)| < \epsilon$.

We write δ_x to emphasise the dependence on the point x. So for each x we may require a different δ. If a function is continuous at every point in its domain, we say that it is continuous. The two definitions are clearly equivalent.

Theorem 1.8. *The two definitions of continuity stated above are equivalent.*

Proof. First suppose that f satisfies Definition 1.7. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in X with limit x. Pick $\epsilon > 0$ and $\delta_x > 0$ such that $|x - x_0| < \delta - x$ implies $|f(x) - f(x_0)| < \epsilon$. Since $x_n \to x$ we may find an $N \in \mathbb{N}$ such that $n \geq N$ implies $|x_n - x| < \delta_x$. Then $|f(x_n) - f(x)| < \epsilon$, but this means that $f(x_n) \to f(x)$, so f is continuous according to Definition 1.6.

Suppose that f does not satisfy Definition 1.7. Then we can find $\epsilon > 0$ such that for every $\delta_x > 0$ with $|x - x_0| < \delta_x$ we have $|f(x) - f(x_0)| \geq \epsilon$. Now choose a sequence $\{x_n\}_{n=1}^{\infty}$ in X with limit $x \in X$. Then given $\delta_x > 0$ we may find an $N \in \mathbb{N}$ such that $|x_n - x| < \delta_x$, but $|f(x_n) - f(x)| \geq \epsilon$. So $\{f(x_n)\}_{n=1}^{\infty}$ does not converge to $f(x)$ and thus f is not continuous by Definition 1.6. $\square$

In practice we usually suppress the x subscript, taking it as read. We can also consider functions which are right and left continuous.

Definition 1.9. We say that f is right continuous at x_0 if $\lim_{x\to x_0^+} f(x)$ exists. If $\lim_{x\to x_0^-} f(x)$ exists, then we say that f is left continuous.

A stronger form of continuity is needed when we consider the problem of integration. This is uniform continuity.

Definition 1.10. A function $f : X \to \mathbb{R}$ is said to be uniformly continuous if given $\epsilon > 0$ we can find a $\delta > 0$ such that whenever $|x - y| < \delta$ we have $|f(x) - f(y)| < \epsilon$.

The point here is that unlike ordinary continuity, δ does not depend on x or y. Only on how far apart they are. Uniform continuity implies continuity, but the converse is false. An arbitrary continuous function on the real line need not be uniformly continuous. However, every continuous function is uniformly continuous if we restrict it to a closed and bounded interval. In order to prove this we introduce an equivalent idea.

Definition 1.11. A function $f : X \subseteq \mathbb{R} \to \mathbb{R}$ is sequentially uniformly continuous if given $x_n, y_n \in X$, $y_n - x_n \to 0$ implies $f(y_n) - f(x_n) \to 0$.

The proof of the following is straightforward and we omit it.

Theorem 1.12. *A function $f : X \to \mathbb{R}$ is sequentially uniformly continuous if and only if it is uniformly continuous.*

Now we will prove our previous assertion. It is one of the most important results in analysis and plays an important role in the theory of the Riemann integral.

Theorem 1.13. *A continuous function on a closed bounded interval $[a, b]$ is uniformly continuous.*

Proof. Suppose that f is not uniformly continuous. It therefore cannot be sequentially uniformly continuous. Choose $r \geq 0$ such that for every $\delta > 0$ there exists $x, y \in [a, b]$ such that $|x-y| < \delta$ and $|f(x)-f(y)| > r$.

For each $N \in \mathbb{N}$, choose $x_n, y_n \in [a, b]$ such that

$$|x_n - y_n| < \frac{1}{n}, \text{ and } |f(x_n) - f(y_n)| \geq r.$$

By the Bolzano-Weierstrass Theorem, $\{x_n\}_{n=1}^{\infty}$ has a convergent subsequence $\{x_{n_K}\}_{K=1}^{\infty}$. Suppose that $x_{n_K} \to x$. Since $\{x_{n_K} - y_{n_K}\}_{K=1}^{\infty}$ is a subsequence of $\{x_n - y_n\}_{n=1}^{\infty}$ and $x_n - y_n \to 0$, so $x_{n_K} - y_{n_K} \to 0$. So we have

$$y_{n_K} = x_{n_K} - (x_{n_K} - y_{n_K}) \to x - 0 = x.$$

But f is continuous on $[a, b]$ and hence at x. So $f(x_{n_K}) \to f(x)$. and $f(y_{n_K}) \to f(x)$ and so $f(x_{n_K}) - f(y_{n_K}) \to 0$. But we have assumed that

$$|f(x_{n_K}) - f(y_{n_K})| \geq r > 0, \tag{1.5}$$

for all $K > 0$. We have a contradiction. So f is sequentially uniformly continuous and hence uniformly continuous. □

The fact that continuous functions on closed and bounded intervals are uniformly continuous is essential to many other results. For example, the proof of Riemann's theorem that every continuous function is Riemann integrable requires it. So does the proof of the Fundamental Theorem of Calculus.

We now turn to another of the big results about continuous functions. This is about maxima and minima.

Theorem 1.14. *A continuous function on a closed, bounded interval $[a, b]$ is bounded. Moreover it attains its maximum and minimum values on $[a, b]$.*

Proof. Suppose that f is unbounded. Then given $n \in \mathbb{N}$, n is not a bound for f and thus there exists $x_n \in [a, b]$ such that $|f(x_n)| > n$. However, we know that $[a, b]$ is closed and bounded, and so the sequence $\{x_n\}_{n=1}^{\infty}$ has a convergent subsequence $\{x_{n_K}\}_{K=1}^{\infty}$. Suppose that $x_{n_K} \to x$ as $K \to \infty$. By continuity of f, $f(x_{n_K}) \to f(x)$. But this is impossible, since $f(x_{n_K}) > n_K$ for each K and $n_K \to \infty$, so the sequence $\{f(x_{n_K})\}_{K=1}^{\infty}$ is not convergent, and hence f is not continuous at x. This is a contradiction and we therefore conclude that f is bounded.

Now suppose that $M = \sup_{x\in[a,b]} f(x)$. For each $n \in \mathbb{N}$ choose $x_n \in [a, b]$ such that $f(x_n) > M - 1/n$. Then $f(x_n) \to M$. $\{x_n\}_{n=1}^{\infty}$ is contained in $[a, b]$, so is bounded and hence has a convergent subsequence $\{x_{n_K}\}_{K=1}^{\infty}$. Suppose $x_{n_K} \to c \in [a, b]$. By continuity, $f(x_{n_K}) \to f(c)$. But the sequence $\{f(x_n)\}_{n=1}^{\infty}$ is convergent, so the sequence $\{f(x_{n_K})\}_{K=1}^{\infty}$ has the same limit. Thus $f(c) = M$, so f reaches its maximum. The case for the minimum is similar. □

We also need to mention the intermediate value property. This is the result which tells us that we can solve certain equations.

Theorem 1.15. *Suppose $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and $f(a)f(b) < 0$. Then there is a $c \in [a, b]$ such that $f(c) = 0$.*

Proof. Without loss of generality, we suppose that $f(a) < 0, f(b) > 0$. Let $A = \{x \in [a, b] : f(x) < 0\}$. Then $a \in A$ and so A is nonempty and bounded above. It therefore has a least upper bound, which we we call c. Choose x_n such that $c - 1/n < x_n \leq c$. Then $f(x_n) < 0$. By continuity, $f(c) = \lim_{n\to\infty} f(x_n) \leq 0$. Now take $y_n = c+(b-c)/n$. Then $y_n \to c$ and by continuity $f(c) = \lim_{n\to\infty} f(y_n) \geq 0$. Hence $f(c) = 0$. The case $f(a) > 0$ and $f(b) < 0$ is similar. □

Corollary 1.16. *Let f be continuous on $[a, b]$. Suppose that $f(a) \neq f(b)$ and that M lies between $f(a)$ and $f(b)$. Then there is a $c \in [a, b]$ such that $f(c) = M$.*

Proof. Apply Theorem 1.15 to the function $g(x) = f(x) - M$. □

Definition 1.17. A function is said to be monotone increasing if for each $x \geq y$ we have $f(x) \geq f(y)$. We say that f is monotone decreasing if $f(y) \leq f(x)$.

1.2. **The Derivative.** The derivative is one of the two major tools of calculus. It is the limit of the Newton quotient.

Definition 1.18. A function $f : X \to \mathbb{R}$, where X is open, is said to be differentiable at x if

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}, \tag{1.6}$$

exists. We say that $f'(x)$ is the derivative of f at x. We also write $\frac{df}{dx}$ for f'. The right hand side of (1.6) is called the Newton quotient for f.

An equivalent formulation is

$$f'(x_0) = \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0}. \tag{1.7}$$

Derivatives are defined on open sets. So one talks about a function being differentiable on an open interval (a, b) rather than on $[a, b]$, because the limit in the definition is not necessarily defined at the end points of an interval. The basic rules of differentiation are well known.

Theorem 1.19. *Let c be constant and f, g be differentiable at x_0. Then*

$$(cf)'(x_0) = cf'(x_0) \tag{1.8}$$

$$(f+g)'(x_0) = f'(x_0) + g'(x_0) \tag{1.9}$$

$$(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0) \tag{1.10}$$

Proof. Once more this an exercise manipulating limits. For example, the product rule is proved as follows.

$$\begin{aligned}
(fg)'(x_0) &= \lim_{x\to x_0} \frac{(fg)(x) - (fg)(x_0)}{x - x_0} \\
&= \lim_{x\to x_0} \left[\frac{f(x)g(x) - f(x)g(x_0) + f(x)g(x_0) - f(x_0)g(x_0)}{x - x_0}\right] \\
&= \lim_{x\to x_0} f(x)\frac{g(x) - g(x_0)}{x - x_0} + \lim_{x\to x_0} g(x_0)\frac{f(x) - f(x_0)}{x - x_0} \\
&= f(x_0)g'(x_0) + f'(x_0)g(x_0).
\end{aligned}$$

□

For a function of two variables, we define the partial derivative in a similar way.

$$\frac{\partial f}{\partial x}(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}. \tag{1.11}$$

$$\frac{\partial f}{\partial y}(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}. \tag{1.12}$$

In practice the partial derivative is computed by treating the other variables as fixed and differentiating with respect to the given one.

The next result is easy to prove and will be used in the proof of the chain rule.

Theorem 1.20. *If f is differentiable at a point x, then it is continuous at x.*

Proof. We can write

$$f(x) = (x - x_0)\left(\frac{f(x) - f(x_0)}{x - x_0}\right) + f(x_0) \tag{1.13}$$

Since f is differentiable at x_0, we have

$$\begin{aligned}\lim_{x\to x_0} f(x) &= \lim_{x\to x_0}\left((x - x_0)\left(\frac{f(x) - f(x_0)}{x - x_0}\right) + f(x_0)\right)\\ &= \lim_{x\to x_0}(x - x_0)\lim_{x\to x_0}\frac{f(x) - f(x_0)}{x - x_0} + f(x_0)\\ &= 0 \times f'(x_0) + f(x_0) = f(x_0).\end{aligned}$$

So f is continuous at x_0. □

Again the converse of this result is false. The function $f(x) = |x|$ is continuous at zero, but is not differentiable there. Indeed Weierstrass proved that there are functions which are continuous everywhere, but differentiable nowhere. We will see Weierstrass' nowhere differentiable function later.

The most important result about the derivative is the chain rule.

Theorem 1.21 (The Chain Rule)**.** *Suppose that g is differentiable at x and f is differentiable at $y = g(x)$. Then*

$$(f \circ g)'(x) = f'(y)g'(x). \tag{1.14}$$

Proof. Write $k = g(x + h) - g(x)$. Since g is differentiable at x, it is continuous there and so as $h \to 0$, $k \to 0$. Now

$$\begin{aligned}\frac{f(g(x+h)) - f(g(x))}{h} &= \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)}\frac{g(x+h) - g(x)}{h}\\ &= \frac{f(y+k) - f(y)}{k}\frac{g(x+h) - g(x)}{h}.\end{aligned}$$

Suppose that at no value of h does $k = 0$. Then taking the limit as $h \to 0$ gives the result. To take care of the case $k = 0$ we let

$$F(k) = \begin{cases}\frac{f(y+k)-f(y)}{k} & k \neq 0\\ f'(y) & k = 0.\end{cases} \tag{1.15}$$

By differentiability of f, as $k \to 0$, $F(k) \to f'(y)$ and so F is continuous at 0. Thus as $h \to 0$, $F(k) \to f'(y)$. So for $k \neq 0$

$$\frac{f(g(x+h)) - f(g(x))}{h} = F(k)\frac{g(x+h) - g(x)}{h}. \tag{1.16}$$

This also holds when $k = 0$ since both sides will be zero. Consequently

$$\frac{f(g(x+h)) - f(g(x))}{h} \to f'(y)g'(x) \tag{1.17}$$

as $h \to 0$. $\square$

Example 1.1. Let us compute the derivative of a reciprocal. We have $f(x) = 1/g(x) = h(g(x))$, where $h(u) = 1/u$. Hence

$$\frac{d}{dx} f(x) = g'(x)h'(u) = -\frac{g'(x)}{(g(x))^2}.$$

Example 1.2. The quotient rule is obtained by combining the chain rule and the product rule:

$$\begin{aligned}\frac{d}{dx}\frac{f(x)}{g(x)} &= \frac{f'(x)}{g(x)} + f(x)\frac{d}{dx}\frac{1}{g(x)} \\ &= \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{g(x)^2} \\ &= \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.\end{aligned}$$

The first application of differentiation that we see is usually to the problem of obtaining maxima and minima.

Definition 1.22. A function $f : X \to \mathbb{R}$ has a local maximum at $c \in X$ if there is a subset $Y \subseteq X$ such that $c \in Y$ and $f(c) > f(x)$ for all $x \in Y$. A point c is a local minimum for f if there is a subset $Y \subseteq X$ such that $c \in Y$ and $f(c) < f(x)$ for all $x \in Y$. If f has a local maximum at c, then c is called a maximimiser. If f has a local minimum at c, Then c is called a minimiser. In general c is called an extreme point.

Theorem 1.23. *Let I be an open interval in $\mathbb{R}$, $f : I \to \mathbb{R}$ be differentiable at $c \in I$. If f attains a local maximum or minimum at c, then $f'(c) = 0$.*

Proof. There are two cases to consider, which turn out to be very similar. So we only prove the case for a local maximum. The proof proceeds by contradiction, so we assume that c is a point where f attains a local maximum and that $f'(c) > 0$. Choose $\delta > 0$ such that for $x \in I$ and $0 < |x - c| < \delta$ we have

$$\left|\frac{f(x) - f(c)}{x - c} - f'(c)\right| < f'(c).$$

Pick an $x > c$ with $|x - c| < \delta$. Then we have

$$-f'(c) < \frac{f(x) - f(c)}{x - c} - f'(c) < f'(c).$$

Which implies

$$\frac{f(x) - f(c)}{x - c} > 0$$

and hence $f(x) > f(c)$, which is a contradiction. Thus $f'(c) \leq 0$.

Suppose then that $f'(c) < 0$. Pick a $\delta > 0$ such that for $x \in I$ and $0 < |x - c| < \delta$ we have

$$\left| \frac{f(x) - f(c)}{x - c} - f'(c) \right| < -f'(c).$$

Pick an $x < c$ with $|x - c| < \delta$. Then

$$f'(c) < \frac{f(x) - f(c)}{x - c} - f'(c) < -f'(c).$$

Which implies

$$\frac{f(x) - f(c)}{x - c} < 0,$$

and hence $f(x) > f(c)$, since $x - c < 0$, which is a contradiction once more. Thus $f'(c) = 0$. The proof for a local minimum is essentially the same. $\square$

A useful corollary of this is called Rolle's Theorem.

Theorem 1.24 (Rolle's Theorem)**.** *Let $[a, b]$ be a closed interval in $\mathbb{R}$ and suppose that f is continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b) = 0$ then then there is a point $c \in (a, b)$ such that $f'(c) = 0$.*

Proof. Continuous functions attain their maximum and minimum values on closed bounded intervals. If $c \in (a, b)$ is an extreme point, then $f'(c) = 0$. Suppose that both the maximum and minimum values occur at $[a, b]$. Then since $f(a) = f(b)$, it follows that f is constant and so $f'(x) = 0$ for all $x \in (a, b)$. $\square$

The main applications of Rolle's Theorem are to prove the Mean Value Theorem and Taylor's Theorem, which are two of the most useful results in analysis.

Theorem 1.25 (Mean Value Theorem)**.** *Let $[a, b]$ be a closed and bounded interval on $\mathbb{R}$ and $f : [a, b] \to \mathbb{R}$ a continuous function which is differentiable on (a, b). Then there is a point $c \in (a, b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof. The proof is an application of Rolle's Theorem. We consider the function

$$g(x) = f(x) - f(a) - \left(\frac{f(b) - f(a)}{b - a} \right) (x - a).$$

Then $g(a) = g(b) = 0$ and

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}.$$

By Rolle's Theorem there is a $c \in (a, b)$ with $g'(c) = 0$, which proves the result. $\square$

The MVT is one of the most powerful results in calculus. Let us consider some simple applications. Later we will see it used to prove a result about the behaviour of limits for sequences of derivatives. One can also use it to prove the Fundamental Theorem of Calculus. It is quite ubiquitous.

Corollary 1.26. *If $[a, b]$ is a closed and bounded interval in $\mathbb{R}$ and f is continuous on $[a, b]$ and differentiable on (a, b), then f is Lipschitz continuous on $[a, b]$.*

Proof. For any $x, y \in (a, b)$ the MVT gives, $|f(x) - f(y)| \leq |f'(c)||x - y|$ for some $c \in (x, y)$. $\square$

The following result is well known from high school calculus, but usually is not given a rigourous proof.

Corollary 1.27. *If f is continuous on $[a, b]$ and differentiable on (a, b), and $f'(x) = 0$ for all $x \in (a, b)$, then f is constant on $[a, b]$.*

Proof. For any $x, y \in (a, b)$, $f(x) - f(y) = f'(c)(x - y) = 0$. Hence $f(x) = f(y)$ for all x, y and so f is constant on (a, b). By continuity it is also constant on $[a, b]$. $\square$

Let us use this to prove uniqueness for the solution of a differential equation.

Proposition 1.28. *The equation $y' = ky, y(0) = y_0$ has a unique solution.*

Proof. We let $y(x) = y_0 e^{kx}$. Then this is clearly a solution of the differential equation. Now suppose that f is any solution of the equation. Consider $h(x) = f(x)e^{-kx}$. Then

$$h'(x) = f'(x)e^{-kx} - ke^{-kx}f(x) = e^{-kx}(f'(x) - kf(x)) = 0.$$

Thus h is constant. Hence $f(x) = Ce^{kx}$. The condition that $f(0) = y_0$ completes the proof. $\square$

There is a more general version of the MVT. It is due to Cauchy and is often called the Cauchy Mean Value Theorem. We refer to it by another common name.

Theorem 1.29 (Generalised Mean Value Theorem)**.** *Suppose that f and g are continuous functions on $[a, b]$, which are differentiable on (a, b) and suppose that $g'(x) \neq 0$ for all $x \in (a, b)$. Then there exists a point $c \in (a, b)$ such that*

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}. \tag{1.18}$$

Proof. This again relies upon Rolle's Theorem. First, observe that if $g(b) - g(a) = 0$, then the Mean Value Theorem tells us that there exists a point $c \in (a, b)$ such that $g'(c) = 0$. However we have assumed that g' is nonzero, so $g(b) - g(a) \neq 0$. Next introduce the function

$$h(x) = f(x)[g(b) - g(a)] - g(x)[f(b) - f(a)].$$

Then

$$\begin{aligned} h(a) &= f(a)[g(b) - g(a)] - g(a)[f(b) - f(a)] \\ &= f(a)g(b) - f(b)g(a) = h(b). \end{aligned}$$

Rolle's Theorem then tells us that there is a $c \in (a, b)$ such that $h'(c) = 0$. Which means that

$$f'(c)[g(b) - g(a)] - g'(c)[f(b) - f(a)] = 0. \tag{1.19}$$

Rearranging gives the result. □

As an application of this result we prove L'Hôpital's rule.

Theorem 1.30. *Suppose that f and g are differentiable on (a, b) and that $g(x) \neq 0$ and $g'(x) \neq 0$ for all $x \in (a, b)$. Suppose further that $\lim_{x \to a^+} f(x) = \lim_{x \to a^+} g(x) = 0$. Then,*

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lim_{x \to a^+} \frac{f'(x)}{g'(x)}, \tag{1.20}$$

provided the right side exists.

Proof. Suppose that

$$\lim_{x \to a^+} \frac{f'(x)}{g'(x)} = L.$$

Then given $\epsilon > 0$ we can find $\delta > 0$ such that if $c \in (a, a + \delta)$ then

$$\left| \frac{f'(c)}{g'(c)} - L \right| < \epsilon.$$

However, by the generalised MVT, if $x \in (a, a + \delta)$ then

$$\left| \frac{f(x)}{g(x)} - L \right| = \left| \frac{f(x) - f(a)}{g(x) - g(a)} - L \right| < \epsilon.$$

□

The extension of this result to the case when

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = \infty$$

can also be established using the generalised MVT.

Remark 1.31. L'Hôpital's rule was actually discovered by the Swiss mathematician Johann Bernoulli, who taught Euler and worked for L'Hôpital. L'Hôpital published the rule in his textbook on calculus, and it became known by his name.

1.2.1. *Inverse Functions.* We first state our definitions.

Definition 1.32. A function $f : X \to Y$ is said to be one to one if for each $y \in Y$ there is at most one $x \in X$ such that $f(x) = y$. We also say that such an f is a bijection. If $f : X \to Y$ is one to one then it has an inverse function $f^{-1} : Y \to X$ which satisfies

$$f(f^{-1}(f)) = f^{-1}(f(x)) = x$$

for all $x \in X$.

Suppose that $f : X \subseteq \mathbb{R} \to \mathbb{R}$ is strictly increasing (or decreasing). Then f is clearly one to one, and hence it has an inverse. If f is continuous, then the inverse function will also be continuous.

Theorem 1.33. *Suppose that $f : X \subseteq \mathbb{R} \to Y$ is a strictly increasing (or decreasing) continuous function. Then the inverse function f^{-1} exists and is continuous and increasing (or decreasing) on $f(X)$.*

Proof. We only deal with the case when f is increasing. We show that f^{-1} is increasing. Assume not. Then we can find $y_1, y_2 \in Y$ with $y_2 > y_1$ and $f^{-1}(y_2) < f^{-1}(y_1)$. But f is increasing, so

$$f(f^{-1}(y_2)) < f(f^{-1}(y_1)),$$

so that $y_2 < y_1$ which is a contradiction.

To prove continuity, take $y_0 \in f(X)$. Then there exists $x_0 \in X$ with $f(x_0) = y_0$. We suppose that y_0 is not an endpoint, so x_0 is not an endpoint and we may find $\epsilon_0 > 0$ such that the interval

$$(f^{-1}(y_0) - \epsilon_0, f^{-1}(y_0) + \epsilon_0) \subset X.$$

Pick $\epsilon < \epsilon_0$. Then there exist $y_1, y_2 \in f(X)$ such that $f^{-1}(y_1) = f^{-1}(y_0) - \epsilon$ and $f^{-1}(y_2) = f^{-1}(y_0) + \epsilon$. Because f is increasing $y_1 < y_0 < y_2$ and the inverse is increasing so for all $y \in (y_1, y_2)$ we have the inequality

$$f^{-1}(y_0) - \epsilon < f^{-1}(y) < f^{-1}(y_0) + \epsilon.$$

Consequently, if $\delta = \min\{y_2 - y_0, y_0 - y_1\}$, then

$$|f^{-1}(y_0) - f^{-1}(y)| < \epsilon$$

whenever $|y_0 - y| < \delta$. So f^{-1} is continuous at y_0.

We can also prove that if y_0 is a left (or right) endpoint, then f^{-1} is left (or right) continuous at y_0. $\square$

The most important result about inverse functions relates the derivative of f and that of f^{-1}.

Theorem 1.34 (The Inverse Function Theorem)**.** *Suppose that f is differentiable and one to one on an open interval I. If $f'(a) \neq 0$, $a \in I$, then f^{-1} exists and is differentiable at $f(a)$ and*

$$\left(f^{-1}\right)'(f(a)) = \frac{1}{f'(a)}.$$

Proof. Since f' is nonzero on I, it follows that f is either increasing or decreasing on I, and hence f is invertible. The inverse is continuous. Since f is decreasing or increasing, for $x \neq a$ it follows that $f(x) \neq f(a)$. Now

$$\begin{aligned}
\lim_{y \to f(a)} \frac{f^{-1}(y) - f^{-1}(f(a))}{y - f(a)} &= \lim_{f(x) \to f(a)} \frac{f^{-1}(f(x)) - f^{-1}(f(a))}{f(x) - f(a)} \\
&= \lim_{x \to a} \left(\frac{x - a}{f(x) - f(a)} \right)^{-1} \\
&= \frac{1}{f'(a)}.
\end{aligned}$$

□

1.3. **Power Series and Taylor Expansions.** A power series about a point x_0 is an expression of the form

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

By the ratio test such a series will converge if

$$\lim_{n \to \infty} \left| \frac{a_{n+1}(x - x_0)^{n+1}}{a_n (x - x_0)^n} \right| = L < 1.$$

Upon rewriting this becomes

$$|x - x_0| \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1. \tag{1.21}$$

We can think of this as determining the values of x for which the series converges.

Definition 1.35. Suppose that for the series $\sum_{n=0}^{\infty} a_n (x - x_0)^n$

$$|x - x_0| \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1 \tag{1.22}$$

for all $|x - x_0| < R$. We call R the radius of convergence of the power series.

Note a power series with radius of convergence R may converge or diverge when $|x - x_0| = R$. One has to check convergence at the end points individually.

Example 1.3. The series $1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n$ is convergent for all $|x| < 1$. Hence the radius of convergence is 1.

For simplicity we will take $x_0 = 0$ in what follows. All results can be transferred to the more general case by making the replacement $x \to x - x_0$

Power series have very nice properties. In particular they converge absolutely within their radius of convergence.

Theorem 1.36. *Let $\sum_{n=o}^{\infty} a_n x^n$ be a power series with radius of convergence R. Then the series converges absolutely for $|x| < R$ and diverges for $|x| > R$.*

Proof. Let $t \in (-R, R)$, then $\sum_{n=0}^{\infty} a_n t^n$ converges and the sequence $a_n t^n \to 0$ and is thus bounded. Let M be a bound. Now pick x with $|x| < |t|$, then

$$|a_n x^n| = |a_n t^n| \left|\frac{x}{t}\right|^n \leq M r^n$$

where $r = |x/t| < 1$. But $\sum_{n=0}^{\infty} M r^n$ is a convergent geometric series, and so $\sum_{n=0}^{\infty} |a_n x^n|$ converges by the comparison test. The other result is similar. □

Power series actually converge uniformly, a result we prove later. An important fact is that we can differentiate power series term by term and this does not change the radius of convergence.

Theorem 1.37. *Let $\sum_{n=0}^{\infty} a_n x^n$ have radius of convergence R. Then the power series $\sum_{n=1}^{\infty} n a_n x^{n-1}$ has radius of convergence R.*

Proof. Suppose that the series $\sum_{n=1}^{\infty} n a_n x^{n-1}$ has radius of convergence $R_d < R$. Choose r, s so that $R_d < r < s < R$. Clearly $\sum_{n=0}^{\infty} a_n s^n$ converges which shows that $a_n s^n \to 0$ and so is bounded by a constant M. Then

$$\begin{aligned} |n a_n r^{n-1}| &= n|a_n| s^{n-1} \left(\frac{r}{s}\right)^{n-1} \\ &\leq \frac{M}{s} n \left(\frac{r}{s}\right)^{n-1}. \end{aligned}$$

Now

$$\lim_{n\to\infty} \frac{(M/s)(n+1)(r/s)^n}{(M/s)n(r/s)^{n-1}} = \frac{r}{s} < 1.$$

Thus the series $\sum_{n=1}^{\infty} \frac{M}{s} n \left(\frac{r}{s}\right)^{n-1}$ is convergent by the ratio test. Thus $\sum_{n=1}^{\infty} n a_n r^{n-1}$ is absolutely convergent, which is a contradiction since $r > R_d$. Hence $R \leq R_d$. Similarly we show that $R_d > R$ leads to a contradiction. (Exercise). Hence $R = R_d$. □

From this we can establish an important corollary.

Theorem 1.38. *Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $R > 0$. Let $f : (-R, R) \to \mathbb{R}$ be defined by*

$$f(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Then f is differentiable on $(-R, R)$ and

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

To prove this result we actually need some more information about the convergence of series. The key is that the series for f and f' both converge uniformly. We will discuss uniform convergence later.

The most commonly encountered power series are functions given by Taylor series expansions.

Definition 1.39. Let f be smooth in a neighbourhood X of a point a. We let the Taylor series for f at a be given by

$$T_f(x) = f(a) + f'(a)(x-a) + \cdots + \frac{1}{n!} f^{(n)}(a)(x-a)^n + \cdots .$$

If the series is convergent for all $x \in X$ and $|T_f(x) - f(x)| = 0$ for all $x \in X$, we say that f is analytic at a. If we truncate the Taylor expansion after n terms, the resulting expression is known as the nth Taylor polynomial.

Even if the Taylor series does not converge, smooth functions can be approximated by Taylor polynomials.

Theorem 1.40 (Taylor's Theorem). *Let I be an open interval in $\mathbb{R}$, $n \in \mathcal{N}$ and $f \in C^{n+1}(I)$. Let $a \in I$ and $x \in I$, with $x \neq a$. Then there is a point ξ between a and x such that*

$$\begin{aligned} f(x) = f(a) + f'(a)(x-a) + \frac{1}{2!} f''(a)(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \\ + \frac{f^{(n+1)}(\xi)}{n!}(x-a)^n. \end{aligned}$$

Proof. The proof uses Rolle's Theorem and is conceptually similar to the proof of the MVT. We define a function

$$\begin{aligned} F(t) = f(x) - f(t) - f'(t)(x-t) - \frac{1}{2!} f'(t)(x-t)^2 - \cdots \\ - \frac{f^{(n)}(t)}{n!}(x-t)^n. \end{aligned} \tag{1.23}$$

Plainly $F(x) = 0$. Since $f \in C^{(n+1)}(I)$ we see that F is differentiable. Now

$$\begin{aligned} F'(t) &= -f'(t) - f''(t)(x-t) + f'(t) - \frac{f'''(t)}{2!}(x-t)^2 + 2\frac{f''(t)}{2!}(x-t) \\ &\quad - \cdots - \frac{f^{(n+1)}t}{n!}(x-t)^n + n\frac{f^{(n)}(t)}{n!}(x-t)^{n-1} \\ &= -\frac{f^{(n+1)}}{n!}(x-t)^n. \end{aligned}$$

Next we introduce the function

$$G(t) = F(t) - \left(\frac{x-t}{x-a}\right)^{n+1} F(a).$$

Obviously $G(a) = 0$ and $G(x) = F(x) = 0$. Then

$$\begin{aligned} G'(t) &= F'(t) \\ &= -\frac{f^{(n+1)}(t)}{n!}(x-t)^n + (n+1)\frac{(x-t)^n}{(x-a)^{n+1}}F(a). \end{aligned}$$

By Rolle's Theorem there is a point ξ between x and a such that $G'(\xi) = 0$. That is

$$\frac{f^{(n+1)}(\xi)}{n!}(x-\xi)^n = (n+1)\frac{(x-\xi)^n}{(x-a)^{n+1}}F(a).$$

Rearranging we get

$$F(a) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}.$$

If we substitute this into (1.23) we have Taylor's Theorem.

□

The other major tool in analysis is the integral. Although the Fundamental Theorem of Calculus was first stated by Newton and Leibnitz, the first rigorous theory of integration was developed by Cauchy, and extended by Riemann. Let us briefly summarise Riemann's theory.

1.4. **The Riemann Integral.** We take an interval $[a, b]$ and partition it as

$$\mathcal{P} = \{x_0, x_1, ..., x_n\},$$

where $x_0 = a$, $x_0 < x_1 < \cdots < x_n$ and $x_n = b$.

Now let f be a bounded function on $[a, b]$ then define

$$M_i = \sup\{f(x) : x \in [x_{i-1}, x_i)\},$$

and

$$m_i = \inf\{f(x) : x \in [x_{i-1}, x_i)\}.$$

We then form the upper and lower Riemann sums

$$U(f, \mathcal{P}) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}) \tag{1.24}$$

and

$$L(f, \mathcal{P}) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}). \tag{1.25}$$

The least upper bound axiom establishes that the upper and lower integrals

$$\overline{\int_a^b} f = \inf\{U(f, \mathcal{P}) : \mathcal{P} \text{ a partition of } [a, b]\} \tag{1.26}$$

and

$$\underline{\int_a^b} f = \sup\{U(f, \mathcal{P}) : \mathcal{P} \text{ a partition of } [a, b]\} \tag{1.27}$$

both exist. We then say that f is Riemann integrable on $[a, b]$ if $\overline{\int_a^b} f = \underline{\int_a^b} f$. The Riemann integral is then equal to the upper (or lower) integral.

It is easy to prove the following results.

Proposition 1.41. *The Riemann integral has the following properties.*

(1) *If c is a constant, $\int_a^b c dx = c(b - a)$.*

(2) $\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx.$

The most important results about the Riemann integral are as follows.

Theorem 1.42 (Riemann's Criterion)**.** *Let f be a bounded function on the closed interval $[a, b]$. Then f is Riemann integrable on $[a, b]$ if and only if, given any $\epsilon > 0$, there exists a partition $\mathcal{P}$ of $[a, b]$ such that $U(f, \mathcal{P}) - L(f, \mathcal{P}) < \epsilon$.*

From this one establishes the first major result.

Theorem 1.43. *Every continuous function on a closed bounded interval $[a, b]$ is Riemann integrable.*

Proof. The function f is continuous on $[a, b]$ and so is bounded. Let $\epsilon > 0$. Since f is continuous it is uniformly continuous and so we can choose $\delta > 0$ such that $x, y \in [a, b]$ with $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon/(b - a)$. Now choose $N \in \mathbb{N}$ such that $N > (b - a)/\delta$. For each $i = 0, 1, ...N$, let $x_i = a + (b - a)i/N$. Then $\mathcal{P} = \{x_0, x_1, ...x_N\}$ is a partition of $[a, b]$, with $|x_i - x_{i-1}| < \delta$. By continuity, f attains its maximum and minimum values on each closed subinterval $[x_{i-1}, x_i]$. Now let

$$f(c_i) = \inf\{f(x) : x \in [x_{i-1}, x_i]\}, \tag{1.28}$$

$$f(d_i) = \sup\{f(x) : x \in [x_{i-1}, x_i]\}. \tag{1.29}$$

Obviously $|d_i - c_i| < \delta$ and $f(d_i) \geq f(c_i)$. By uniform continuity

$$f(d_i) - f(c_i) < \frac{\epsilon}{(b - a)}.$$

So we have

$$\begin{aligned} U(f,\mathcal{P})-L(f,\mathcal{P}) &= \sum_{i=1}^{N} f(d_i)(x_i-x_{i-1}) - \sum_{i=1}^{N} f(c_i)(x_i-x_{i-1}) \\ &= \sum_{i=1}^{N} (f(d_i)-f(c_i))(x_i-x_{i-1}) \\ &< \sum_{i=1}^{N} \frac{\epsilon}{b-a}(x_i-x_{i-1}) \\ &= \frac{\epsilon}{b-a}\sum_{i=1}^{N}(x_i-x_{i-1}) = \epsilon. \end{aligned}$$

Thus by Riemann's criterion, f is integrable on $[a,b]$.

$\square$

1.4.1. *Calculating Integrals By Riemann Sums.* It is possible to explicitly compute a surprisingly large class of integrals by evaluating Riemann sums. For monotone functions, the construction of upper and lower sums is straightforward. One simply picks sample points at the ends of each subinterval. We restrict our attention to $[0,1]$. We can extend to the interval $[a,b]$ by a linear change of variable.

Example 1.4. We integrate $f(x)=x^2$ on $[0,1]$ Since f is increasing we can take $\mathcal{P}=\{0,1/n,2/n,...,n/n\}$ and note that

$$\sum_{i=1}^{n} i^2 = \frac{1}{6}n(n+1)(2n+1). \tag{1.30}$$

Now we observe that

$$\begin{aligned} m_i(f,\mathcal{P}) &= \inf\{x^2 : x\in[\frac{i-1}{n},\frac{i}{n})\} \\ &= \frac{(i-1)^2}{n^2} \\ M_i(f,\mathcal{P}) &= \sup\{x^2 : x\in[\frac{i-1}{n},\frac{i}{n})\} \\ &= \frac{i^2}{n^2}. \end{aligned}$$

Then

$$\begin{aligned} L(f,\mathcal{P}) &= \sum_{i=1}^{n} \frac{(i-1)^2}{n^2}\left(\frac{i}{n}-\frac{(i-1)}{n}\right) \\ &= \frac{1}{n^3}\sum_{i=1}^{n}(i-1)^2. \end{aligned}$$

Also

$$U(f,\mathcal{P}) = \sum_{i=1}^{n} \frac{i^2}{n^2}\left(\frac{i}{n} - \frac{(i-1)}{n}\right)$$
$$= \frac{1}{n^3}\sum_{i=1}^{n} i^2.$$

Using (1.30) we get

$$U(f,\mathcal{P}) - L(f,\mathcal{P}) = \frac{1}{6}n(n+1)(2n+1)\frac{1}{n^3} - \frac{(n-1)n(2n-1)}{6n^3} = \frac{1}{n}.$$

By Riemann's Criterion, f is Riemann integrable if for any $\epsilon > 0$ we can find a partition $\mathcal{P}$ such that $U(f,\mathcal{P}) - L(f,\mathcal{P}) < \epsilon$. Clearly we can do this by taking $n > 1/\epsilon$. So f is Riemann integrable. Further

$$\int_0^1 f(x)dx = \sup\{L(f,\mathcal{P}), \mathcal{P} \text{ a partition of } [0,1]\}$$
$$= \sup_{n\geq 1}\left\{\frac{(n-1)n(2n-1)}{6n^3}\right\}$$
$$= \sup_{n\geq 1}\left\{\frac{1}{6n^2} - \frac{1}{2n} + \frac{1}{3}\right\} = \frac{1}{3}.$$

Example 1.5. Let $a \neq 0$ and consider $f(x) = e^{ax}$ on $[0,1]$. The function is monotone and we take the same partition as in the previous example. Then

$$m_k(f,\mathcal{P}) = \inf\left\{e^{ax} : x \in \left[\frac{k-1}{n}, \frac{k}{n}\right)\right\} = e^{(k-1)a/n} \tag{1.31}$$
$$M_k(f,\mathcal{P}) = \sup\left\{e^{ax} : x \in \left[\frac{k-1}{n}, \frac{k}{n}\right)\right\} = e^{ka/n} \tag{1.32}$$

Then

$$L(f,\mathcal{P}) = \sum_{k=1}^{n} m_k(f,\mathcal{P})(x_k - x_{k-1})$$
$$= \frac{1}{n}(1 + e^{a/n} + \cdots + e^{(n-1)a/n})$$

and

$$U(f,\mathcal{P}) = \sum_{k=1}^{n} M_k(f,\mathcal{P})(x_k - x_{k-1})$$
$$= \frac{1}{n}(e^{a/n} + e^{2a/n} + \cdots + e^{an/n}).$$

So

$$U(f,\mathcal{P}) - L(f,\mathcal{P}) = \frac{e^a - 1}{n}.$$

This can be made smaller than ϵ by picking $n > \epsilon/(e^a - 1)$. Thus by Riemann's Criterion, f is Riemann integrable on $[0, 1]$. We can explicitly evaluate the upper and lower sums by noticing that they are sums of geometric progressions with common ratio e^a. Hence

$$
\begin{aligned}
L(f, \mathcal{P}) &= \frac{1}{n}(1 + e^{a/n} + \cdots + e^{(n-1)a/n}) \\
&= \frac{1}{n}\frac{(1 - e^a)}{(1 - e^{a/n})}.
\end{aligned}
$$

So we have

$$
\begin{aligned}
\int_0^1 e^{ax} dx &= \sup_n \left\{ \frac{1}{n}\frac{(1 - e^a)}{(1 - e^{a/n})} \right\} \\
&= \lim_{u \to 0} \frac{u(1 - e^a)}{1 - e^{au}} \\
&= \frac{1}{a}(e^a - 1)
\end{aligned}
$$

where we put $u = 1/n$ and used L'Hôpital's rule to evaluate the limit.

We can actually prove that bounded monotone functions are Riemann integrable.

Theorem 1.44. *Suppose that $f : [0, 1] \to \mathbb{R}$ is a bounded monotone increasing function. Then f is Riemann integrable on $[0, 1]$.*

Proof. With the previous partition of $[0, 1]$ we have, using the monotonicity of f,

$$
U(f, \mathcal{P}) - L(f, \mathcal{P}) = \frac{1}{n}(f(1) - f(0)). \tag{1.33}
$$

Since f is bounded, then $f(0)$ and $f(1)$ are finite, we can make this smaller than any $\epsilon > 0$ by suitable choice of n. So f is Riemann integrable. $\square$

It is possible to evaluate many integrals by means of Riemann sums-in particular, we can integrate any polynomial- but it is clearly a laborious procedure. Fortunately we have a far more powerful means of doing integration. The key is the following result, which is at the heart of modern science.

Theorem 1.45 (Fundamental Theorem of Calculus)**.** *If f is a continuous function on $[a, b]$, then for all $x \in [a, b]$*

$$
\frac{d}{dx}\int_a^x f(t)dt = f(x).
$$

Proof. We define the function $F(x) = \int_a^x f(t)dt$. Since f is continuous, it is bounded. Thus there is an $M > 0$ such that $|f(t)| \leq M$ for all

$t \in [a, b]$. Then

$$\begin{aligned}|F(x) - F(y)| &= \left|\int_y^x f(t)dt\right| \\ &\leq \int_y^x |f(t)|dt \\ &\leq M|x - y|.\end{aligned}$$

Consequently, F is Lipschitz continuous on $[a, b]$ and hence continuous. Now

$$\begin{aligned}\frac{F(x) - F(y)}{x - y} - f(y) &= \frac{1}{x - y}\left(F(x) - F(y) - (x - y)f(y)\right) \\ &= \frac{1}{x - y}\int_y^x (f(t) - f(y))dt.\end{aligned}$$

By uniform continuity of f, given $\epsilon > 0$, we may find $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$. We choose such an ϵ and δ to obtain

$$\begin{aligned}\left|\frac{F(x) - F(y)}{x - y} - f(y)\right| &\leq \frac{1}{|x - y|}\int_y^x |f(t) - f(y)|dt \\ &< \frac{1}{|x - y|}\epsilon(x - y) = \epsilon\end{aligned}$$

as $x > y$. Thus F is differentiable and $F' = f$. □

In other words, integration is essentially the inverse of differentiation. From this we can establish the well known second form of the fundamental theorem.

Corollary 1.46 (The Fundamental Theorem of Calculus II)**.** *Let f be a Riemann integrable function on $[a, b]$. Then if $F' = f$ on (a, b) the integral is given by*

$$\int_a^b f(x)dx = F(b) - F(a). \tag{1.34}$$

Proof. Suppose that $G(x) = \int_a^x f(t)dt$ and $F'(x) = f(x)$. It follows that $G - F$ is a constant, since $G' = f$. Hence $G(b) - F(b) = G(a) - F(a)$. But $G(a) = 0$. Hence $G(b) = \int_a^b f(x)dx = F(b) - F(a)$. □

There is a mean value theorem for the Riemann integral which is often useful.

Theorem 1.47 (Mean Value Theorem for Integrals)**.** *Suppose that f and g are continuous on $[a, b]$ and $g(x) \geq 0$, for all $x \in [a, b]$. Then there exists $c \in [a, b]$ such that*

$$\int_a^b f(x)g(x)dx = f(c)\int_a^b g(x)dx. \tag{1.35}$$

Proof. By continuity f is bounded. Suppose that for all $t \in [a,b]$ $m \le f(t) \le M$. Then

$$m\int_a^b g(x)dx \le \int_a^b f(x)g(x)dx \le M\int_a^b g(x)dx.$$

Let $F(t) = f(t)\int_a^b g(t)dt$. By the Intermediate Value Theorem, there is a $c \in [a,b]$ such that

$$F(c) = f(c)\int_a^b g(t)dt = \int_a^b f(x)g(x)dx.$$

□

Notice that if $g = 1$ and $F' = f$ then we have the existence of a $c \in [a,b]$ such that

$$\int_a^b f(x)dx = F(b) - F(a) = F'(c)(b-a) \tag{1.36}$$

which is the mean value theorem. Actually the mean value theorem can be used to prove the fundamental theorem of calculus. This is an exercise.

1.4.2. *Integration Rules.* Integration is intrinsically more difficult than differentiation. Useful rules for evaluating integrals exist however. Integration by parts is simply the product rule of differentiation backwards. Specifically

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x).$$

Integrating both sides gives the integration by parts rule

$$\int_a^b f(x)g'(x)dx = [f(x)g(x)]_a^b - \int_a^b f'(x)g(x)dx. \tag{1.37}$$

The most important technique for evaluating integrals is the use of substitutions. This is the chain rule in reverse. The chain rule says that $(f\circ g)'(x) = f'(g(x))g'(x)$. Thus letting $u = g(x)$ gives

$$\int_a^b f'(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du. \tag{1.38}$$

We can use integration by parts to show how Taylor's Theorem follows from the Fundamental Theorem of Calculus. Assume that f is continuously differentiable $n+1$ times. We know that

$$f(x) - f(a) = \int_a^x f'(t)dt. \tag{1.39}$$

We are going to integrate by parts. Notice however that instead of using the obvious anti-derivative of 1, we are going to use $t - x$, which is also an anti-derivative of 1. So that

$$\begin{aligned} f(x) - f(a) &= [(t-x)f'(t)]_a^x - \int_a^x (t-x)f'(t)dt \\ &= (x-a)f'(a) + \int_a^x (x-t)f'(t)dt \\ &= (x-a)f'(a) + \frac{(x-a)^2}{2}f''(x) + \frac{1}{2}\int_a^x (x-t)^2 f''(t)dt. \end{aligned}$$

Repeating this n times gives

$$\begin{aligned} f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2}f''(x) + \cdots \\ + \frac{1}{n!}(x-a)^n f^{(n)}(a) + \frac{1}{n!}\int_a^x (x-t)^n f^{(n)}(t)dt. \end{aligned}$$

This gives us the useful form for the remainder in the Taylor series expansion

$$R_n(a,x) = \frac{1}{n!}\int_a^x (x-t)^n f^{(n)}(t)dt.$$

Using the mean value theorem for integrals we can show that this is the same as the derivative form we found earlier.

1.4.3. *Improper Riemann Integrals.* It is often the case that we wish to consider an integral of a function over a set where the function is discontinuous.

Definition 1.48. Let $f : [a,b] \to \mathbb{R}$ be continuous on $(a,b]$, but f is discontinuous at a. Then the improper Riemann integral of f over $[a,b]$ is define by

$$\int_a^b f(x)dx = \lim_{X\to a}\int_X^b f(x)dx, \tag{1.40}$$

provided the limit exists. Similarly, if the discontinuity is at $x = b$ then

$$\int_a^b f(x)dx = \lim_{X\to b}\int_a^X f(x)dx, \tag{1.41}$$

provided the limit exists.

Example 1.6. Consider $f(x) = 1/\sqrt{x}$ on $[0,1]$. Then f is continuous on $(0,1]$ with a discontinuity at 0. Thus the improper Riemann integral of f over $[0,1]$ is

$$\begin{aligned}\int_0^1 f(x)dx &= \lim_{X\to 0}\int_X^1 \frac{dx}{\sqrt{x}}\\ &= \lim_{X\to 0} 2\sqrt{x}]_X^1\\ &= \lim_{X\to 0}(2\sqrt{1}-\sqrt{X}) = 2.\end{aligned}$$

For integrals on unbounded domains we can use the same idea.

Definition 1.49. The improper Riemann integral of f over $\mathbb{R}$ is defined by

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty}\int_{-R}^{0} f(x)dx + \lim_{T\to\infty}\int_0^T f(x)dx, \tag{1.42}$$

provided the limits exist.

One has to be careful to distinguish between Definition 1.49 and the Cauchy Principal value.

Definition 1.50. The quantity

$$\text{P.V.}\int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty}\int_{-R}^{R} f(x)dx,$$

is known as the Cauchy Principal value of the integral, provided that the limit exists.

Example 1.7. The improper Riemann integral $\int_{-\infty}^{\infty} xdx$ does not exist, but

$$\begin{aligned}\text{P.V.}\int_{-\infty}^{\infty} xdx &= \lim_{R\to\infty}\left[\frac{x^2}{2}\right]_{-R}^{R}\\ &= \frac{1}{2}(R^2-R^2) = 0.\end{aligned}$$

1.5. **Sequences of Functions.** It is very common to have to deal with a sequence of functions. For example, a power series with partial sums $f_n(x) = \sum_{k=0}^{n} a_k x^k$ defines a sequence of functions. An important question is what happens as $n\to\infty$. Let us introduce the notion of convergence of a sequence of functions.

Definition 1.51. We say that a sequence of functions $\{f_n\}_{n=1}^{\infty}$ converges pointwise to a function f on a set $X\subseteq\mathbb{R}$ if $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x\in X$.

A much stronger type of convergence is uniform convergence.

Definition 1.52. A sequence of functions $\{f_n\}_{n=1}^{\infty}$ on a set $X\subseteq\mathbb{R}$ converges uniformly to f on X if for any $\epsilon>0$ we can find $N\in\mathbb{N}$ such that $n\geq N$ implies $|f_n(x)-f(x)|<\epsilon$ for all $x\in X$.

The first result is a trivial exercise.

Lemma 1.53. *If $f_n \to f$ uniformly on X, then $f_n \to f$ pointwise.*

The converse of this result is false. Pointwise convergent sequences do not have to converge uniformly. Uniformly convergent sequences have nice properties. An important one is that they preserve continuity. Pointwise convergence does *not* do this.

Theorem 1.54. *If $\{f_n\}_{n=1}^{\infty}$ is a uniformly convergent sequence of continuous functions on $X \subseteq \mathbb{R}$, with $f_n \to f$ then f is continuous on X.*

Proof. Since f_k is continuous at $x \in X$, given $\epsilon > 0$, we may choose $\delta > 0$ such that for all y satisfying $0 < |x - y| < \delta$ we have

$$|f_k(x) - f_k(y)| < \epsilon/3.$$

By uniform convergence, we may choose $N \in \mathcal{N}$ such that $k \geq N$ implies

$$|f(x) - f_k(x)| < \epsilon/3,$$

for all $x \in X$. Consequently, given $x \in X$, then for all $y \in X$ satisfying $0 < |x - y| < \delta$ we have

$$\begin{aligned}|f(x) - f(y)| &= |f(x) - f_k(x) + f_k(x) - f_k(y) + f_k(y) - f(y)| \\ &\leq |f(x) - f_k(x)| + |f_k(x) - f_k(y)| + |f_k(y) - f(y)| \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon.\end{aligned}$$

Thus f is continuous at x. □

There are various tests for uniform convergence. For series we have the following powerful result.

Theorem 1.55 (Weierstrass M-Test)**.** *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions on X such that $|f_n(x)| \leq M_n$ all $x \in X$ and $\sum_{n=1}^{\infty} M_n < \infty$. Then the series $\sum_{n=1}^{\infty} f_n(x)$ is uniformly convergent.*

Proof. Let $S_N(x) = \sum_{n=1}^{\infty} f_n(x)$ and suppose that $|f_n(x)| \leq M_n$. Then for all $N \geq M$

$$\begin{aligned}|S_N(x) - S_M(x)| &= |\sum_{n=M+1}^{N} f_n(x)| \\ &\leq \sum_{n=M+1}^{N} |f_n(x)| \\ &\leq \sum_{n=M+1}^{N} M_n \to 0,\end{aligned}$$

as $N, M \to \infty$. So the series S_N converges independently of x and hence is uniformly convergent. □

Example 1.8. The M test is generally easy to use. To illustrate, consider the series

$$f(x)=\sum_{n=1}^{\infty}\frac{\cos(nx)}{n^2+1}. \tag{1.43}$$

Letting $f_n(x)=\dfrac{\cos(nx)}{n^2+1}$, we immediately see that

$$|f_n(x)|\leq\frac{1}{n^2+1}, \tag{1.44}$$

and by the comparison test $\sum_{n=1}^{\infty}\frac{1}{n^2+1}<\infty$. Hence the series (1.43) is uniformly convergent and so f is a continuous function.

As an application we prove a result about power series.

Theorem 1.56. *Let $\sum_{n=0}^{\infty}a_nx^n$ be a power series with radius of convergence R. Let $0<r<R$. Then the series converges uniformly on $[-r,r]$.*

Proof. Let $\sum_{n=0}^{\infty}a_nx^n$ be convergent for $|x|<R$. Then it is absolutely convergent. Pick $x=x_0>r$ and $x_0<R$ and we have $\sum_{n=0}^{\infty}a_nx_0^n$ is convergent, hence $a_nx_0^n\to 0$. So there is an $M>0$ such that $|a_nx_0^n|\leq M$. Then

$$\begin{aligned}|a_nx^n|&\leq|a_n|r^n\\&=|a_nx_0^n|\left|\frac{r}{x_0}\right|^n\\&\leq M\left|\frac{r}{x_0}\right|^n.\end{aligned}$$

Now $\sum_{n=0}^{\infty}M\left|\frac{r}{x_0}\right|^n$ converges and hence the power series converges uniformly by the Weierstrass M test. □

It is important to note that this theorem does not say that a power series converges uniformly on $(-R,R)$. Indeed the series $\sum_{n=0}^{\infty}x^n$ converges on $(-1,1)$ but the convergence is not uniform. It does converge uniformly on $[-r,r]$ for $r<1$. The point is we cannot necessarily extend the uniform convergence to the entire interval of convergence. We will make use of the M-test later.

2. The Origins of Complex Analysis

The purpose of these notes is to introduce the main ideas of complex variable theory. Techniques for dealing with functions of a complex variable are among the most important in mathematics.

The French mathematician Jacques Hadamard once said that the shortest distance between two points on the real line often passes through the complex plane. What he meant by that remark was that introducing complex variables often makes problems with real variables much easier to handle.

Complex variable theory is one of the most important branches in mathematics. The range of applications is limitless. It is even essential to such apparently unrelated areas as the study of prime numbers and the design of electric circuits. It is therefore essential for any mathematician to have at least a working knowledge of the subject.

2.1. The Solution of the Cubic.

One often hears that complex numbers were introduced to allow us to solve equations like $x^2+1=0$. There are more than a few youtube videos in which this claim is made, but it just isn't true. It was only after complex numbers were discovered that it occurred to anyone that you could solve that particular quadratic. Prior to the 16th century, if you had asked a mathematician to solve $x^2+1=0$, they would have told you that it was a foolish question because it has no solutions.

In a sense, complex numbers forced themselves into mathematics through the solution not of quadratic but of cubic equations. Their discovery was an inevitable accident that was waiting to happen. When they were at last found, they simply would not go away, but it took around two centuries for people to begin to fully understand them.

The solution of the general quadratic equation $ax^2+bx+c=0$ is very easy to obtain. We simply complete the square and take square roots. The problem of obtaining the roots of a cubic equation is quite a bit more difficult, but there is a formula for it, which we will now derive. It is ultimately due to a mathematician named Scipione del Ferro, though it is usually called Cardano's formula. The quite bizarre story of the solution of the cubic is explained in the appendix.

Suppose that we wish to find the roots of the cubic equation

$$ay^3+by^2+cy+d=0.$$

Obviously the coefficient of y^3 can always be made equal to one just by dividing the equation by a, so we may as well take $a=1$. The trick is to reduce the problem to solving a quadratic equation and this is accomplished by the following steps.

First we knock out the quadratic term to produce what is called a *depressed* cubic. Given a polynomial of degree n it is always possible to knock out the term of degree $n-1$ by a linear substitution $y=x-\lambda$.

One simply substitutes the term into the polynomial and chooses λ such that the term of degree $n-1$ disappears. To do this for a cubic we set $y = x - b/3$. Observe that this substitution gives - after we omit the rather tedious business of expanding terms and performing the various cancellations - the expression

$$\begin{aligned}(x-\frac{b}{3})^3 + b(x-\frac{b}{3})^2 + c(x-\frac{b}{3}) + d &= x^3 + (c - \frac{b^2}{3})x + \frac{2\,b^3}{27} - \frac{b\,c}{3} + d \\ &= x^3 + px - q,\end{aligned}$$

where $p = c - \frac{b^2}{3}$ and $q = -(\frac{2\,b^3}{27} - \frac{b\,c}{3} + d)$. Thus our cubic can be expressed as

$$x^3 + px = q.$$

This is the depressed cubic. Next we make the change of variables

$$x = w - \frac{p}{3w}.$$

This produces

$$w^3 - p\,w - \frac{p^3}{27\,w^3} + \frac{p^2}{3\,w} + pw - \frac{p^2}{3w} = q,$$

which is of course

$$w^3 - \frac{p^3}{27w^3} = q. \tag{2.1}$$

Multiplying through by w^3 gives

$$w^6 - qw^3 - \frac{p^3}{27} = 0.$$

This is a quadratic equation for w^3. The quadratic formula then produces the result

$$w^3 = \frac{q \pm \sqrt{q^2 + 4p^3/27}}{2}. \tag{2.2}$$

It is now an easy matter to extract the roots of the cubic by calculating the values of w and then obtaining x then finally y. One has the nice easy to remember formula for the roots of a cubic:

$$\begin{aligned}y = &\left(\frac{-(\frac{2\,b^3}{27} - \frac{b\,c}{3} + d) \pm \sqrt{(-(\frac{2b^3}{27} - \frac{b\,c}{3} + d))^2 + \frac{4}{27}(c - \frac{b^2}{3})^3}}{2}\right)^{1/3} - b/3 \\ &- \frac{(c - \frac{b^2}{3})^3}{27} \\ &\times \left(\frac{-(\frac{2\,b^3}{27} - \frac{b\,c}{3} + d) \pm \sqrt{(-(\frac{2\,b^3}{27} - \frac{b\,c}{3} + d))^2 + \frac{4}{27}(c - \frac{b^2}{3})^3}}{2}\right)^{-1/3}.\end{aligned}$$

Actually it takes a bit of work to show that this produces exactly three roots for any cubic, but we will not go into that. The formula is obviously too complicated for any reasonable person to remember, but it is coded into Mathematica. The **Solve** command in Mathematica will gives the roots of any cubic or quartic using what in effect the del Ferro formula for the cubic and a formula due to Ferrari for the quartic.

There are some curious features of this formula which only make sense when complex numbers are introduced. Remember that in the sixteenth century, not only were complex numbers unknown, but negative numbers were not generally accepted either. However, years after del Ferro had died, when Giralamo Cardano (see the appendix) applied the formula to find the roots of a cubic which he knew had positive integers as solutions, something decidedly odd happened, or at least it appeared that way to him.

Because negative numbers were not generally accepted, Cardano's classification of cubics looks peculiar to modern eyes. He considered the problem of solving $x^3 = 15x + 4$. Notice that this is the same as $x^3 - 15x - 4 = 0$, but Cardano would never have written it that way because of the minus signs in front of 15 and 4. So what happened when Cardano used his formula? It is easy to check that $x = 4$ is a root of the given cubic. Yet Cardano's application of del Ferro's formula returned the solution

$$x = (2 + \sqrt{-121})^{1/3} + (2 - \sqrt{-121})^{1/3}.$$

This was greatly puzzling. To us, there is no difficulty because if we perform the complex arithmetic, this simply reduces to $x = 4$ as expected. To us $\sqrt{-121} = 11i$ and it is easy to check that $(2 \pm i)^3 = 2 \pm 11i$. So

$$x = (2 + \sqrt{-121})^{1/3} + (2 - \sqrt{-121})^{1/3} = 2 + i + 2 - i = 4.$$

To Cardano however, this was extremely strange. What was he to make of an expression like $\sqrt{-121}$? No matter how often he checked, he always got the same result. After much thought he concluded that the only thing he could do was to assume that the square root of a negative number could be treated as an ordinary square root. He then performed the algebraic reduction that we would do and extracted 4 as the answer.

He was leaping into the unknown. He hoped that the square roots of negative quantities might be useful as an intermediate step in some calculation, because he believed that such 'unreal' quantities should always disappear at the end, leaving a real answer. In some sense this is not a completely silly idea, because when complex numbers occur in problems with real solutions, the imaginary terms will eventually cancel. It's just that not every problem is so nice as to have a real valued answer.

2.1.1. *The solution of higher order polynomial equations.* Before turning to our main subject, it would be remiss not to discuss the method for finding the roots of a quartic. This problem was solved by Lodovico Ferrari, who was a student of Cardano. As with the cubic, we start by knocking out the term of degree $n-1$. In this instance it is the cubic term.

So we start with $y^4+ay^3+by^2+cy+d=0$ and set $y=x-a/4$. Then after we do the simplifications, we end up with the quartic equation

$$x^4+\left(b-\frac{3a^2}{8}\right)x^2+\left(\frac{a^3}{8}-\frac{ab}{2}+c\right)x-\frac{3a^4}{256}+\frac{a^2b}{16}-\frac{ac}{4}+d$$
$$=^{def} x^4+px^2+qx+r=0. \tag{2.3}$$

This quartic can be reduced to a cubic equation by the following steps.

First we think of the two terms involving x^4 and x^2 as coming from a quadratic in x^2. So we complete the square. This gives

$$(x^2+p)^2-px^2-p^2+qx+r=0 \tag{2.4}$$

Or

$$(x^2+p)^2=px^2+p^2-qx-r. \tag{2.5}$$

Ideally we would like the right hand side of the expression to also be a perfect square, so that we can simply take the square root of both sides. Ferrari had the clever idea of introducing another term z into the equation which he could use to cancel terms he didn't want. The trick is to consider not $(x^2+p)^2$ but rather $(x^2+p+z)^2$. If (2.5) holds then

$$\begin{aligned}(x^2+p+z)^2&=(x^2+p)^2+2z(x^2+p)+z^2\\&=px^2+p^2-qx-r+2z(x^2+p)+z^2\\&=(p+2z)x^2-qx+p^2-r+2zp+z^2.\end{aligned} \tag{2.6}$$

What we want is to choose z so that

$$(p+2z)x^2-qx+p^2-r+2zp+z^2=(\alpha x+\beta)^2 \tag{2.7}$$

for some α and β. If we can do this then we have reduced the quartic to the equation

$$(x^2+p+z)^2=(\alpha x+\beta)^2. \tag{2.8}$$

which is of course a quadratic equation, when we take the square root of both sides. Notice a quadratic has two roots, and there are in fact two quadratics, namely

$$x^2+p+z=\pm(\alpha x+\beta)$$

so we get four roots for our quartic.

The problem remains how to choose z? A quadratic is a perfect square if and only if it has a single repeated root. This happens precisely when the discriminant in the quadratic formula is equal to zero. Thus we require the discriminant for the quadratic

$$(p+2z)x^2 - qx + p^2 - r + 2zp + z^2 \tag{2.9}$$

to equal zero. The discriminant of $ax^2 + bx + c$ is $b^2 - 4ac$. So if we set the discriminant of (2.9) equal to zero we get

$$q^2 - 4(p+2z)(p^2 - r + 2zp + z^2) = 0. \tag{2.10}$$

Notice that (2.10) is a cubic equation in z! We can solve cubics by means of del Ferro's formula. Thus we can solve an arbitrary quartic equation. The process requires us first to solve a cubic, then solve a quadratic, but both of these can be done by the corresponding formulae. Obviously the process is rather laborious. In practice if we want the roots of a quartic we would use a package like Mathematica, or we use trial and error or perhaps Newton's method. We will not present the actual formula for the roots of a quartic because it is horrendously complicated.

It might be suspected that the general quintic would be solvable by similar means. That is, to solve

$$x^5 + ax^4 + bx^3 + cx^2 + dx + e = 0$$

we should knock out the x^4 term (which can be done by setting $x \to x - a/5$) then reducing the problem to solving a quartic, then a cubic then a quadratic.

Alas this is not the case. In 1798, Gauss conjectured that a general solution of the quintic by radicals was impossible. A solution by radicals is one that relies only upon the basic arithmetic operations of addition, subtraction, multiplication and the extraction of nth roots. For example the quadratic formula

$$x_{\pm} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \tag{2.11}$$

giving the roots of $ax^2 + bx + c = 0$ uses only addition, subtraction, mutiplication, division and the extraction of a square root. The cubic formula uses the basic arithmetic operations as well as the extraction of square roots and cube roots. The quartic formula involves the extraction of fourth roots as well as the same operations used for a cubic. Gauss suspected that for a general polynomial of degree 5 (or higher), there could not be any formula for the roots using the same operations as for a quartic, plus taking fifth (or indeed any other) roots.

The first serious attempt to prove this was by Paolo Ruffini in 1799. Ruffini's proof was criticised for being unreadably complicated and for making assumptions about how the roots of the polynomial could actually be expressed in terms of radicals, which he did not justify.

Special cases of higher order polynomials can actually be solved by radicals. For example $x^5 - x = 0$ can be solved easily; the eighth degree polynomial equation $x^8 + ax^4 + b = 0$ is just a quadratic in x^4. Ruffini basically ignored these special cases.

Niels Henrik Abel was the first to produce a complete proof, which appeared in 1824. Evariste Galois (at the age of 18), produced a memoir that presented a detailed theory of equations and their solution, which greatly superseded all that went before it. However, in 1832 Galois managed to get himself killed in a duel[1] at the age of 20 and his work was not published until 1846.

A Technical Digression.

The material that follows up to the section on complex numbers can be safely ommited. It is intended to give a rough idea of why we cannot find a general formula for the roots of polynomials using only the standard operations. We also explain what we can do about it.

The reason why no method exists for an arbitrary quintic lies in the theory of *Galois groups.* Technically, the reason is that *the Galois group of the splitting field for a polynomial of degree 5 is not solvable.* We cannot explain this in complete detail, but the outline is as follows.

A group is a nonempty set G which has a multiplication defined on it. So given two elements $g, h \in G$, the product $gh \in G$. There must also be an identity element e, such that $eg = ge = g$ for every element $g \in G$. Finally, every element g must have an inverse g^{-1}. So $gg^{-1} = g^{-1}g = e$.

For example, the collection of all invertible $n \times n$ matrices is a group. Multiplication is matrix multiplication. The identity matrix is the identity element and the matrix inverse is the group inverse. Group theory is an enormous branch of mathematics with numerous applications.

Another example of a group is the set of positive real numbers $\mathbb{R}^+$. The identity is 1, since $1 \times x = x \times 1 = x$ for all $x > 0$. The inverse of x is $1/x$ and obviously the product of two positive numbers is positive.

We recall that a *Field* is a set $\mathbb{F}$ together with two operations called addition and multiplication, which we usually write as $a + b$ and ab. These operations are required to satisfy

(i) Associativity: $a + (b + c) = (a + b) + c$, and $a(bc) = (ab)c$.

(ii) Commutativity: $a + b = b + a$, and $ab = ba$.

(iii) There exist distinct elements 0 and 1 in $\mathbb{F}$ such that $a + 0 = a$ and $1a = a$.

[1]The exact reason for the duel is not known. It appears to have had something to do with a woman, Stéphanie-Félicie Poterin du Motel. Who the duel was fought with is also not known.

(iv) For every a in $\mathbb{F}$, there exists an element in $\mathbb{F}$, denoted $-a$, called the additive inverse of a, such that $a + (-a) = 0$.

(v) For every $a \neq 0$ in $\mathbb{F}$, there exists an element in $\mathbb{F}$, denoted by a^{-1} or $1/a$, called the multiplicative inverse of a, such that $a(1/a) = 1$.

(vi) Distributivity: $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$.

$\mathbb{R}$ and $\mathbb{C}$ are the most important examples of fields. Given a field, $\mathbb{F}$ it is often possible to extend to a larger field $\mathbb{E}$ which contains $\mathbb{F}$. This is known as a field extension. The most familiar example is extending $\mathbb{R}$ to $\mathbb{C}$.

Loosely speaking, the Galois group of a field extension is the collection of certain kinds of transformations (technically called automorphisms), of the field extension, that leave the original field elements unchanged. In practice, these transformations turn out to be permutations.

Think of a simple example. Take $p(x) = x^2 + 1$. The coefficients of the polynomial are real numbers, but the roots are not. The *Splitting Field* of p is the smallest field extension in which p can be factorised into linear terms. In our case we know $p(x) = (x-i)(x+i)$. So the splitting field will be the real numbers with $\pm i$ appended to it somehow.

For a polynomial with coefficients in some field $\mathbb{F}$, (which is usually either the real numbers or the complex numbers), we construct the splitting field of the polynomial by defining it to consist of numbers of the form $z = a_0 + \sum_{i=1}^{n} a_i y_i$ where the numbers $y_1, ..., y_n$ are the roots of the polynomial and the a_i are elements of the field $\mathbb{F}$. One can prove the splitting field is indeed a field extension. In the example above the splitting field consists of number of the form $z = a + ib - ic$ where a, b, c are real. (Do these look familiar?)

Now think of the number of ways that a set of five objects can be rearranged. The collection of all permutations of a set of five elements is an example of a *symmetric group*, which in this case is called S_5. This turns out to be the Galois group for the splitting field of a general polynomial of degree 5.

In order to be able to solve the general equation by radicals, Galois proved that we have to be able to find a chain of nested *subgroups* which we label G^1, G^2,G^3, G^4, with G^4 sitting inside $G = S_5$, G^3 sitting inside G^4 etc. A subgroup of a group G is a subset which is also a group.

This chain should terminate with a subgroup consisting only of the identity element e. Groups with this property are called solvable groups. The name solvable comes from the connection between the existence of these groups and our ability so solve an equation.

Roughly speaking, it turns out that the existence of this chain corresponds to the process of reducing a quintic to a series of lower order polynomials. Remember how the quartic was solved by reducing to a cubic, which we solved by reducing to a quadratic.

If this chain of nested subgroups exists for a given polynomial $p(x)$, then it is possible to solve $p(x) = 0$ by radicals. The problem is that one can show, (and Galois did), that no such chain of subgroups exists for S_5. So S_5 is not a solvable group. This fact means that the general quintic cannot be solved by radicals.

Since polynomials of degree five cannot be solved by radicals, it follows that there is no formula for the solution of general higher degree polynomials by radicals either, since sooner or later the procedure for reducing the order of the polynomial will require us to solve a quintic. So the nonsolubility of the quintic is a road block to the solution of the higher order polynomials. However, as noted above, lots of special cases can indeed be solved by radicals. It is just that there is no formula that works for every possible polynomial of degree five, (or six, or seven,....).

This can all be summed up as follows.

Theorem 2.1. *A polynomial equation is solvable by radicals if and only if the Galois group of the splitting field is solvable.*

Of course, to really understand what all of this actually means is an entire subject in itself. Galois theory is a major branch of modern algebra.

This is not the end of the story however. in 1858 Charles Hermite showed that although it is not possible to solve the quintic by radicals, it is possible to do so by other means. In particular the quintic can be solved by using so called *elliptic functions*. This is a subject well outside our scope.

There is even a general formula[2] for the roots of a polynomial of degree n in terms of what are called Theta functions, but the truth is that this is usually of no real value because of its sheer complexity. It requires the evaluation of what are called hyperelliptic integrals. It is remarkable that such a formula exists, but most of the time it is easier to just find the roots numerically.

2.2. **Complex Numbers.** So the del Ferro formula for the solution of the cubic actually generated complex numbers. This is important to understand historically. Cardano did not sit down and invent complex numbers. Rather he used a formula for solving a real problem and it gave him complex numbers.

The best that anyone could do was try to pretend that this was just some strange quirk and these bizarre quantities could always be dismissed. Whenever they appeared, the correct response was to just

[2]It is called Thomae's formula, after Carl Thomae who published it in 1870.

do the arithmetic and eventually they would go away. This viewpoint took a long time to be overturned.

The first mathematician to make serious progress with Complex numbers was Rafael Bombelli. We do not know the date of his birth, but he was baptised on January 20, 1526 and died sometime in 1572. He showed that real values could be consistently obtained from the formula for the roots of a cubic by treating quantities like $\sqrt{-121}$ as actual numbers subject to straightforward rules. He introduced the concept of what we call imaginary numbers in a book in 1572 and showed considerable understanding of their properties. For example he realised that $\sqrt{-1}$ is neither positive, nor negative and stated the basic rules of complex arithmetic. Yet even after Bombelli, mathematicians continued to think of the $\sqrt{-1}$ as being something that was convenient for calculations, but did not actually exist in the same way that ordinary numbers do. The fact that we call the square roots of negative numbers imaginary (a term invented by René Descartes), is really an historical artifact of this long held, but false, belief.

The key to accepting complex numbers ultimately lay in understanding their geometry. From primary school the student is taught to think of all numbers as lying on some infinitely long line. Mathematicians still refer to the real numbers $\mathbb{R}$ as 'the line.' The insight of Caspar Wessel(1745-1818), Jean-Robert Argand (1768-1822), Carl Friedrich Gauss(1777-1855) and others was that complex numbers are to be thought of as points on a plane. The real numbers are simply a line through the middle of that plane, exactly as the x axis is a line running through the Cartesian plane.

If we multiply a real number x by -1, the effect is to 'rotate' x through 180 degrees. More precisely, if we think of the line connecting 0 with $-x$, then this is the line connecting 0 and x rotated through 180 degrees. Argand realised that multiplying a number z by i is to effectively rotate z through 90 degrees. Caspar Wessel actually had many of the same insights as Argand many years earlier, but he published his work in Danish and it was largely ignored. Nevertheless, this kind of geometrical thinking led to the ultimate acceptance of complex numbers.

The complex numbers are formally constructed by considering pairs of real numbers (a, b), $a, b \in \mathbb{R}$ and defining arithmetic operations as follows.

$$(a, b) + (c, d) = (a + c, b + d), \quad (a, b)(c, d) = (ac - bd, ad + bc).$$

The real numbers are identified as a subset of the complex numbers by the relation $a \in \mathbb{R} \to (a, 0)$. This just means that the real numbers lie on the x axis in the complex plane.

Following the definition of multiplication and the identification of $a \in \mathbb{R}$ with the complex number $(a, 0)$, we make the following observation.

$$(0,1)^2 = (0,1)(0,1) = (0-1, 0+0) = (-1,0).$$

Of course we identify $(-1,0)$ with the real number -1. This means that $(0,1)^2 = (-1,0)$. In other words the complex number $(0,1)$ is the square root of -1.

Traditionally we write a complex number (a,b) in the form $z = a+ib$. This means that we are identifying the complex number $(0,1)$ with the symbol i. More commonly we write $i = \sqrt{-1}$.

Definition 2.2. For a complex number $z = a+ib$ we have the following.

(1) Given any complex number z we define the *real part of* z to be a and the *imaginary part* to be b. We write $\Re(z) = Re(z) = a$ and $\Im(z) = Im(z) = b$.

(2) The *complex conjugate* $\overline{z}$ of a complex number z is defined by $\overline{z} = a - ib$.

(3) The modulus of a complex number z is $|z| = \sqrt{a^2+b^2}$.

2.2.1. *The algebra of complex numbers.* The algebra of complex numbers is quite straightforward. It is very similar to that of real numbers with some important distinctions. Indeed in many ways it is easier. There are some differences however. The most obvious is that it is impossible to order the complex numbers.

Theorem 2.3. *It is not possible to assign a consistent ordering to the complex numbers as is done with the real numbers. That is, one cannot say that a given complex number is greater or smaller than another complex number.*

Proof. Assume that we could order the complex numbers. Then either $i > 0$ or $i < 0$. Let us assume that $i > 0$. Now multiply both sides of this inequality by i. Since $i > 0$ the inequality sign remains the same. Hence

$$i.i > i.0 \implies -1 > 0.$$

This is a contradiction, so $i \not> 0$. If we assume that $i < 0$ then we can derive a contradiction in the same manner. Since $i < 0$, then multiplying by i will reverse the inequality sign. Hence

$$i < 0 \implies i.i > i.0 \implies -1 > 0.$$

We have another contradiction. Thus $i \not< 0$ either. Clearly $i \neq 0$. Our conclusion follows. $\square$

Manipulating complex numbers is often best achieved by introducing the polar form and using Euler's formula. Actually, we should call this the Euler-Cotes formula, since nearly forty years before Euler, Roger Cotes stated the relation

$$i\theta = \ln(\cos\theta + i\sin\theta),$$

though he does not appear to have made use of it.

His proof was not completely rigorous by modern standards, but his result is still clearly equivalent to the more common version $e^{i\theta} = \cos\theta + i\sin\theta$. However at the time, neither the complex numbers nor the connection between the natural logarithm and the exponential function were properly understood. Euler's work dramatically altered the status of complex variables.

Proposition 2.4 (Euler's formula)**.** *Let θ be a real number. Then $e^{i\theta} = \cos\theta + i\sin\theta$*

Proof. There are a number of ways to do this. The simplest is to use the fact that the equation $\frac{dy}{d\theta} = iy$, $y(0) = 1$ has a unique solution. Now let $y(\theta) = e^{i\theta}$. Then $\frac{d}{d\theta}e^{i\theta} = ie^{i\theta}$, so $y' = iy$. Similarly, if $h(\theta) = \cos\theta + i\sin\theta$, then since

$$\frac{d}{d\theta}(\cos\theta + i\sin\theta) = -\sin\theta + i\cos\theta = i(\cos\theta + i\sin\theta),$$

we have $h' = ih$. Since $y(0) = 1 = h(0)$, and the solution of the ODE is unique, it follows that y and h must be the same function. $\square$

Since $(e^{i\theta})^n = e^{in\theta}$ we have an immediate corollary, namely the famous result attributed to De Moivre- despite the fact that it appears nowhere in his writings.

Corollary 2.5 (De Moivre's Theorem)**.** *For all real θ and any integer n*

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta).$$

If n is not an integer, one needs to be careful. We will see why shortly.

Since we are dealing with points on a plane, we can convert from Cartesian coordinate x, y to polar coordinates, r and θ. We let $r = \sqrt{x^2 + y^2}$. Then for some angle θ

$$x = r\cos\theta, \quad y = r\sin\theta. \tag{2.12}$$

Thus

$$z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}.$$

The choice of r is unambiguous. It is simply the modulus of the complex number z. However we have not yet specified what the *argument* θ is. This is an extremely important point. The trigonometric functions are 2π periodic, hence there is no unique value for θ. We have to make a choice.

Definition 2.6. The *principal value of the argument of a complex number $z = x + iy$* is the unique value of $\theta \in (-\pi, \pi]$ which solves the simultaneous equations

$$\cos\theta = \frac{x}{\sqrt{x^2 + y^2}}, \quad \sin\theta = \frac{y}{\sqrt{x^2 + y^2}}. \tag{2.13}$$

This choice of θ is denoted $\mathrm{Arg}(z)$. The set of all solutions of the equations (2.13) is called the argument of the complex number z. That is

$$\arg(z) = \left\{\theta \in \mathbb{R} : \cos\theta = \frac{x}{\sqrt{x^2+y^2}}, \quad \sin\theta = \frac{y}{\sqrt{x^2+y^2}}\right\}. \tag{2.14}$$

Every complex number $z = x + iy$ can be written in polar form

$$z = re^{i\theta} = r(\cos\theta + i\sin\theta),$$

where $r = |z| = \sqrt{x^2+y^2}$ and $\theta = \mathrm{Arg}(z) \in (-\pi, \pi]$ is the principle value of the argument of z. This representation comes from the equations

$$x = r\cos\theta, \ y = r\sin\theta.$$

2.3. **Determining the Polar Form.** To determine the polar form of a complex number we have to solve these equations for r and θ. Finding r is easy. To find θ we have to know what quadrant the number is in and keep in mind what the signs of $\cos\theta$ and $\sin\theta$ are in those quadrants. We should note that the command Arg[z] in Mathematica will produce the principle value of the argument of any complex number.

Recall that in the first quadrant $\cos\theta$ and $\sin\theta$ are positive. In the second quadrant $\sin\theta$ is positive and $\cos\theta$ is negative. In the third quadrant $\sin\theta$ and $\cos\theta$ are both negative, whereas $\tan\theta$ is positive and in the fourth quadrant $\sin\theta$ is negative and $\cos\theta$ is positive.

Example 2.1. We find the polar form of $z = 3 + 4i$. This number is in the first quadrant. We see $r = \sqrt{3^2+4^2} = 5$. To find the argument observe that we want $\theta \in [0, \pi/2]$ (we are in the first quadrant) such that

$$3 = 5\cos\theta, \ 4 = 5\sin\theta. \tag{2.15}$$

Clearly we require $\tan\theta = 4/3$. So we can take $\theta = \tan^{-1}(4/3) \approx 0.927295$. Since $\pi/2 \approx 1.57$, we see that this gives a value of θ in the right range. So

$$z = 5e^{i\tan^{-1}(4/3)}.$$

This suggests that if $z = x + iy$ is in the first quadrant, the principle value of the argument is just

$$\mathrm{Arg}(z) = \tan^{-1}(y/x).$$

Example 2.2. We find the polar form of $z = 3 - 4i$. This number is in the fourth quadrant. As before we see that $r = \sqrt{3^2+4^2} = 5$. To find the argument observe that we want $\theta \in [-\pi/2, 0]$ (we are in the fourth quadrant) such that

$$3 = 5\cos\theta, \ -4 = 5\sin\theta. \tag{2.16}$$

Clearly we require $\tan\theta = -4/3$. So we can take $\theta = -\tan^{-1}(4/3) \approx -0.927295$. Since $-\pi/2 \approx -1.57$, we see that this gives a value of θ in the right range. So

$$z = 5e^{-i\tan^{-1}(4/3)}.$$

(Note that since $3-4i$ is the complex conjugate of $3+4i$, discussed in the previous example, the result we have just obtained could easily be found just by taking the complex conjugate of the previous result).

If $z = x+iy$ is in the fourth quadrant, the argument is again just

$$\mathrm{Arg}(z) = \tan^{-1}(y/x).$$

Example 2.3. Now we find the polar form of $z = -3+4i$. This is in the second quadrant. Again $|z| = \sqrt{3^2+4^2} = 5$ and we want

$$-3 = 5\cos\theta,\ \ 4 = 5\sin\theta \tag{2.17}$$

with $\theta \in [\pi/2, \pi]$. Now taking $\theta = \tan^{-1}(y/x) = -\tan^{-1}(4/3)$ will not give the right value of θ, as it does not return a value in the right range. (See previous example). However $\tan\theta$ is periodic with period π. i.e. $\tan(\theta+\pi) = \tan\theta$. So let us take $\theta = \pi - \tan^{-1}(4/3)$. We will make use of two very useful trigonometric identities.

$$\cos(\tan^{-1}(x)) = \frac{1}{\sqrt{x^2+1}} \tag{2.18}$$

and

$$\sin(\tan^{-1}(x)) = \frac{x}{\sqrt{x^2+1}}. \tag{2.19}$$

We use these to perform the following calculations.

$$\begin{aligned} 5\cos(\pi - \tan^{-1}(4/3)) &= 5\cos(\pi)\cos(\tan^{-1}(4/3)) + \sin\pi\sin(\tan^{-1}(4/3)) \\ &= -5\cos(\tan^{-1}(4/3)) \\ &= -3 \end{aligned}$$

and

$$\begin{aligned} 5\sin(\pi - \tan^{-1}(4/3) &= 5\sin\pi\cos(\tan^{-1}(4/3)) - 5\cos\pi\sin(\tan^{-1}(4/3)) \\ &= 5\sin(\tan^{-1}(4/3)) \\ &= 4. \end{aligned}$$

Thus this is the right choice of θ and we have

$$z = 5e^{i(\pi - \tan^{-1}(4/3))}.$$

The lesson of this example is that if $z = x+iy$ is in the second quadrant, we find $\mathrm{Arg}(z)$ by

$$\mathrm{Arg}(z) = \pi + \tan^{-1}(y/x).$$

What about a complex number in the third quadrant?

Example 2.4. Now we find the polar form of $z = -3 - 4i$. This is in the third quadrant. As before $|z| = \sqrt{3^2 + 4^2} = 5$ and we want

$$-3 = 5\cos\theta,\ -4 = 5\sin\theta \tag{2.20}$$

with $\theta \in [-\pi, -\pi/2]$. Now taking $\theta = \tan^{-1}(y/x) = \tan^{-1}(4/3) \approx 0.9273$ does not give the right value of θ, because it does not return a value in the right range. We use the periodicity of tan again.

$$\begin{aligned}
5\cos(-\pi + \tan^{-1}(4/3)) &= 5\cos(-\pi)\cos(\tan^{-1}(4/3)) \\
&\quad - \sin(-\pi)\sin(\tan^{-1}(4/3)) \\
&= -5\cos(\tan^{-1}(4/3)) \\
&= -3.
\end{aligned}$$

and

$$\begin{aligned}
5\sin(-\pi + \tan^{-1}(4/3)) &= 5\sin(-\pi)\cos(\tan^{-1}(4/3)) \\
&\quad + 5\cos(-\pi)\sin(\tan^{-1}(4/3)) \\
&= -5\sin(\tan^{-1}(4/3)) \\
&= -4.
\end{aligned}$$

Thus this is the right choice of θ and we have

$$z = 5e^{i(-\pi + \tan^{-1}(4/3))}.$$

The lesson of this example is that if $z = x + iy$ is in the third quadrant, we find $\mathrm{Arg}(z)$ by

$$\mathrm{Arg}(z) = \tan^{-1}(y/x) - \pi.$$

So in the first and fourth quadrants, we have $\theta = \tan^{-1}(y/x)$, in the second quadrant we add π and in the third quadrant we subtract π.

Exercise 2.1. As an exercise show that the following complex numbers have the given polar forms.

(i) $2 + 2i = 2\sqrt{2}e^{i\pi/4}$,

(ii) $2 - 2i = 2\sqrt{2}e^{-i\pi/4}$,

(iii) $-1 + \sqrt{3}i = 2e^{2\pi i/3}$

(iv) $-1 + 4i = \sqrt{17}e^{i(\pi - \tan^{-1}(4))}$

(v) $-5 + 5i = 5\sqrt{2}e^{3\pi i/4}$

(vi) $-\sqrt{6} - \sqrt{2}i = 2\sqrt{2}e^{-5\pi i/6}$.

What if you forget the simple rule for finding the principle value of the argument? Then use your calculator and the equations

$$x = r\cos\theta,\ y = r\sin\theta$$

to determine θ. See if $\theta = \tan^{-1}(y/x)$ satisfies *both* equations. If it does, then you have the right value. If it doesn't, then add and subtract π from your answer, and see which one satisfies *both* equations. Remember, the correct value of θ must satisfy both these equations, not just one of them. Be aware that if you use a calculator, you might get a bit of rounding error when you check the equations.

The choice of θ we have made here for the principle value is the most common one made in the literature. It is obvious that the function $\mathrm{Arg}(z)$ as a function of z is discontinuous. The choice we have made means that for a real number $z > 0$ we have $\mathrm{Arg}(z) = 0$. What about numbers on the negative real axis? If $z \in \mathbb{R}$ and $z < 0$ then $\mathrm{Arg}(z) = \pi$. So far so good.

However what about a number in the third quadrant? That is a number with both negative real part and negative imaginary part? For such numbers $-\pi < \mathrm{Arg}(z) < -\frac{\pi}{2}$. As we approach the negative real axis from below, $\mathrm{Arg}(z) \to -\pi$. But when we hit the negative axis the argument becomes π. So the value of $\mathrm{Arg}(z)$ jumps by 2π. Consequently the function $\mathrm{Arg}(z)$ has jump discontinuities. But the jump discontinuities do not lie at a single point or an isolated set of points. Every point on the negative real axis is a point of discontinuity for $\mathrm{Arg}(z)$. The negative real axis is known as a *branch cut* for the argument function. Some authors define the principal value of the argument to lie between 0 and 2π. For this choice, the branch cut lies on the positive real axis.

There is no way of avoiding this problem. We have to make some choice for the argument and placing the branch cut on the negative real axis is the most sensible choice we can make. However we do need to be a little bit careful when dealing with arguments of complex numbers. There are a number of subtleties which can cause problems for the careless student.

Consider this very well known apparent conundrum. We know that $i = \sqrt{-1}$. So we can surely write

$$-1 = i^2 = \sqrt{-1}\sqrt{-1} = \sqrt{(-1)(-1)} = \sqrt{1} = 1.$$

So apparently $1 = -1$. Of course this is false. Seeming paradoxes like this were one of the reasons mathematicians took so long to accept the existence of complex numbers. So what exactly has gone wrong here? To understand the problem we need to realise that the usual index laws for real numbers don't quite work the same way for complex numbers. To see why, we have to ask what exactly an expression like z^α where both z and α are complex, actually means? To answer this question we have to define the complex logarithm.

So we let $z = re^{i\theta}$ and we want to define $\ln z$. By definition the natural logarithm of z is a number $\ln z$ with the property that

$$e^{\ln z} = z.$$

If we set

$$\ln z = \ln(re^{i\theta}) = \ln r + i\theta,$$

where θ is the principal value of the argument of z, then clearly

$$e^{\ln z} = e^{\ln r + i\theta} = e^{\ln r}e^{i\theta} = re^{i\theta} = z.$$

So this choice for the natural logarithm works. But why should we use the principal value of the argument? Obviously $z = re^{i\theta} = re^{i(\theta+2k\pi)}$ for any integer k. Given this, why could we not define the logarithm of z to be

$$\ln z = \ln(re^{i(\theta+2\pi}) = \ln r + i(\theta + 2\pi)$$

or

$$\ln z = \ln(re^{i(\theta-26\pi}) = \ln r + i(\theta - 26\pi)?$$

For example, using the second we would have

$$e^{\ln r + i(\theta-26\pi)} = re^{i\theta}e^{-26i\pi} = re^{i\theta} = z,$$

since $e^{-26i\pi} = \cos(-26\pi) + \sin(-26\pi) = 1$. This works just as well as using the principal value of the argument.

The fact is that a complex number z has infinitely many possible choices for its logarithm, all of which are perfectly valid. This suggests that something rather different is going on when we deal with certain kinds of complex functions.

We have always thought of a function as being a rule which takes one number for its argument and produces one number for its output. By definition functions are single valued. We want to keep this definition, because it works. Here we have a function which is naturally set valued. That is, to each complex number z the logarithm is a set of numbers.

This is not so unfamiliar, when we realise that lots of real functions are like this. For example the square root of a positive real number has two possible real values. So does the fourth root. From our earliest encounter with the square root we have become used to making a choice as to which square root to take, depending on the particular problem we are solving. With the argument of a complex number and the complex logarithm, we are in exactly the same situation. We have to make a choice of which argument and which logarithm we are going to take.

The notion of a set valued, or *multi valued* function is however rather useful. So we make it formal.

Definition 2.7. Let $\mathcal{P}(\mathbb{C})$ denote the set of all subsets of $\mathbb{C}$. A set valued, or alternatively a multi valued function on $\mathbb{C}$ is a mapping from $\mathbb{C}$ to $\mathcal{P}(\mathbb{C})$. That is

$$f : \mathbb{C} \to \mathcal{P}(\mathbb{C}).$$

Many functions are actually set valued.

Example 2.5. The following are examples of set valued functions.

$$f(z) = \sqrt{z}, \quad g(z) = z^{1/3}.$$

Multi-valued functions can arise because of the periodicity of $e^{i\theta}$. That is, because $e^{i\theta} = e^{i(\theta+2k\pi)}$ for all integers k. What we have to do is choose a *branch* of a given function and work with that. This really amounts to choosing a value for k and $k = 0$ is the most usual choice. When $k = 0$ is chosen, we are working with the principal branch.

For example, with the square root of a positive real number, we may choose one of two branches: $+\sqrt{z}$ and $-\sqrt{z}$. How does this correspond to a choice of k? Let $z = re^{i(\theta+2k\pi)}$, for $\theta \in (-\pi, \pi]$. Here we can take $k = 0$ or $k = 1$. Other choices of k will give us the same values we get from these.

If we take $k = 0$ we have

$$\sqrt{z} = (re^{i\theta})^{1/2} = \sqrt{r}e^{i\theta/2}.$$

Conversely, if we take $k = 1$ we get

$$\sqrt{z} = (re^{i(\theta+2\pi)})^{1/2} = \sqrt{r}e^{i\theta/2}e^{i\pi} = -\sqrt{r}e^{i\theta/2}.$$

Thus the two branches of the square root function correspond to different choice for the argument of z.

The apparent paradox where we showed that "$-1 = 1$" can be resolved if we understand how to work with powers of a complex number. As we just saw, an arbitrary power of a complex number itself turns out to be a set valued function.

We will define the natural logarithm as a set of complex numbers and also define a single valued branch of the natural logarithm function as follows.

Definition 2.8. Let $z = x + iy$ be a complex number. The natural logarithm of z is a set valued function defined by

$$\log_e(z) = \ln|z| + i\arg(z), \tag{2.21}$$

where $\ln|z|$ is the natural logarithm of the modulus of $|z|$, (remember $|z|$ is real, so this is just the ordinary natural logarithm) and $\arg(z)$ is the set of all arguments of z. The *principal branch of the logarithm of* z is defined to be the choice of $\log_e(z)$ obtained by taking the principal branch of the argument of z. That is

$$\ln z = \ln|z| + i\mathrm{Arg}(z), \tag{2.22}$$

where $\mathrm{Arg}(z)$ is the principal value of the argument of z.

So for example, since i sits on the imaginary axis, which is at right angles to the real axis, the argument can be taken to be $\pi/2$. Therefore the natural logarithm of i is the set

$$\log_e(i) = \ln|i| + \{(\pi/2 + 2k\pi)i, k \in \mathbb{Z}\} = \{(\pi/2 + 2k\pi)i, k \in \mathbb{Z}\}.$$

For the principal value of the log we just take $k = 0$. Hence

$$\ln i = \frac{\pi}{2} i.$$

When we write $\ln z$ we always mean the principal value of the natural logarithm, so it is always single valued. An important point which is often missed by those unfamiliar with the theory is that in general $\ln(z_1 z_2) \neq \ln z_1 + \log z_2$. The reason of course, is to do with the need to use the 'right' argument.

Remember that if we compute $\ln z$, the value is

$$\ln z = \ln|z| + iArg(z),$$

where $Arg(z)$ is the principle value of the argument of z. That is $Arg(z) \in (-\pi, \pi]$.

Now take $z_1 = z_2 = e^{2\pi i/3}$. Clearly $z_1 z_2 = e^{4\pi i/3}$. But by the periodicity of e^{ix} we observe that

$$z_1 z_2 = e^{4\pi i/3} = e^{4\pi i/3 - 2\pi i} = e^{-2\pi i/3}.$$

By definition then, $\ln(z_1 z_2) = -\frac{2\pi i}{3}$ since we must use the principle value of the argument. Consequently

$$\ln(z_1 z_2) = -\frac{2\pi i}{3} \neq \ln z_1 + \ln z_2 = \frac{2\pi i}{3} + \frac{2\pi i}{3} = \frac{4\pi i}{3}.$$

The problem is that adding the arguments together can give a result that does not lie in $(-\pi, \pi]$. If we think in terms of sets however, we can retain additivity for the logarithm. It is a straightforward exercise to show that

$$\log_e(z_1 z_2) = \log_e(z_1) + \log_e(z_2) \tag{2.23}$$

as sets.

Differentiation kills constants however. Thus even though $\ln(f_1 f_2)$ is not additive, the derivative is, since

$$\frac{d}{dz}\ln(f_1 f_2) = \frac{(f_1 f_2)'}{f_1 f_2} = \frac{f_1' f_2 + f_1 f_2'}{f_1 f_2} = \frac{f_1'}{f_1} + \frac{f_2'}{f_2} = \frac{d}{dz}\ln f_1 + \frac{d}{dz}\ln f_2.$$

The fact that the log is a set valued function rather than a single valued function is crucial when we come to define powers of a complex number. Positive integral powers of complex numbers are uniquely defined. For example if $z = 1 + i$, then

$$z^2 = (1+i)(1+i) = 1 + 2i - 1 = 2i$$

and this answer is the only possible value of z^2. Because positive integer powers of complex numbers are uniquely defined, we have no problem in defining the exponential of a complex number, since we know how to compute a complex exponential by means of a power series.

Definition 2.9. For any $z \in \mathbb{C}$ we have

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}. \tag{2.24}$$

This will give us a unique value for e^z for any given z. Therefore, the natural way to define z^α is by $z = e^{\alpha \ln z}$. But why do we have to use the principal value of the logarithm? Will choosing a different branch of the logarithm give a different answer? Yes, in general it will. Therefore, in general a complex power is a set valued object.

Definition 2.10. Let α and z be complex numbers. Then z^α is the set of complex numbers defined by

$$z^\alpha = e^{\alpha \log_e(z)} = \{e^{\alpha \ln|z|} e^{i\alpha(\theta + 2k\pi)}, k \in \mathbb{Z}\}, \tag{2.25}$$

where α is the principal value of the argument of z. The principal value of z^α is obtained by taking $k = 0$.

Let us ask the obvious question first. If α is an integer, what happens? Do we only get one possible value for the αth power of a complex number z? Observe that if $\alpha = n$ is an integer, then $n \log_e(z) = n \ln|z| + in \arg(z)$. So let the principal value of the argument of z be θ. Then

$$n \arg(z) = \{n\theta + 2kn\pi, k \in \mathbb{Z}\}.$$

Thus according to our definition the set of powers z^n is

$$z^n = \left\{\exp(n \ln|z|) e^{in\theta} e^{2kn\pi i}, k \in \mathbb{Z}\right\}.$$

But $e^{2kn\pi i} = 1$ for all n. Thus for each $k \in \mathbb{Z}$, $e^{in\theta} e^{2kn\pi i} = e^{in\theta}$. Hence

$$z^n = \left\{\exp(n \ln|z|) e^{in\theta}\right\}.$$

That is, the set of powers z^n has only one element in it, since for each choice of k we get exactly the same number. Thus our definition gives us a unique value of z^n if n is an integer. Which is a relief, because we'd be in trouble if it didn't.

However, if the power α is not an integer, we do not get a unique answer.

Example 2.6. Calculate the principal value of i^i. Also calculate the value of i^i as a set.

Solution. We write $i = e^{\pi/2i}$ for the principal value. Then the principal value is $i^i = e^{i(\pi/2)i} = e^{-\pi/2}$.

For the set we know that $i = e^{i(\pi/2 + 2k\pi)}$ for all $k \in \mathbb{Z}$. Thus

$$i^i = \{e^{-\pi/2 - 2k\pi}, k \in \mathbb{Z}\}.$$

So we see that there are infinitely many possible values for i^i, indexed by integers k. Of course the principal value is obtained by setting $k = 0$.

What now can we say about the index laws for complex numbers? With real numbers we have the familiar rules $(ab)^m = a^m b^m$ and $(a^\alpha)^\beta = a^{\alpha\beta}$. However for complex numbers it is not quite so straightforward, unless we think in terms of the rule as applying to sets. As *sets* we can say that, for example, that if z, w and α are complex numbers then

$$(zw)^\alpha = z^\alpha w^\alpha.$$

What this means is that the two sides of this relationship are equal, if we interpret it as a statement about sets. All the elements of $(zw)^\alpha$ may be obtained by multiplying elements from the sets z^α and w^α.

Let us consider an example.

Example 2.7. Calculate $((-i)(i))^i$. Show that as sets $((-i)i)^i = (-i)^i i^i$.
Solution. Clearly $(-i)(i) = 1$. So that $((-i)(i))^i = 1^i$. But $1 = e^{2k\pi i}$ for all $k \in \mathbb{Z}$. Thus

$$1^i = \{e^{-2k\pi}, k \in \mathbb{Z}\}.$$

Notice that if we take $k = 0$, then we see that the principal value of $1^i = 1$.

Now we also know that

$$i^i = \{e^{-\pi/2-2n\pi}, n \in \mathbb{Z}\}.$$

In addition, we can write $-i = e^{i(-\pi/2+2j\pi)}$, $j \in \mathbb{Z}$. Thus

$$(-i)^i = \left\{e^{(i)i(-\pi/2+2j\pi)}, j \in \mathbb{Z}\right\} = \left\{e^{\pi/2-2j\pi}, j \in \mathbb{Z}\right\}.$$

Consequently

$$(-i)^i i^i = \left\{e^{\pi/2-2j\pi}, j \in \mathbb{Z}\right\}\left\{e^{-\pi/2-2n\pi}, n \in \mathbb{Z}\right\} = \left\{e^{2(n-j)\pi}, \quad n, j \in \mathbb{Z}\right\}.$$

Now for each $k \in \mathbb{Z}$ we can find $n, j \in \mathbb{Z}$ with $k = j - n$. So as sets

$$\left\{e^{-2k\pi}, k \in \mathbb{Z}\right\} = \left\{e^{2(n-j)\pi}, \quad n, j \in \mathbb{Z}\right\}.$$

That is, every element in the set on the left is also in the set on the right. In other words as sets, $((-i)i)^i = (-i)^i i^i$.

As long as we are consistent with our choice of branches, we will not run into trouble. If we consider again the paradox

$$-1 = i^2 = \sqrt{-1}\sqrt{-1} = \sqrt{(-1)(-1)} = \sqrt{1} = 1, \qquad (2.26)$$

we see now why this is wrong. Which $\sqrt{-1}$ do we mean? There are two, since $(-i)^2 = -1$ also. So in (2.26) we are equating a number, namely i^2, with the product of two sets, namely $\sqrt{-1}\sqrt{-1}$, which of course makes no sense.

However, if we are consistent with our choice of square roots, we have no problem. Let us take $i = \sqrt{-1} = e^{\pi/2i}$ and do this calculation again. We get

$$-1 = (e^{\pi/2i})^2 = e^{\pi/2i}e^{\pi/2i} = e^{\pi i} = \cos\pi + i\sin\pi = -1$$

and there is no contradiction. If instead we took $-i = e^{-\pi/2i}$ then

$$-1 = (e^{-\pi/2i})^2 = e^{-\pi/2i}e^{-\pi/2i} = e^{-\pi i} = \cos(-\pi) + i\sin(-\pi) = -1,$$

and again, the contradiction does not appear.

Although this may appear rather complicated, it rarely causes difficulties in practice. As long as we make consistent choices when deciding what arguments to use, we will avoid problems.

2.3.1. *Roots of unity.* One of the profound consequences of the discovery of complex numbers is the fact that every polynomial of degree n has n complex roots. Consequently the polynomial $z^n = 1$ must have n roots. These numbers are known as the nth roots of unity. They are easy to find. We make use of our previous discussion to see what is going on.

Example 2.1. Suppose we wish to find the 4th roots of unity. There are of course 4 of them.

We happily write $1 = e^{2k\pi i}$, for $k \in \mathbb{Z}$. Then the equation $z^4 = 1$ is the same as $z^4 = e^{2k\pi i}$. This plainly has a solution $z = e^{k\pi i/2}$ for each k. The key of course is that if we choose the values of k, then we get different solutions. However because of the periodicity of e^{ix} we will only get four solutions.

Let us label the solutions corresponding to a choice of k by z_k. So $k = 0$ gives $z_1 = e^0 = 1$. $k = 1$ gives $z_1 = e^{\pi i/2} = i$. $k = 2$ gives $z_2 = e^{-\pi i} = -1$ and $k = 3$ gives $z_3 = e^{3\pi i/2} = -i$.

What happens if we take $k = 4$? Then $z_4 = e^{2\pi i} = 1$. We have this root already. Increasing the value of k will just cycle through the solutions we have already found, over and over again.

One interesting feature of the roots of unity comes from the fact that the equation $z^n - 1 = 0$ has a very simple factorisation. We know that

$$z^n - 1 = (z-1)(1 + z + z^2 + \cdots + z^{n-1}).$$

Let us take ω to be an nth root of unity not equal to one. Then $\omega - 1 \neq 0$. However $\omega^n - 1 = 0 = (\omega - 1)(1 + \omega + \omega^2 + \cdots + \omega^{n-1})$. Consequently, if ω is an nth root of unity and $\omega \neq 1$ then

$$1 + \omega + \omega^2 + \cdots + \omega^{n-1} = 0.$$

This turns out to be quite a useful fact.

2.4. **Some complex valued series.** The convergence of complex valued series is handled in exactly the same way as the convergence of real valued series.

Definition 2.11. Let $\{z_n\}_{n=1}^{\infty}$ denote a sequence of complex numbers. If the sequence of partial sums $S_N = \sum_{n=1}^{N} z_n$ is convergent, then we say that the series $S = \sum_{n=1}^{\infty} z_n$ is convergent and $S = \lim_{N\to\infty} S_N$.

We may use the same tests for convergence of complex valued series as we use for real valued series. For example, the ratio test works exactly the same way.

Theorem 2.12. *Let $\{z_n\}_{n=1}^{\infty}$ be a sequence of complex numbers. Suppose that*

$$\lim_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right| < 1.$$

Then the series $\sum_{n=1}^{\infty} z_n$ is convergent. Suppose conversely that

$$\lim_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right| > 1.$$

Then the series $\sum_{n=1}^{\infty} z_n$ is divergent. If however

$$\lim_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right| = 1$$

then no conclusion about the convergence of the series can be drawn.

The easiest complex functions to handle and indeed the most important, turn out to be those given by power series. Power series involving complex numbers behave exactly the same way as power series involving real numbers. Let us start with the simplest such series. Namely the geometric series. Surprisingly perhaps, this turns out to be remarkably important.

We want to sum $1 + z + z^2 + z^3 + \cdots$ for $|z| < 1$. We already know that $1 + z + z^2 + z^3 + \cdots = \frac{1}{1-z}$. Now we also know that we can write $z = re^{i\theta}$. So we have for $0 < r < 1$

$$K(r,\theta) = \sum_{n=0}^{\infty} r^n e^{ni\theta} = 1 + re^{i\theta} + r^2e^{2i\theta} + r^3e^{3i\theta} + \cdots = \frac{1}{1-re^{i\theta}}.$$

Using Euler's formula gives

$$\begin{aligned}\frac{1}{1-re^{i\theta}} &= \frac{1}{1-r\cos\theta - ir\sin\theta}\\ &= \frac{1}{1-r\cos\theta - ir\sin\theta}\left(\frac{1-r\cos\theta + ir\cos\theta}{1-r\cos\theta + ir\sin\theta}\right)\\ &= \frac{1-r\cos\theta + ir\sin\theta}{r^2 - 2r\cos\theta + 1}. \qquad (2.27)\end{aligned}$$

Let $K_1(r,\theta) = \frac{1-r\cos\theta}{r^2-2r\cos\theta+1}$ and $K_2(r,\theta) = \frac{r\sin\theta}{r^2-2r\cos\theta+1}$. The function K_1 will turn out to be very important for the problem of solving the Laplace equation on the unit disc. We will investigate this shortly.

Other familiar series are those for $e^z, \sin z$ and $\cos z$. The series for e^z we have introduced. For the trigonometric functions we have the following elementary result.

Theorem 2.13. *For all* $z \in \mathbb{C}$

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \tag{2.28}$$

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} \tag{2.29}$$

Using this result we can give another proof of Euler's formula.

$$\begin{aligned} e^{iz} &= 1 + iz - \frac{z^2}{2!} - i\frac{z^3}{3!} + \frac{z^4}{4!} + i\frac{z^5}{5!} - \frac{z^6}{6!} + \cdots \\ &= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots + i(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots) \\ &= \cos z + i \sin z. \end{aligned}$$

As this true for all complex z, we have actually improved on our previous proof which only held for real values of z.

Functions defined by power series are the essence of complex variable theory. We will see that they are the only ones which are differentiable. Other important examples are $\cosh z$ and $\sinh z$.

Theorem 2.14. *For all* $z \in \mathbb{C}$

$$\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} \tag{2.30}$$

$$\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}. \tag{2.31}$$

This can be deduced from the definition

$$\cosh z = \frac{1}{2}(e^z + e^{-z}), \ \sinh z = \frac{1}{2}(e^z - e^{-z}) \tag{2.32}$$

and the series for e^z.

3. Differentiation and the Cauchy-Riemann equations

We consider a function $f : \mathbb{C} \to \mathbb{C}$ and ask whether it is differentiable. The derivative of f is defined in exactly the manner we would expect.

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}, \tag{3.1}$$

provided the limit exists.

First we define what we mean by a limit. Actually the definition is identical to the real case.

Definition 3.1. Let $f : S \subseteq \mathbb{C} \to \mathbb{C}$. We say that $\lim_{z \to z_0} f(z) = L$ if given any $\epsilon > 0$ we can find $\delta > 0$ such that $|z - z_0| < \delta$ implies $|f(z) - L| < \epsilon$.

The usual properties of limits hold. That is Theorem 1.2 holds with x replaced by a complex variable z.

The familiar functions from analysis behave just as we would expect with respect to differentiation. So for example

$$\frac{d}{dz} e^z = e^z, \quad \frac{d}{dz} \ln z = \frac{1}{z}, \quad \frac{d}{dz} \sin z = \cos z, \tag{3.2}$$

etc. However differentiation for functions of a complex variable is in a real sense, a stronger property than for functions of a real variable. The fact that a function of a complex variable is differentiable implies a good deal more about the function than is the case for functions of a real variable.

It is possible to have a function of a real variable which is differentiable once, but not twice. For example, the function

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} \sin(3^n x)$$

is continuous by the Weierstrass M-test. To see this note that each term in the series is continuous and the series converges uniformly, since

$$\left| \frac{1}{2^n} \sin(3^n x) \right| < \frac{1}{2^n}$$

and

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1 < \infty. \tag{3.3}$$

The sum of a uniformly convergent series of continuous functions is continuous. However f is not differentiable anywhere. This is not straightforward to prove, but essentially boils down to the fact that the series for f' diverges. However f is continuous and hence integrable on $[a, x]$ and by the Fundamental Theorem of Calculus the function

$$F(x) = \int_a^x f(t) dt \tag{3.4}$$

is differentiable, with $F'(x) = f(x)$. But F'' does not exist because f is not differentiable. So we have a function with one derivative but no more.

In the complex plane this is not possible. A function $f : S \to \mathbb{C}$ which is differentiable once is automatically infinitely differentiable. This requires quite a lot of work and new material to show. But the reason, roughly speaking, is that it is *harder* for a function of a complex variable to be differentiable than it is for a function of a real variable. Why this should be the case is easy to understand. Consider the definition of the derivative in (3.1). It involves the familiar limit. The difference here is the number of ways we can approach the limit.

For a function of a real variable, a limit can be approached in only two directions. From the left and the right. We know that $\lim_{x\to a} f(x)$ exists if and only if $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x)$. That is to say, the left and right limits must both exist and be equal.

In the case of a complex variable, we have a lot more freedom to move, because we are on a plane. There are infinitely many directions that a given point in $\mathbb{C}$ can be approached. So the fact that a limit $\lim_{z\to a} f(z)$ exists in $\mathbb{C}$ is a much stronger property for a function to have than is the case when we are working on the real line.

Fortunately there is a very straightforward test to decide whether a given function of a complex variable is differentiable. Before we introduce the result, we need to point out a rather important feature of functions of a complex variable $z = x + iy$. Since we are dealing with functions $f : \mathbb{C} \to \mathbb{C}$, f itself has a real and imaginary part. Thus for any complex function f there exist real valued functions u and v of two real variables such that

$$f(z) = f(x + iy) = u(x, y) + iv(x, y). \tag{3.5}$$

The function u is called the real part of f and v is the imaginary part of f.

Example 3.1. Let $f(z) = z^2$. Then since $z = x+iy$, $z^2 = x^2 - y^2 + 2ixy$. Hence $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$.

With this result in mind, we can state conditions on u and v which guarantee that f is differentiable.

Theorem 3.2. *If $f(z) = f(x + iy) = u(x, y) + iv(x, y)$ where f is a complex function defined on an open set $S \subseteq \mathbb{C}$ and at some point $z_0 = x_0 + iy_0 \in S$ the partial derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ all exist, are continuous and satisfy the Cauchy-Riemann equations at z_0,*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{3.6}$$

then f is differentiable at z_0. Conversely, if f is differentiable at z_0 then the given partial derivatives exist and the Cauchy-Riemann equations are satisfied at $z = x + iy$.

Proof. The proof of this result is in two parts. We prove the second part first. Since $f'(z)$ exists, we calculate the limit in two different ways. In the first, we approach the limit on real axis. In the second we approach along the imaginary axis.

So we set $z + h = x + h + iy$. Then

$$\begin{aligned} f'(z) &= \lim_{h\to 0} \frac{f(z+h) - f(z)}{h} \\ &= \lim_{h\to 0} \frac{u(x+h,y) + iv(x+h,y) - u(x,y) - iv(x,y)}{h} \\ &= \lim_{h\to 0} \frac{u(x+h,y) - u(x,y)}{h} + i \lim_{h\to 0} \frac{v(x+h,y) - v(x,y)}{h} \\ &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}. \end{aligned}$$

The limit certainly exists because f is differentiable.

Now we approach the limit on the imaginary axis. So we set $z+ik = x + i(y+k)$. We then have

$$\begin{aligned} f'(z) &= \lim_{k\to 0} \frac{f(z+ik) - f(z)}{ik} \\ &= \lim_{k\to 0} \frac{u(x,y+k) + iv(x,y+k) - u(x,y) - iv(x,y)}{ik} \\ &= \lim_{k\to 0} \frac{u(x,y+k) - u(x,y)}{ik} + i \lim_{k\to 0} \frac{v(x,y+k) - v(x,y)}{ik} \\ &= -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}. \end{aligned}$$

Since the two limits must be equal, we see that

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}. \tag{3.7}$$

Equating real and imaginary parts shows that u and v satisfy the Cauchy-Riemann equations.

The proof of the other part is somewhat harder. We use a technical lemma from multivariable calculus. Suppose that for a given function u $\partial u/\partial x$ and $\partial u/\partial y$ exist at (x,y) and are continuous there. Then

$$u(x+h,y+k) = u(x,y) + h\left(\frac{\partial u}{\partial x} + A(h,k)\right) + k\left(\frac{\partial u}{\partial y} + B(h,k)\right), \tag{3.8}$$

where $A(h,k), B(h,k) \to 0$ as $h,k \to 0$. This result really only says that $u(x+h,y+k)$ is approximated by the first derivative terms of its Taylor series when h and k are small. It is nothing more than the two dimensional analogue of the statement that for h small $f(x+h) \approx f(x) + f'(x)h$.

We now wish to show that if the Cauchy-Riemann equations are satisfied, then $f'(z)$ exists. Let $z = z_0 + h + ik$. Then

$$
\begin{aligned}
f(z)-f(z_0) &= u(x+h,y+k)+iv(x+h,y+k)-u(x_0,y_0)-iv(x_0,y_0)\\
&= h\left(\frac{\partial u}{\partial x}+A(h,k)\right)+k\left(\frac{\partial u}{\partial y}+B(h,k)\right)\\
&\quad + ih\left(\frac{\partial v}{\partial x}+A_1(h,k)\right)+ik\left(\frac{\partial v}{\partial y}+B_1(h,k)\right),
\end{aligned}
$$

by our technical lemma applied to both u and v. The terms A, B, A_1, B_1 are the error terms from approximating u and v by their first partial derivatives. Then we observe that using the Cauchy-Riemann equations we have

$$
\begin{aligned}
f(z)-f(z_0) &= (h+ik)\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}+\theta_z\\
&= (z-z_0)\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)+\theta_z,
\end{aligned}
$$

in which θ_z denotes the remainder term. Because of the technical lemma $\lim_{z\to z_0}\theta_z/(z-z_0)=0$. (Check this!) Consequently

$$
\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}. \tag{3.9}
$$

In other words $f'(z)$ exists. This proves the theorem. □

3.1. **Examples of Differentiable Functions.** Let us show how the Cauchy-Riemann equations work in practice.

Example 3.1. We consider $f(z)=e^{iz}$. Then

$$
f(x+iy)=e^{i(x+iy)}=e^{-y}e^{ix}=e^{-y}\cos x+ie^{-y}\sin x.
$$

Then $u(x,y)=e^{-y}\cos x, v(x,y)=e^{-y}\sin x$. Then

$$
u_x=-e^{-y}\sin x=v_y, u_y=-e^{-y}\cos x=-v_y.
$$

These hold for all x, y, so f is differentiable everywhere.

Example 3.2. Let $f(z)=z^4$. Then

$$
f(x+iy)=(x+iy)^4=x^4-6x^2y^2+y^4+i(4x^3y-4xy^3).
$$

So

$$
u(x,y)=x^4-6x^2y^2+y^4
$$

and

$$
v(x,y)=4x^3y-4xy^3.
$$

Then

$$
u_x=4x^3-12xy^2=v_y
$$

and

$$
u_y=4y^3-12x^2y=-v_x.
$$

So the Cauchy-Riemann equations are satisfied and hence f is differentiable. In fact they are satisfied everywhere, so f is differentiable everywhere.

We also note that we have

$$f'(z) = u_x + iv_x = 4x^3 - 12xy^2 + i(12x^2y - 4y^3) = 4z^3.$$

Example 3.3. Let $f(z) = \sin z$. Then

$$f(x+iy) = \sin(x+iy) = \sin x \cos(iy) + \sin(iy)\cos x.$$

Now

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$
$$\cos x = \frac{e^{ix} + e^{-ix}}{2}.$$

So it follows that

$$\sin(iy) = -\frac{e^{y} - e^{-y}}{2i} = i\sinh y$$

and

$$\cos(iy) = \frac{e^{y} + e^{-y}}{2} = \cosh y.$$

So

$$f(x+iy) = \sin x \cosh y + i \cos x \sinh y.$$

Hence

$$u(x,y) = \sin x \cosh y, v(x,y) = \cos x \sinh y.$$

Then

$$u_x = \cos x \cosh y = v_y$$

and

$$u_y = \sin x \sinh y = -v_x.$$

These hold for all x, y. Hence $f(z)$ is differentiable everywhere.

It is also not hard to check that

$$f'(z) = u_x + iv_x = \cos x \cosh y - i \sin x \sinh y = \cos(x+iy) = \cos z.$$

Example 3.2. Let us consider the function $f(z) = e^{-|z|^2}$. Note that $|z|^2 = x^2 + y^2$. Thus $e^{-|z|^2} = e^{-x^2-y^2}$. Hence $u(x,y) = e^{-x^2-y^2}$ and $v(x,y) = 0$. Thus

$$\frac{\partial u}{\partial x} = -2xe^{-x^2-y^2}, \quad \frac{\partial u}{\partial y} = -2ye^{-x^2-y^2}, \quad \frac{\partial v}{\partial x} = 0, \quad \frac{\partial v}{\partial y} = 0.$$

Thus at $x = y = 0$ we have $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. But the Cauchy-Riemann equations are satisfied nowhere else. So the function is not differentiable anywhere else.

3.2. **Properties of Differentiable Functions.** In real analysis a function f is analytic in some domain $D \subseteq \mathbb{R}$ if it is equal to its Taylor expansion throughout D. A function on $\mathbb{R}$ which is once differentiable need not be twice differentiable, so a differentiable function which is differentiable on $\mathbb{R}$ is is not in general analytic.

However, as noted above, if a complex valued function is differentiable *once* in some region $D \subseteq \mathbb{C}$, then it is differentiable infinitely many times and is equal to its Taylor series expansion throughout D. Thus in the complex plane, differentiable functions are automatically analytic. This is actually a consequence of the Cauchy integral formula for a differentiable function that we will prove later. So in complex analysis, differentiable functions and analytic functions turn out to be the same thing.

So how do we formally define the term analytic? We could use the following definition from real analysis:

Definition A. A function $f : \Omega \to \mathbb{C}$ is said to be *analytic* in Ω if for every $z \in \Omega$ we can write $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ for some $z_0 \in \Omega$. That is, f is equal to its Taylor expansion throughout Ω.

However many textbooks on complex variable theory use the following definition.

Definition 3.3. A function $f : \Omega \to \mathbb{C}$ is said to be *analytic* in Ω if it is differentiable throughout Ω.

There are problems with this, because it can be confusing to the student, who is learning real and complex analysis at the same. In many books the term *holomorphic* is preferred for a function of a complex variable which is differentiable. But this now gives us three words which in complex analysis refer to the same set of functions.

We will refer to differentiable functions and also use the term analytic, but will endeavour to be clear about precisely what is meant in each context. What must be kept in mind is that by differentiable, we mean a function whose real and imaginary parts satisfy the Cauchy-Riemann equations.

The next two results are applications of the Cauchy-Riemann equations. The first is very familiar from one variable calculus.

Theorem 3.4. *Let f be differentiable on the open set S and let $f'(z) = 0$ on S. Then f is a constant throughout S.*

Proof. Since f is differentiable and $f'(z) = 0$, we have by the Cauchy-Riemann equations $u_x = u_y = 0$ and $v = v_y = 0$ and hence u and v are constant. So $f = u + iv$ is constant. Here we use u_x to denote $\partial u/\partial x$ etc. □

We can also say the following. Recall that $\Re(f)$ is the real part of f and $\Im(f)$ is the imaginary part of f.

Theorem 3.5. *Suppose that f is differentiable on an open set $S \subseteq \mathbb{C}$. If any one of $|f|$, $\Re(f)$ or $\Im(f)$ are constant on S, then f is constant.*

Proof. We only do the second. The other two parts are exercises. Since $\Re(f) = u$ is a constant, $u_x = u_y = 0$. Thus by the Cauchy-Riemann equations $v_x = -u_y = 0, v_y = u_x = 0$ So v is also a constant. □

We also note the following definition.

Definition 3.6. A function f which is differentiable for all $z \in \mathbb{C}$ is said to be *entire*.

3.3. **Analytic Functions and Laplace's Equation.** One of the major consequences of the Cauchy-Riemann equations is the fact that the real and imaginary parts of a differentiable function are solutions of Laplace's equation.

Theorem 3.7. *Suppose that $f(z) = u + iv$ is a differentiable function on an open set $S \subseteq \mathbb{C}$. Then u and v are harmonic. That is, they satisfy the two dimensional Laplace equation on S treated as a subset of $\mathbb{R}^2$.*

Proof. The proof of this is a very easy exercise. The Laplace equation in two dimensions is $\Delta\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$. We know that $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ So therefore $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 v}{\partial x \partial y} = \dfrac{\partial}{\partial y}\dfrac{\partial v}{\partial x} = -\dfrac{\partial^2 u}{\partial y^2}$, since $\dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y}$. Thus

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

The proof for v is identical. □

We thus have a way of generating infinitely many solutions of the Laplace equation.

Example 3.4. It is easy to see that $f(z) = z^n$ is differentiable. So that the real and imaginary parts of z^n are harmonic. Taking $n = 2$ we see that $z^2 = x^2 - y^2 + 2ixy$. So $u = x^2 - y^2$ and $v = 2xy$ are harmonic. This is easy to check. $u_{xx} = 2$, $u_{yy} = -2$. So $u_{xx} + u_{yy} = 0$. The calculation for v is similar.

Given a harmonic function u, a harmonic function v with the property that $f = u + iv$ is an analytic function, is said to be a *harmonic conjugate* of u. Similarly, if v is harmonic, a a harmonic function u such that $f = u+iv$ is analytic is said to be a harmonic conjugate of v. Harmonic conjugates are not unique, since, if v is a harmonic conjugate of u, then $v + c$ is also a harmonic conjugate, for any constant c.

Example 3.3. Let $u(x, y) = x^2 - y^2$. Suppose that

$$f(x + iy) = u(x, y) + iv(x, y)$$

is differentiable. What is v? Well we know that $u_x = v_y$ and $u_y = -v_x$. Since $u_x = 2x$ and $u_y = -2y$, it follows that $v_y = 2x$. Integrating with respect to y gives $v(x,y) = 2xy + k(x)$, where k is an arbitrary function of x. Now $v_x = 2y + k'(x) = -u_y = 2y$. So that $k' = 0$ and k is a constant.

Hence $v(x,y) = 2xy + C$ where C is a constant. Thus

$$f(x+iy) = x^2 - y^2 + 2ixy + iC = (x+iy)^2 + iC.$$

Thus $f(z) = z^2 + iC$.

We saw earlier that if we take $z = re^{i\theta}$ then

$$\frac{1}{1-z} = \frac{1}{1-re^{i\theta}} = \frac{1 - r\cos\theta + ir\sin\theta}{r^2 - 2r\cos\theta + 1}. \tag{3.10}$$

Notice that $1/(1-z)$ is differentiable. So the real and imaginary parts must be harmonic. Here however we have written the function in terms of polar coordinates.

Lemma 3.8. *The two dimensional Laplace equation when expressed in polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$ is*

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0. \tag{3.11}$$

Proof. We set $x = r\cos\theta$ and $y = r\sin\theta$. Applying the chain rule to calculate the derivatives $\partial^2/\partial x^2$ and $\partial^2/\partial y^2$ in terms of $\partial^2/\partial r^2$, $\partial^2/\partial\theta^2$ and $\partial/\partial r$ gives the result. □

We therefore deduce the following.

Proposition 3.9. *The functions*

$$K_1(r,\theta) = \frac{1 - r\cos\theta}{r^2 - 2r\cos\theta + 1}, \tag{3.12}$$

$$K_2(r,\theta) = \frac{r\sin\theta}{r^2 - 2r\cos\theta + 1}, \tag{3.13}$$

are solutions of the two dimensional Laplace's equation in polar coordinates.

The function K_1 is essentially what is known as the *Poisson kernel* on the disc. The Poisson kernel allows us to solve a very important class of boundary value problems for the Laplace equation. This is discussed in Differential Equations.

4. Integration in the Complex Plane

We begin with the straightforward remark that if $h_1, h_2 : [a,b] \to \mathbb{R}$ are integrable functions then the integral of $h(t) = h_1(t) + ih_2(t)$ is defined by setting

$$\int_a^b h(t)dt = \int_a^b h_1(t)dt + i\int_a^b h_2(t)dt.$$

This is not surprising, but it is important in what follows.

When studying complex integration we are primarily interested in integration along paths, or contours. A *contour integral* is an integral along some contour in the complex plane. In particular we will be interested in contour integrals taken around closed paths. For such integrals the most important result is the Cauchy Theorem. This is also probably the most important result in complex analysis. In this section however, we will introduce some of the basic facts about contour integrals.

We start with the definition of a contour.

Definition 4.1. A simple, smooth contour between two points $z_0, z_1 \in \mathbb{C}$, is a parameterised curve $\gamma : [a,b] \to \mathbb{C}$ such that

(1) $\gamma'(t)$ exists and is continuous for all $t \in [a,b]$

(2) $\gamma(a) = z_0$, $\gamma(b) = z_1$.

(3) γ does not pass through any point twice.

An integral of a complex function can be defined by analogy with the theory of Riemann integrals. However we will sidestep this and define our integral in such a way that it does the job we want it to do. The motivation as we shall see, comes from physics.

Definition 4.2. Let $f : D \subseteq \mathbb{C} \to \mathbb{C}$ be a continuous function. Suppose that $z_0, z_1 \in D$. Let $\gamma : [a,b] \to S$ be a simple smooth contour, with $\gamma(a) = z_0$ and $\gamma(b) = z_1$. Then the contour integral $\int_\gamma f(z)dz$ is defined by

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt. \tag{4.1}$$

Where does this idea come from? In physics, the work done on a body by a force $\mathbf{F}$ over a total displacement $\mathbf{d}$ is

$$W = \mathbf{F} \cdot \mathbf{d}.$$

Roughly we can think of it as the total energy expended. If a body is moved around a closed curve however, the total work done might actually be zero, but the physics need not concern us.

Now let the force at time t acting at the point $\mathbf{x}(t)$ be $\mathbf{F}(\mathbf{x}(t))$. The change in total work done in moving from $\mathbf{x}(t)$ to $\mathbf{x}(t+\Delta t)$ is

$$W(t+\Delta t) - W(t) \approx \mathbf{F}(\mathbf{x}(t)) \cdot [\mathbf{x}(t+\Delta t) - \mathbf{x}(t)].$$

So that

$$\frac{W(t+\Delta t) - W(t)}{\Delta t} \approx \frac{\mathbf{F}(\mathbf{x}(t)) \cdot [\mathbf{x}(t+\Delta t) - \mathbf{x}(t)]}{\Delta t}.$$

Taking the limit of both sides as $\Delta t \to 0$ gives

$$W'(t) = \mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t).$$

Thus the total work done moving from $\mathbf{x}(a)$ to $\mathbf{x}(b)$ is

$$W = \int_a^b \mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t) dt.$$

This is known as a *line integral.* This definition is for vectors $\mathbf{F}$ and curves $\mathbf{x}(t)$ on $\mathbb{R}^n$. The definition of a contour integral is taken from this. It turns out to be an extremely useful idea.

An important observation is that there is more than one path between z_0 and z_1. Thus in general there should be more than one possible value for the integral of f between two points in the complex plane. Integration along two different paths should give different values for the integrals. This is correct. However there is an important case when this is not true.

It turns out that if f has an anti-derivative F, then the value of the contour integral of f from z_0 to z_1, does not depend upon the path we take. It is said to be *path independent.* If the integral between z_0 and z_1 is path independent then for any contour γ between z_0 and z_1 we can write

$$\int_\gamma f(z)dz = \int_{z_0}^{z_1} f(z)dz.$$

Thus our usual idea of an integral is actually a contour integral, where the contour is $\gamma(t) = t$.

Example 4.1. Consider the function $f(z) = z^2$. Take $z_0 = 0$ and $z_1 = 1+i$. Take the path $\gamma(t) = (1+i)t$, $t \in [0,1]$. Clearly $\gamma(0) = 0$ and $\gamma(1) = 1+i$. We have $f(\gamma(t)) = ((1+i)t)^2$. Also $\gamma'(t) = 1+i$. So by definition

$$\begin{aligned}
\int_\gamma z^2 dz &= \int_0^1 f(\gamma(t))\gamma'(t)dt \\
&= \int_0^1 (1+i)^2 t^2 (1+i)dt \\
&= (1+i)^3 \int_0^1 t^2 dt = \frac{(1+i)^3}{3} = -\frac{2}{3} + \frac{2}{3}i.
\end{aligned}$$

We will see below that this integral can be obtained in a rather easier manner since it is in fact path independent.

Exercise. Show that if $f(z) = z^2$ and $\gamma(t) = t^2 + it, t \in [0,1]$, then

$$\int_0^1 f(\gamma(t))\gamma'(t)dt = -\frac{2}{3} + \frac{2}{3}i.$$

Example 4.1. Let $f(z) = z^2$ and take the contour $\gamma(t) = e^{it}$, $t \in [0, \pi]$. Then by our definition

$$\begin{aligned}\int_\gamma f(z)dz &= \int_0^\pi f(\gamma(t))\gamma'(t)dt \\ &= \int_0^\pi (e^{it})^2 ie^{it}dt,\end{aligned}$$

since $\gamma'(t) = ie^{it}$. Hence

$$\begin{aligned}\int_\gamma f(z)dz &= i\int_0^\pi e^{3it}dt \\ &= i\frac{1}{3i}\left[e^{3it}\right]_0^\pi \\ &= \frac{1}{3}(e^{3\pi i} - 1) \\ &= -\frac{2}{3}.\end{aligned}$$

4.0.1. *Properties of Contour Integrals.* A result from one dimensional calculus which is often useful gives the length of a contour.

Proposition 4.3. *Let $\gamma : [a,b] \to \mathbb{C}$ be a simple smooth contour in $\mathbb{C}$. Then the length $L(\gamma)$ of the contour between $t = a$ and $t = b$ is*

$$L(\gamma) = \int_a^b |\gamma'(t)|dt \tag{4.2}$$

where $|z|$ is the modulus of z.

Proof. The length of a curve $(x(t), y(t)), t \in [a,b]$ in $\mathbb{R}^2$ is

$$L = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}dt.$$

This is essentially an application of Pythagoras' Theorem and the method of Riemann sums. Applying this to the contour γ gives the result. □

Another technical lemma which is extremely useful is the so called ML inequality.

Lemma 4.4 (The ML inequality). *Suppose that a complex valued function $f : D \subseteq \mathbb{C} \to \mathbb{C}$ has a maximum value of M on the contour γ. Suppose also that the length of γ is L. Then*

$$\left|\int_\gamma f(z)dz\right| \leq ML. \tag{4.3}$$

Proof. We have

$$\begin{aligned}\left|\int_\gamma f(z)dz\right| &= \left|\int_a^b f(\gamma(t))\gamma'(t)dt\right|\\ &\le \max|f(\gamma(t))|_{t\in[a,b]}\int_a^b|\gamma'(t)|dt\\ &= ML.\end{aligned}$$

□

As with integration of functions of a real variable, there is a fundamental theorem of contour integration. However unlike the real variable version of the theorem, it is not used as often. The most important result about contour integration is in fact Cauchy's theorem, which we discuss in the next chapter. Nevertheless, the Fundamental Theorem does have important consequences.

4.1. **The Fundamental Theorem of Contour Integration.**

Theorem 4.5 (The fundamental theorem of contour integration)**.** *Suppose that $F' = f$, $D \subseteq \mathbb{C}$ is open and $f : D \subseteq \mathbb{C}$ is continuous and γ is a contour in D with end points z_0 and z_1. Then*

$$\int_\gamma f(z)dz = F(z_1) - F(z_0) \tag{4.4}$$

Proof. Suppose that $\gamma(t), t \in [a,b]$ is a smooth contour which connects z_0 and z_1. So that $\gamma(a) = z_0$ and $\gamma(b) = z_1$. Then by the chain rule

$$\begin{aligned}\frac{d}{dt}F(z(t)) &= \frac{dF}{dz}\frac{dz}{dt}\\ &= f(\gamma(t))\gamma'(t),\end{aligned} \tag{4.5}$$

since $F' = f$ and $z(t) = \gamma(t)$. From the definition of a contour integral we see that

$$\begin{aligned}\int_\gamma f(z)dz &= \int_a^b f(\gamma(t))\gamma'(t)dt\\ &= \int_a^b \frac{d}{dt}F(\gamma(t))dt\\ &= [F(\gamma(t))]_a^b = F(\gamma(a)) - F(\gamma(b))\\ &= F(z_1) - F(z_0).\end{aligned} \tag{4.6}$$

This is true for any contour γ and hence the result is proven. □

Example 4.2. If we use the Fundamental Theorem to redo the integral of $f(z) = z^2$ between 0 and $1+i$ we find that $F(z) = z^3/3$ so that

$$\int_0^{1+i} f(z)dz = \left[\frac{z^3}{3}\right]_0^{1+i} = \frac{(1+i)^3}{3},$$

which is the same answer we found before.

Example 4.3. Let $f(z) = ze^z$, $\gamma(t) = t + it^2, t \in [0,1]$. We set up the integral $\int_\gamma f(z)dz$.

Since $f(\gamma(t)) = (t + it^2)e^{t+it^2}$ and $\gamma'(t) = 1 + it$, by the definition

$$\begin{aligned}\int_\gamma f(z)dz &= \int_0^1 f(t+it^2)(1+it)dt \\ &= \int_0^1 (1+it)(t+it^2)e^{t+it^2}dt.\end{aligned}$$

In principle we can evaluate this integral, but it is easier to use the Fundamental Theorem. Since $\gamma(0) = 0, \gamma(1) = 1 + i$ and

$$ze^z = \frac{d}{dz}(z-1)e^z, \tag{4.7}$$

we have

$$\begin{aligned}\int_\gamma f(z)dz &= \int_0^{1+i} ze^z dz \\ &= [(z-1)e^z]_0^{1+i} \\ &= 1 + ie^{1+i}.\end{aligned}$$

4.2. **Integration Along Several Contours.** A very useful idea when dealing with contour integrals is that of integrating along a series of contours in succession. Suppose that $\gamma_1(t), t \in [a,b]$ and $\gamma_2(t), t \in [b,c]$ define contours with $\gamma_1(b) = \gamma_2(a)$. Then $\gamma = \gamma_1 + \gamma_2$ defines a contour and it is not hard to show that

$$\int_\gamma f = \int_{\gamma_1} f + \int_{\gamma_2} f.$$

In fact we make this notion more general. If $\gamma_1, ..., \gamma_k$, are contours, then $\gamma = \gamma_1 + \cdots + \gamma_k$ is a contour and we define

$$\int_\gamma f = \int_{\gamma_1} f + \cdots + \int_{\gamma_k} f.$$

Example 4.4. A very important contour is the one which runs along the real axis from $z = -R$ to $z = R$ and then moves in the semi- circular arc of radius R back to its starting point at $z = -R$.

The contour along the real axis can be parameterised by setting $\gamma_1(t) = t, \quad -R \le t \le R$. The semicircle part of the contour can be parameterised by $\gamma_2(t) = Re^{it}, \quad 0 \le t \le \pi$. Then by definition if $\gamma = \gamma_1 + \gamma_2$ we have

$$\begin{aligned}\int_\gamma f(z)dz &= \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz \\ &= \int_{-R}^R f(t)dt + \int_0^\pi f(Re^{it})Rie^{it}dt. \end{aligned}\tag{4.8}$$

This particular contour is one of the most commonly used in complex analysis.

An important result which flows from the fundamental theorem is the following.

Theorem 4.6. *Let $f : D \subseteq \mathbb{C} \to \mathbb{C}$ be a continuous differentiable function. Then the following statements are equivalent.*

(1) *There exists a differentiable function $F : D \to \mathbb{C}$ such that $F' = f$ throughout D.*

(2) *$\int_\gamma f = 0$ for every closed contour γ contained in D.*

(3) *The integral $\int_\gamma f$ depends only on the endpoints of γ for any contour γ in D.*

Proof. First we prove that (1) implies (2). Let γ be a closed contour connecting z_0 to itself. Then by the existence of an antiderivative F for f we get $\int_\gamma f = F(z_0) - F(z_0) = 0$.

To show that (2) implies (3) we let γ_1 and γ_2 be two contours connecting z_0 and z_1. Let $\gamma = \gamma_1 - \gamma_2$. Then γ is a contour which starts at z_0 and moves along γ_1 to z_1 then moves back to z_0 along γ_2. Hence it is a closed contour. By part (2) we have

$$\int_\gamma f = 0 = \int_{\gamma_1} f - \int_{\gamma_2} f.$$

Which tells us that $\int_{\gamma_1} f = \int_{\gamma_2} f$. This tells us that the contour integral connecting z_0 and z_1 is independent of the path we take, which is the content of (3).

Now we have to show that (3) implies (1). Fix some point $z_0 \in D$ and take a second point $z_1 \in D$. We wish to prove that there is an F such that $F'(z_1) = f(z_1)$. We do this by constructing the function. Fix some contour γ connecting z_0 and z_1. Then define

$$F(z_1) = \int_\gamma f.$$

Now consider another point $z_1 + h$ contained within D. We can connect z_1 and $z_1 + h$ by means of a line segment $\lambda(t) = z_1 + ht$, for $t \in [0, 1]$. We must have

$$F(z_1 + h) = \int_\gamma f + \int_\lambda f.$$

Thus

$$F(z_1 + h) - F(z_1) = \int_\gamma f + \int_\lambda f - \int_\gamma = \int_\lambda f.$$

This tells us that

$$\frac{F(z_1 + h) - F(z_1)}{h} = \frac{1}{h} \int_\lambda f.$$

From the definition of a contour integral we can write

$$\int_\lambda g(z)dz = \int_0^1 g(\lambda(t))\lambda'(t)dt = \int_0^1 g(z_1 + ht)hdt.$$

Now take $g(z) = f(z_1)$. Since z_1 is fixed, $f(z_1)$ is a constant, so we have

$$\int_\lambda \frac{f(z_1)}{h}dz = \int_0^1 h\frac{f(z_1)}{h}dt = f(z_1).$$

Thus

$$\frac{F(z_1+h) - F(z_1)}{h} - f(z_1) = \frac{1}{h}\int_\lambda (f(z) - f(z_1))dz.$$

Now f is continuous. So given ϵ there exists a $\delta > 0$ such that

$$|z - z_1| < \delta \implies |f(z) - f(z_1)| < \epsilon.$$

We therefore choose $|h| < \delta$, so that

$$\left|\frac{f(z) - f(z_1)}{h}\right| < \frac{\epsilon}{|h|}.$$

By the ML inequality since the length of the contour λ is $|h|$, we obtain

$$\left|\int_\lambda (f(z_1) - f(z_1))dz\right| \leq |h|\frac{\epsilon}{|h|} = \epsilon.$$

This shows that for $|h| < \delta$

$$\left|\frac{F(z_1+h) - F(z_1)}{h} - f(z_1)\right| < \epsilon.$$

Since ϵ is arbitrary we deduce that

$$\lim_{h\to 0}\frac{F(z_1+h) - F(z_1)}{h} = f(z_1).$$

Therefore $F'(z_1) = f(z_1)$. Which is of course (1). So (3) implies (1) and the proof is complete.

□

Notice that the second part of the theorem tells us that if f has an antiderivative, then $\int_\gamma f = 0$ for any closed contour γ. This is a special case of Cauchy's Theorem, which will be discussed in detail below.

However, it is not always the case that an antiderivative can be found that is valid throughout the entire region. The next example turns out to be crucial, though it looks innocuous.

Example 4.5. Integrate the function $f(z) = \dfrac{1}{z}$ around a circular contour of radius R centered at the origin.

Solution. Although $\ln z$ is an antiderivative for f, neither f nor the logarithm are continuous at $z = 0$, which is a point contained with the circle of radius R. So we cannot conclude that the integral around the closed circle is equal to zero. In fact is is not. To calculate the integral we have to parameterise the contour. Handling circles is easy. We just set $\gamma(t) = Re^{it}, 0 \leq t < 2\pi$. Then by definition for any f the contour integral is $\int_\gamma f(z)dz = \int_0^{2\pi} f(Re^{it})Rie^{it}dt$. So the value of the integral is

$$\begin{aligned}\int_\gamma \frac{dz}{z} &= \int_0^{2\pi} \frac{1}{Re^{it}} Rie^{it} dt \\ &= i \int_0^{2\pi} dt = 2\pi i. \end{aligned} \tag{4.9}$$

Of interest in this example is that fact that the function $1/z$ has a very special kind of singularity at $z = 0$. It is a *pole* and this function actually plays a crucial part in the theory of integration of functions of a complex variable. More precisely, this is a simple example of the Cauchy *residue theorem.* The residue theorem is a consequence of Cauchy's theorem. So Cauchy's Theorem and its consequences will be the subject of the next chapter.

5. Cauchy's Theorem and its Consequences

In many ways, the real heart of complex analysis lies in Cauchy's integral theorem, its extensions and applications. This profound and beautiful result displays the power of complex variables. It tells us that the integral of a differentiable function around a closed simple path is always equal to zero. This seemingly simple observation has profound consequences. For one, it gives us an explicit formula for the computation of contour integrals, which in turn allows us to evaluate many real integrals.

Actually, the consequences of Cauchy's Theorem go way beyond merely evaluating integrals. We will be able to show that a function which is once differentiable in some $\Omega \subseteq \mathbb{C}$ is actually infinitely differentiable in Ω and equal to its Taylor series expansion throughout Ω. We will be able to prove Liouville's theorem, which says that a bounded everywhere complex differentiable function is a constant. This in turn leads to the fundamental theorem of algebra. There are many more applications, which will hopefully justify to the reader the rather grandiose claim that Cauchy's Theorem is one of the most profound in mathematics.

The full version of Cauchy's Theorem is actually very difficult to prove. Proofs of the theorem for Discs and rectangular regions, or for more general so called *star shaped domains* may be obtained with a considerable amount of work. The general result on an arbitrary simply connected domain, however, is highly non trivial to prove. We refer the reader to any advanced text on complex analysis for the result, which we merely state. We will however prove a simple version of the theorem.

Theorem 5.1 (Cauchy's Theorem)**.** *Let f be differentiable in a simply connected region D. Then for any simple closed contour γ in D,*

$$\int_\gamma f(z)dz = 0. \tag{5.1}$$

The full result is often called the Cauchy-Goursat theorem, since Goursat established a proof under less stringent assumptions than Cauchy did. We however will use the term Cauchy's Theorem.

A version of Cauchy's theorem for a triangle is relatively straightforward to prove and from this we can prove a version of the Cauchy theorem on any *star domain.*

Definition 5.2. A simply connected open subset $D \subseteq \mathbb{C}$ is said to be a *star domain* or *star shaped domain* with *star centre* z_* if given any point $z \in D$ the line segment connecting z_* and z is contained wholly within D.

There are many examples of star domains. The simplest is an open disc. The star centre in this case is just the centre of the disc. The interior on an ellipse is also a star domain. So is a rectangular region, or a region bounded by a triangle. With this in mind we can now state the following.

Theorem 5.3 (Cauchy's Theorem for a triangle)**.** *Let f be a differentiable function in a domain D. Let T be a triangle contained wholly within D. Suppose that the boundary of T is denoted by ∂T. Then $\int_{\partial T} f = 0$.*

Proof. See the book by Stewart and Tall [5] for the details. The basic idea is rather ingenious and is due to E.H. Moore. The procedure is to divide the triangle up into n subtriangles and use the ML Inequality to estimate the size of the contour integral on each subtriangle. This allows one to estimate $\left|\int_{\partial T} f\right|$. In particular if $|\partial T|$ is the length of the outer triangle, then one can show that for any $\epsilon > 0$ the estimate

$$\left|\int_{\partial T} f\right| \leq (\frac{1}{4})^n \epsilon |\partial T|^2 \tag{5.2}$$

holds. Since this holds for all $\epsilon > 0$ we must have $\int_{\partial T} f = 0$. □

Theorem 5.4. *If f is differentiable in any star shaped domain D with star centre z_*, the function $F(z) = \int_{z_*}^{z} f(u)du$ is an antiderivative for f.*

Proof. We have to show that F is differentiable and $F'(z) = f(z)$. Consider a triangle with vertices z_*, z_1 and $z_1 + h$, where h is small enough so that all points lie in D. Now consider the contour consisting of the straight edges of the triangle, traverses from z_* to z_1 then to $z_1 + h$ then back to z_*. Since this is a closed contour and f is analytic, the Cauchy theorem for a triangle tells us that

$$\int_{z_*}^{z_1} f + \int_{z_1}^{z_1+h} f + \int_{z_1+h}^{z_*} f = 0.$$

Now observe that

$$\int_{z_*}^{z_1} f + \int_{z_1}^{z_1+h} f + \int_{z_1+h}^{z_*} f = F(z_1) + \int_{z_1}^{z_1+h} f - \int_{z_*}^{z_1+h} f$$

$$= F(z_1) - F(z_1 + h) + \int_{z_1}^{z_1+h} f = 0 \tag{5.3}$$

Or

$$F(z_1 + h) - F(z_1) = \int_{z_1}^{z_1+h} f,$$

which implies that

$$\frac{F(z_1 + h) - F(z_1)}{h} = \frac{1}{h}\int_{z_1}^{z_1+h} f. \tag{5.4}$$

We now calculate the limit as $h \to 0$ to show that $F' = f$. The calculation of this limit is in fact exactly the same argument as in the last part of the proof of 4.6 so we omit it. □

The consequence of this result is the following version of Cauchy's Theorem.

Theorem 5.5 (Cauchy's Theorem on a star domain)**.** *If f is a differentiable function in a star domain D, then for all closed contours γ in D the integral $\int_\gamma f = 0$. Moreover, the integral of f between any two points in D is independent of path.*

Proof. This is a straightforward corollary to the previous theorem and theorem 4.6. □

5.1. **Double Integrals and Green's Theorem.** Another proof uses Green's Theorem in the plane from multi-variable calculus. Before presenting this, we revise some material on double integrals.

Let $f(x, y)$ be a continuous function defined over some region

$$D = \{(x, y) \subseteq \mathbb{R}^2, a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}.$$

Then

$$\begin{aligned}\int\int_D f(x,y)dydx &= \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx \\ &= \int_a^b \left(\int_{g_1(x)}^{g_2(x)} f(x,y)dy \right) dx.\end{aligned}$$

Equally, if we can write

$$D = \{(x, y) \subseteq \mathbb{R}^2, h_1(y) \leq x \leq h_2(y), c \leq y \leq d\},$$

then

$$\begin{aligned}\int\int_D f(x,y)dydx &= \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)dxdy \\ &= \int_c^d \left(\int_{h_1(y)}^{h_2(y)} f(x,y)dx \right) dy.\end{aligned}$$

The point is that if f is continuous and hence integrable, the order of integration does not matter.

Example 5.1. Evaluate $\int_1^4 \int_{-1}^2 (2x + 6x^2y)dydx$.

Solution Using the definition we have

$$\begin{aligned}\int_1^4\int_{-1}^2(2x+6x^2y)dydx &= \int_1^4\left(\int_{-1}^2(2x+6x^2y)dy\right)dx\\ &= \int_1^4\left[2xy+3x^2y^2\right]_{-1}^2dx\\ &= \int_1^4(6x+9x^2)dx\\ &= \left[3x^2+3x^3\right]_1^4=234.\end{aligned}$$

As an exercise, show that evaluating $\int_{-1}^2\int_1^4(2x+6x^2y)dxdy$ produces the same answer.

Example 5.2. Evaluate $I=\int_0^2\int_{x^2}^{2x}(x^3+4y)dydx$.

Solution. Again we use the definition as an iterated integral to obtain

$$\begin{aligned}I=\int_0^2\int_{x^2}^{2x}(x^3+4y)dydx &= \int_0^2\left(\int_{x^2}^{2x}(x^3+4y)dy\right)dx\\ &= \int_0^2\left[x^3y+2y^2\right]_{x^2}^{2x}dx\\ &= \int_0^2(8x^2-x^5)dx\\ &= \left[\frac{8}{3}x^3-\frac{1}{6}x^6\right]_0^2=\frac{32}{3}.\end{aligned}$$

We can reverse the order of integration and the result will be the same. We notice that if $x^2=2x$ then $x=0$ or 2 and $y=2x$ implies $x=1/2y$ and $y=x^2$ gives $x=\sqrt{y}$. So the integral is

$$\begin{aligned}I &= \int_0^4\int_{1/2y}^{\sqrt{y}}(x^3+4y)dxdy\\ &= \int_0^4\left[\frac{1}{4}x^4+4xy\right]_{1/2y}^{\sqrt{y}}dy\\ &= \int_0^4\left(4y^{3/2}-\frac{y^4}{64}-\frac{7y^2}{4}\right)dy\\ &= \frac{32}{3}.\end{aligned}$$

For certain regions we can work with different coordinate systems. Circular regions are best described by polar coordinates. We set

$$x=r\cos\theta,\ y=r\sin\theta,\quad 0\le a\le r\le b<\infty, 0\le\theta_1\le\theta\le\theta_2\le 2\pi.$$

The most important thing to remember when doing integration in polar coordinates is that

$$dxdy=rdrd\theta.$$

In general if we make a change of variables $x = x(u,v), y = y(u,v)$ then in the new coordinates

$$dxdy = \left|\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}\right| dudv = \left|\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right| dudv. \tag{5.5}$$

Taking the absolute value means that the result is not changed if we swap the columns of the matrix. The matrix appearing in the determinant,

$$J(u,v) = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \tag{5.6}$$

is called the Jacobian (after Carl Jacobi, 1804-1851). So

$$dxdy = |\det J(u,v)| dudv.$$

The Jacobian represents the derivative of a function from $\mathbb{R}^2 \to \mathbb{R}^2$. More precisely, if

$$f(x,y) = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix},$$

then the function

$$L(h) = \begin{pmatrix} f_1(a,b) \\ f_2(a,b) \end{pmatrix} + \begin{pmatrix} \frac{\partial f_1}{\partial x}(a,b) & \frac{\partial f_1}{\partial y}(a,b) \\ \frac{\partial f_2}{\partial x}(a,b) & \frac{\partial f_2}{\partial y}(a,b) \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}, \tag{5.7}$$

is the best linear approximation to

$$f(a+h) = \begin{pmatrix} f_1(a+h_1, b+h_2) \\ f_2(a+h_1, b+h_2) \end{pmatrix},$$

when $\|h\|$ is small. Here $\dfrac{\partial f_1}{\partial x}(a,b)$ is $\dfrac{\partial f_1}{\partial x}$ evaluated at the point (a,b) etc.

This idea for the meaning of the derivative is actually the same as we use in the one variable case, where

$$f(a+h) \approx f(a) + f'(a)h,$$

when h is small. We think of the derivative in this way in higher dimensions because the one dimensional definition

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h},$$

does not make sense in the case when f and h are vectors.

As an example, consider the case when we change variables by letting $x = u + v$ and $y = u - v$. Then

$$J(u,v) = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

so that $|\det(J(u,v))| = 2$ and $dxdy = 2dudv$.

Exercise. Use the relations $x = r\cos\theta$ and $y = r\sin\theta$ and the Jacobian to show that $dxdy = rdrd\theta$.

Solution

$$J(r,\theta) = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}.$$

Thus $|\det(J(r,\theta))| = r\cos^2\theta + r\sin^2\theta = r$. Hence $dxdy = rdrd\theta$.

Example 5.3. Integrate the function $f(x,y) = \sqrt{x^2+y^2}$ over the semi-circle of radius 1 with centre $(0,0)$.

Solution

We work in polar coordinates. The semi-circle is the region $0 \le r \le 1$ and $0 \le \theta \le \pi$. We also have $f(x,y) = \sqrt{x^2+y^2} = r$. So that if D denotes the semi-circle we have

$$\begin{aligned} \int\int_D f(x,y)dxdy &= \int_0^1 \int_0^\pi r^2 drd\theta \\ &= \int_0^1 \pi r^2 dr \\ &= \left[\frac{1}{3}\pi r^3\right]_0^1 \\ &= \frac{1}{3}\pi. \end{aligned}$$

It should be clear that the area of the region D is given by

$$A = \int\int_D dxdy. \tag{5.8}$$

We also define the average value of a function f over a region D by

$$\bar{f} = \frac{1}{A}\int\int_D f(x,y)dxdy.$$

These concepts are important in physics and engineering, where we often need to calculate quantities like the centre of mass for some body, but we will not discuss these problems.

An important result in multi-variable calculus relates a line integral to a double integral. It is known as Green's Theorem in the plane. There are also versions which holds in three dimensions, but we will not discuss them.

We can integrate a function $f(x,y)$ over a curve in the plane in the following way. We suppose the curve C is given by $y = g(x)$, $x \in [a,b]$. Then we define

$$\int_C f(x,y)dx = \int_a^b f(x,g(x))dx.$$

Example 5.4. Integrate $f(x,y) = y^2$ over the curve $y = \cos x$, for $x \in [0,\pi/2]$.

Solution

The definition gives

$$\begin{aligned}\int_C f(x,y)dx &= \int_0^{\pi/2} \cos^2 x dx \\ &= \int_0^{\pi/2} \frac{1}{2}(1+\cos(2x))dx \\ &= \left[\frac{x}{2} + \frac{1}{4}\sin(2x)\right]_0^{\pi/2} \\ &= \frac{\pi}{4}.\end{aligned}$$

As in the case of the contour integral, these so called line integrals are often taken over closed curves. Typically we think of this as integrating over a curve $g_1(x)$ and a second curve $g_2(x)$ where the curves satisfy $g_1(a) = g_2(a)$ and $g_1(b) = g_2(b)$. Suppose that the curve is traversed from right to left along g_1 and then from left to right along g_2 in order to get back to the starting point, then we would have

$$\oint_C f(x,y)dx = \int_a^b f(x, g_1(x))dx - \int_a^b f(x, g_2(x))dx. \tag{5.9}$$

The minus sign in front of the second integral is because we are moving *backwards* to arrive back at the start point. If the curve is traversed anti-clockwise, we say that the orientation is positive.

The following result was proved by George Green in 1828 in a paper on the theory of electricity and magnetism. Green was an important mathematician and physicist who was almost entirely self taught before entering university, having only completed a year at school.

Theorem 5.6 (Green's Theorem)**.** *Let D be a simply connected region in $\mathbb{R}^2$ and let C be its piecewise smooth boundary, which is traversed counterclockwise. Let P and Q be continuous with continuous first partial derivatives in a disk containing D. Then*

$$\oint_C Pdx + Qdy = \int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy. \tag{5.10}$$

Proof. Our proof is taken from Grossman [1], p941. We suppose that D can be described by $D = \{(x,y) : a \le x \le b, g_1(x) \le g_2(x)\}$ or $D = \{(x,y) : h_1(y) \le x \le h_2(y), c \le y \le d\}$. Then

$$\begin{aligned}\int\int_D \frac{\partial P}{\partial y} dxdy &= \int_a^b \left(\int_{g_1(x)}^{g_2(x)} \frac{\partial P}{\partial y} dy\right) dx \\ &= \int_a^b [P(x,y)]_{g_1(x)}^{g_2(x)} \\ &= \int_a^b [P(x, g_2(x)) - P(x, g_1(x))]dx\end{aligned}$$

Comparing this to (5.9) we have

$$\oint_C Pdx = -\int_a^b [P(x, g_2(x)) - P(x, g_1(x))]dx$$

so that

$$\int\int_D \left(-\frac{\partial P}{\partial y}\right) dxdy = \oint_C Pdx.$$

Repeating this calculation with the second description of the region D gives

$$\int\int_D \frac{\partial Q}{\partial x} dxdy = \oint_C Qdy.$$

Combining these gives the result. □

Example 5.1. Evaluate the line integral $\oint_C (xydx + (x-y)dy)$ where C is the boundary of the rectangle $\{(x,y) : 0 \leq x \leq 1, 1 \leq y \leq 3\}$.
Solution We let $P(x,y) = xy, Q(x,y) = x - y$. Then $\frac{\partial Q}{\partial x} = 1$ and $\frac{\partial P}{\partial y} = x$. So

$$\begin{aligned}\oint_C (xydx + (x-y)dy) &= \int_0^1 \int_1^3 (1-x)dydx \\ &= \int_0^1 (1-x)y\Big|_1^3 dx \\ &= \int_0^1 2(1-x)dx \\ &= 1.\end{aligned}$$

Example 5.2. Evaluate $\oint_C ((x^3 + y^3)dx + (2y^3 - x^3)dy)$ where C is the unit circle.
Solution We compute the partial derivatives

$$\frac{\partial P}{\partial y} = 3y^2, \ \frac{\partial Q}{\partial x} = -3x^2. \tag{5.11}$$

So that

$$\begin{aligned}\oint_C ((x^3 + y^3)dx + (2y^3 - x^3)dy) &= -3\int\int_{x^2+y^2\leq 1} (x^2 + y^2)dxdy \\ &= -3\int_0^{2\pi}\int_0^1 r^3 drd\theta\end{aligned}$$

since $dxdy = rdrd\theta$. So we have

$$\begin{aligned}\oint_C ((x^3+y^3)dx + (2y^3 - x^3)dy) &= -3\int_0^{2\pi} \frac{r^4}{4}\Big|_0^1 d\theta \\ &= -\frac{3}{4}\int_0^{2\pi} d\theta \\ &= -\frac{3\pi}{2}.\end{aligned}$$

With Green's Theorem we can prove the following version of Cauchy's Theorem.

Theorem 5.7 (Cauchy). *Let D be a simply connected domain in $\mathbb{C}$, and let $f(z)$ be a differentiable function on D. Then for any simple, closed, piecewise smooth curve γ in D*

$$\int_\gamma f(z)dz = 0.$$

Proof. The idea is to set up the integral $\int_\gamma f(z)dz$ in terms of u and v. Since $z = x + iy$ then $dz = dx + idy$. Thus

$$\begin{aligned}\int_\gamma f(z)dz &= \int_\gamma (u+iv)(dx+idy) \\ &= \int_\gamma (udx - vdy) + i\int_\gamma (vdx + udy). \qquad (5.12)\end{aligned}$$

To this last expression we can now apply Green's Theorem. If Ω denotes the interior of γ then by Green's Theorem

$$\int_\gamma (udx - vdy) = \int\int_\Omega \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dxdy \qquad (5.13)$$

But $-\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$ by the Cauchy Riemman equations. So that

$$\int_\gamma (udx - vdy) = 0. \qquad (5.14)$$

We also have

$$\int_\gamma (vdx + udy) = \int\int_\Omega \left(\frac{\partial u}{\partial x} - \frac{\partial y}{\partial y}\right) dxdy, \qquad (5.15)$$

Since $\frac{\partial u}{\partial x} = \frac{\partial y}{\partial y}$ by the Cauchy-Riemann equations it is clear that

$$\int_\gamma (vdx + udy) = 0. \qquad (5.16)$$

Thus $\int_\gamma f(z)dz = 0$. □

More general versions of Cauchy's Theorem require more sophisticated methods and make use of the Jordan curve theorem. The reader is advised to consult a more advanced text if she wishes to learn more.

Chapters 8 and 9 of Stewart and Tall provides a good starting point for more advanced work. We will conclude this section with an important generalisation of the Cauchy Theorem. This is Cauchy's Theorem for a system of contours.

Theorem 5.8 (Cauchy's integral theorem for a system of contours)**.** *Let D be an arbitrary domain in the complex plain and let f be a differentiable function on D. Let $C, \gamma_1, ..., \gamma_n$ be a system of $n+1$ closed contours contained in D which satisfy the following conditions:*

(1) *The interior of C contains every contour $\gamma_1, ..., \gamma_n$.*

(2) *For every $k = 1, ..., n$ the exterior of γ_k contains γ_j, $j \neq k$.*

(3) *D contains the multiply connected domain*

$$\Omega = I(C) - \overline{I(\gamma_1)} - \cdots - \overline{I(\gamma_n)}$$

with boundary $C \cup \gamma_1 \cup \cdots \cup \gamma_n$.

Then

$$\int_C f(z)dz = \int_{\gamma_1} f(z)dz + \cdots + \int_{\gamma_n} f(z)dz$$

where all the contours are taken in the same direction. (e.g. all counterclockwise).

Note. The technical conditions here are not difficult to understand. The first says that each smaller contour γ_i is wholly contained in the larger, outer contour C. The second condition just means that no two of the interior contours overlap each other. The third just means that the domain D actually contains the entire set of curves, with no bits missing. This will always be true if D is simply connected.

A proof of this result can be found in Silverman [2].

We however are more interested in what can be done with this most remarkable theorem.

5.2. **Applications of Cauchy's Theorem.** As a first application of the theorem, we will establish the values of the so called Fresnel integrals.

Example 5.3. Evaluate the Fresnel integrals. That is, show that

$$\int_0^\infty \cos(t^2)dt = \int_0^\infty \sin(t^2)dt = \frac{\sqrt{\pi}}{2\sqrt{2}}. \tag{5.17}$$

Example 5.4. *Solution.* We consider the function $f(z) = e^{iz^2}$. This is entire, so we may certainly use Cauchy's Theorem. As always with a contour integral, the trick is to choose the right contour. In this case we consider $\gamma = \gamma_1 + \gamma_2 - \gamma_3$ where

(1) $\gamma_1(t) = t \qquad 0 \leq t \leq R,$

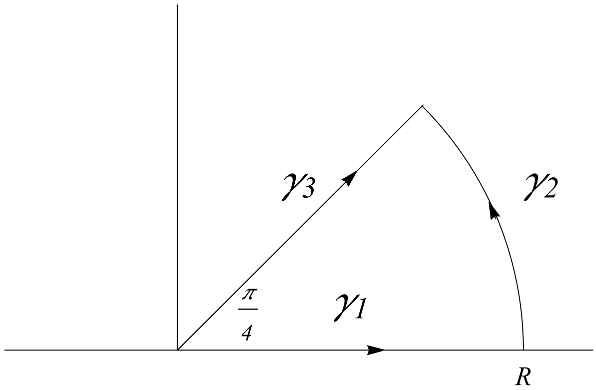

FIGURE 1. A wedge shaped Contour.

(2) $\gamma_2(t) = Re^{it} \quad 0 \le t \le \frac{\pi}{4}$,

(3) $\gamma_3(t) = te^{i\frac{\pi}{4}} \quad 0 \le t \le R$.

This contour runs along the real axis to $z = R$, then moves in a circular arc from $z = R$ through an angle of $\pi/4$ radians (45 degrees). Then it moves along the straight line segment from $z = Re^{i\pi/4}$ back to the origin. See Figure 1.

By Cauchy's theorem $\int_\gamma f = 0$. At first glance this does not appear to be terribly useful. However we observe the following. For all $R > 0$

$$\begin{aligned}\int_\gamma f &= \int_{\gamma_1} f + \int_{\gamma_2} f - \int_{\gamma_3} f \\ &= \int_0^R e^{it^2} dt + \int_0^{\pi/4} e^{iR^2 e^{2it}} Rie^{it} dt - \int_0^R e^{it^2 e^{i\pi/2}} e^{i\pi/4} dt = 0.\end{aligned}$$

Since $e^{i\pi/2} = i$ and $e^{i\pi/4} = 1/\sqrt{2} + i/\sqrt{2}$ Cauchy's Theorem tells us that for all $R > 0$

$$\int_0^R e^{it^2} dt + \int_0^{\pi/4} e^{iR^2 e^{2it}} Rie^{it} dt = (\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}) \int_0^R e^{-t^2} dt. \quad (5.18)$$

Now we examine the middle integral.

$$\begin{aligned}\left| \int_0^{\pi/4} e^{iR^2 e^{2it}} Rie^{it} dt \right| &\le \int_0^{\pi/4} \left| e^{iR^2 e^{2it}} Rie^{it} \right| dt \\ &= R \int_0^{\pi/4} \left| e^{iR^2 e^{2it}} \right| dt \\ &= R \int_0^{\pi/4} e^{-R^2 \sin 2t} dt \le \frac{\pi}{4R}(1 - e^{-R^2}). \quad (5.19)\end{aligned}$$

Where we have used the well known estimate that $\sin t \ge 2t/\pi$ for $0 < t < \pi/2$. The evaluation of the last integral in (5.19) is straightforward. Hence

$$0 \le \left| \lim_{R\to\infty} \int_0^{\pi/4} e^{iR^2 e^{2it}} Rie^{it} dt \right| \le \lim_{R\to\infty} \frac{\pi}{4R}(1 - e^{-R^2}) = 0.$$

So $\lim_{R\to\infty} \int_0^{\pi/4} e^{iR^2e^{2it}} Rie^{it}dt = 0$.

So we take the limit as $R \to \infty$ in (5.18) to obtain

$$\lim_{R\to\infty} \left(\int_0^R e^{it^2} dt + \int_0^{\pi/4} e^{-R^2 e^{2it}} Rie^{it} dt \right) = \lim_{R\to\infty} (\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}) \int_0^R e^{-t^2} dt. \tag{5.20}$$

Which gives

$$\int_0^\infty e^{it^2} dt = (\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}) \int_0^\infty e^{-t^2} dt. \tag{5.21}$$

Since $\int_0^\infty e^{-t^2} dt = \frac{1}{2}\sqrt{\pi}$ this implies

$$\int_0^\infty \left(\cos(t^2) + i\sin(t^2)\right) dt = (\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}})\frac{1}{2}\sqrt{\pi}.$$

Equating the real and imaginary parts gives

$$\int_0^\infty \cos(t^2) dt = \frac{\sqrt{\pi}}{2\sqrt{2}}$$
$$\int_0^\infty \sin(t^2) dt = \frac{\sqrt{\pi}}{2\sqrt{2}}.$$

Observe that since $\cos(t^2)$ is even, then $\int_{-\infty}^\infty \cos(t^2)dt = 2\int_0^\infty \cos(t^2)dt$. The same is true for $\sin(t^2)$. With a change of variables we therefore obtain the more general result that for $a > 0$

$$\int_{-\infty}^\infty \cos(at^2)dt = \int_{-\infty}^\infty \sin(at^2)dt = \sqrt{\frac{\pi}{2a}}. \tag{5.22}$$

These integrals play a role in a number of areas, especially optics.

Example 5.5. Use Cauchy's Theorem to show that

$$\int_{-\infty}^\infty \frac{1-\cos t}{t^2} dt = \pi.$$

Solution. For this integral we consider the function $f(z) = (1 - e^{iz})/z^2$. The motivation for this is that for real z the real part of f is the integrand we are interested in. This is actually an interesting example to consider, because of what happens with the imaginary part of the integral. We proceed as follows.

Here the contour we choose is in four parts. The problem is that f has an obvious singularity at $z = 0$. However the singularity is isolated and so we choose a contour which does not include $z = 0$. Our contour is defined as follows. $\gamma = \gamma_1 - \gamma_\epsilon + \gamma_2 + \gamma_R$ where

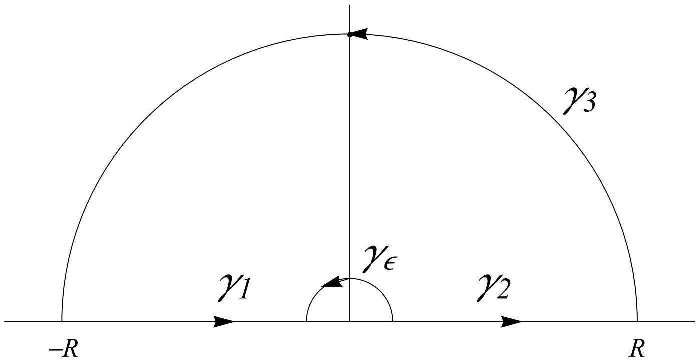

FIGURE 2. An indented Semicircular Contour.

(1) $\gamma_1(t) = t \quad -R \leq t < \epsilon$

(2) $\gamma_\epsilon(t) = \epsilon e^{it} \quad 0 \leq t \leq \pi$

(3) $\gamma_2(t) = t \quad \epsilon \leq t \leq R$

(4) $\gamma_R(t) = Re^{it} \quad 0 \leq t \leq \pi$.

This contour starts at $z = -R$ moves along the real axis to $z = -\epsilon$, then moves in a circular arc to $z = \epsilon$ (thus missing the singularity at $z = 0$, then moves along the real axis to $z = R$ then moves in a circular arc from $z = R$ back to $z = R$. So it consists of a line segment, a semi-circle, another line segment, then another semi-circle. It is closed and f is analytic inside and on the contour, since the singularity at $z = 0$ is excluded. See Figure 2

Cauchy's Theorem gives

$$\int_{-R}^{-\epsilon} \frac{1-e^{it}}{t^2} dt - \int_{\gamma_\epsilon} \frac{1-e^{iz}}{z^2} dz + \int_{\epsilon}^{R} \frac{1-e^{it}}{t^2} dt + \int_{\gamma_R} \frac{1-e^{iz}}{z^2} dz = 0.$$

First we consider the integral over the contour γ_R. We easily obtain the estimate

$$\left| \frac{1-e^{iz}}{z^2} \right| \leq \frac{1+|e^{iz}|}{|z|^2} = \frac{2}{|z|^2},$$

by the triangle inequality. On the semicircle of radius R $|z| = R$, so that

$$\left| \frac{1-e^{iz}}{z^2} \right| \leq \frac{2}{R^2}.$$

Now since the length of the semi circle of radius R is πR, the ML inequality gives

$$\left| \int_{\gamma_R} \frac{1-e^{iz}}{z^2} dz \right| \leq \frac{2\pi}{R}. \tag{5.23}$$

So $\displaystyle\int_{\gamma_R} \frac{1-e^{iz}}{z^2}dz \to 0$ as $R \to \infty$.

From this we can conclude that

$$\int_{-\infty}^{-\epsilon} \frac{1-e^{it}}{t^2}dt + \int_{\epsilon}^{\infty} \frac{1-e^{it}}{t^2}dt = -\int_{\gamma_\epsilon} \frac{1-e^{iz}}{z^2}dz.$$

By Taylor's Theorem, $e^{iz} = 1 + iz - \frac{z^2}{2!} - i\frac{z^3}{3!} + e^{\xi_z}\frac{z^4}{4!}$ for some ξ_z. As $z \to 0$ it is clear that $\xi_z \to 0$, since $e^0 = 1$. Therefore the function

$$f(z) = -\frac{i}{z} + \frac{1}{2} + i\frac{z}{3!} - e^{\xi_z}\frac{z^2}{4!} = -\frac{i}{z} + A(z)$$

where $A(z) = \dfrac{1}{2} + i\dfrac{z}{3!} - e^{\xi_z}\dfrac{z^2}{4!}$ is an analytic function. So

$$\begin{aligned}\int_{\gamma_\epsilon} \frac{1-e^{iz}}{z^2}dz &= \int_{\gamma_\epsilon} \left(-\frac{i}{z} + A(z)\right) dz \\ &= -i\int_0^\pi \frac{i\epsilon e^{it}}{\epsilon e^{it}}dt + \int_0^\pi A(\epsilon e^{it})i\epsilon e^{it}dt \\ &= \pi - \int_0^\pi A(\epsilon e^{it})i\epsilon e^{it}dt. \end{aligned} \tag{5.24}$$

Now

$$\int_0^\pi A(\epsilon e^{it})i\epsilon e^{it}dt = \epsilon\int_0^\pi \left(\frac{1}{2} + i\frac{\epsilon e^{it}}{3!} - e^{\xi_{\epsilon e^{it}}}\frac{\epsilon^2 e^{2it}}{4!}\right) ie^{it}dt.$$

So

$$\begin{aligned}\lim_{\epsilon\to 0}\int_0^\pi A(\epsilon e^{it})i\epsilon e^{it}dt &= \lim_{\epsilon\to 0}\epsilon\int_0^\pi \left(\frac{1}{2} + i\frac{\epsilon e^{it}}{3!} - e^{\xi_{\epsilon e^{it}}}\frac{\epsilon^2 e^{2it}}{4!}\right) ie^{it}dt \\ &= \lim_{\epsilon\to 0}\epsilon\int_0^\pi \frac{1}{2}dt + \lim_{\epsilon\to 0}\epsilon^2\int_0^\pi \frac{ie^{it}}{3!}dt \\ &\quad - \lim_{\epsilon\to 0}\epsilon^3\int_0^\pi \frac{e^{\xi_{\epsilon e^{it}}}e^{2it}}{4!}dt = 0, \end{aligned} \tag{5.25}$$

since all three integrals are finite and bounded. The first two integrals can be computed directly. For the final, observe that ξ_z is the error term in the approximation of e^{iz} on the interval $[0, \pi]$ by a Taylor polynomial. So by Taylor's Theorem $|e^{\xi_z}| \le e^0 = 1$. This gives

$$\begin{aligned}\left|\int_0^\pi \frac{e^{\xi_{\epsilon e^{it}}}e^{2it}}{4!}dt\right| &\le \int_0^\pi \left|\frac{e^{\xi_{\epsilon e^{it}}}e^{2it}}{4!}\right| dt \\ &\le \int_0^\pi \frac{dt}{4!} = \frac{1}{24}.\end{aligned}$$

This is all we need to do the integral because

$$\lim_{\epsilon\to 0}\left(\int_{-\infty}^{-\epsilon}\frac{1-e^{it}}{t^2}dt+\int_{\epsilon}^{\infty}\frac{1-e^{it}}{t^2}dt\right)=\lim_{\epsilon\to 0}\int_{\gamma_\epsilon}\frac{1-e^{iz}}{z^2}dz.$$

So

$$\int_{-\infty}^{\infty}\frac{1-e^{it}}{t^2}dt=\int_{-\infty}^{\infty}\frac{1-\cos t-i\sin t}{t^2}dt=\pi. \tag{5.26}$$

Equating the real parts gives the result. What about the imaginary parts? Can we conclude that

$$\int_{-\infty}^{\infty}\frac{\sin t}{t^2}dt=0? \tag{5.27}$$

Actually no, because the integral does not exist. The singularity at zero is not integrable. In fact what we have computed is the *Cauchy Principle Value* of the integral. In fact we have also computed the Cauchy Principle Value of the integral we set out to determine as well. It just so happens that this integral actually exists, whereas the second does not. We will discuss this in more detail below.

The reader may observe that even though we are able to evaluate the integrals in question, there is quite a lot of work involved in each one. It would be nice if there was an easier way to do integrals. In fact there is. For our next application of Cauchy's Theorem we introduce a method of evaluating an enormous range of integrals. This is the famous Cauchy integral formula.

5.3. **The Cauchy Integral Formula.**

Theorem 5.9. *Let f be differentiable in an open set Ω which contains a closed disc $D_R=\{z\in\mathbb{C}:|z-z_0|\le R\}$. If C_R is the boundary of D where the path is assumed to be traversed counterclockwise then for $|z-z_0|<R$*

$$f(z)=\frac{1}{2\pi i}\int_{C_R}\frac{f(\xi)}{\xi-z}d\xi. \tag{5.28}$$

Proof. We define the function

$$F(\xi)=\frac{f(\xi)-f(z)}{\xi-z}$$

and observe that it is differentiable in the domain $D_R-\{z\}$. We consider a circle S_ϵ of radius ϵ about the point z, traversed counterclockwise. The circle S_ϵ is contained inside C_R. The key to the proof lies in the fact that

$$\int_{C_R}F(\xi)d\xi=\int_{S_\epsilon}F(\xi)d\xi. \tag{5.29}$$

This is a simple consequence of Cauchy's Theorem for a system of contours applied to the contours C_R and S_ϵ, since S_ϵ is wholly contained within C_R.

Since $\lim_{\xi \to z} F(\xi) = f'(z)$ it follows that F is bounded inside S_ϵ. That is, there is an $M > 0$ such that $|F(\xi)| \leq M$ for all ξ. We observe that the length of the circle S_ϵ is $2\pi\epsilon$. So the ML Inequality gives

$$\left| \int_{S_\epsilon} F(\xi) d\xi \right| \leq M(2\pi\epsilon). \tag{5.30}$$

Now because

$$\left| \int_{C_R} F(\xi) d\xi \right| = \left| \int_{S_\epsilon} F(\xi) d\xi \right| \leq 2M\pi\epsilon, \tag{5.31}$$

for all $\epsilon > 0$ and ϵ is arbitrary, it follows that

$$\int_{C_R} F(\xi) d\xi = 0. \tag{5.32}$$

Next we will show that

$$\int_{C_R} \frac{1}{\xi - z} d\xi = 2\pi i. \tag{5.33}$$

To this end let C_R be parameterised by $\gamma(t) = z + Re^{it},\ 0 \leq t \leq 2\pi$. Then

$$\begin{aligned} \int_{C_R} \frac{1}{\xi - z} d\xi &= \int_0^{2\pi} \frac{1}{z + Re^{it} - z} Rie^{it} dt \\ &= i \int_0^{2\pi} dt = 2\pi i. \end{aligned}$$

Next observe that

$$\int_{C_R} F(\xi) d\xi = \int_{C_R} \frac{f(\xi) - f(z)}{\xi - z} d\xi = 0.$$

Hence

$$\begin{aligned} \int_{C_R} \frac{f(\xi)}{\xi - z} d\xi = \int_{C_R} \frac{f(z)}{\xi - z} d\xi &= f(z) \int_{C_R} \frac{1}{\xi - z} d\xi \\ &= 2\pi i f(z). \end{aligned}$$

This of course says that

$$f(z) = \frac{1}{2\pi i} \int_{C_R} \frac{f(\xi)}{\xi - z} d\xi.$$

This establishs the result.

$\square$

The consequences of this result are very important. Properly understood, complex analysis is actually a good deal easier than real analysis. The reasons are not hard to understand. Recall our discussion of the term analytic: In real variable theory an analytic function is one which is equal to its Taylor series expansion. That is, a real function $f(x)$ which is analytic at x_0, must be infinitely differentiable at x_0 and must be equal to the power series $T_f(x) = \sum_{n=0}^{\infty} f^{(n)}(x-x_0)^n/n!$ in some interval of positive length containing $x = x_0$.

There are many examples of functions of a real variable which are once differentiable, but not twice. There are also functions which are infinitely differentiable, but not analytic. For example the function defined by $f(x) = \exp(-1/x^2)$ for $x \neq 0$ and $f(x) = 0$ for $x = 0$, is infinitely differentiable at $x = 0$. In fact it is not hard to show that $f^{(n)}(0) = 0$ for all n. Thus the Taylor series expansion of f about 0 is $T_f(x) = 0$ for all x. Clearly the Taylor series for f is equal to f only at $x = 0$ and not at any other point. So there is no interval containing $x = 0$ on which f equals its Taylor expansion. Therefore, although f is infinitely differentiable, it is not analytic.

So for real functions, being analytic is a very strong property. A real valued function can be once differentiable, but not twice differentiable. However for complex differentiable functions, this is not possible. If $f(z)$ is a function of a complex variable, and $f'(z)$ exists, then it must be infinitely differentiable and it must equal its Taylor series. To see why this is so, we need to study the Cauchy integral formula.

We know that if f is differentiable and C is a circle containing z then

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\xi)}{\xi - z}d\xi.$$

If we can differentiate the integral, then we get

$$f'(z) = \frac{1}{2\pi i}\int_C \frac{f(\xi)}{(\xi - z)^2}d\xi.$$

Differentiating n times would give

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(\xi)}{(\xi - z)^{n+1}}d\xi.$$

Thus if the integral can be differentiated, then we obtain an expression for the nth derivative of f for all n. This is exactly what happens.

Corollary 5.10. *If f is differentiable in an open set D, then f has infinitely many derivatives in D. Moreover if $C \subset D$ is a circle contained in D whose interior is also contained in D then*

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(\xi)}{(\xi - z)^{n+1}}d\xi,$$

for all z in the interior of C. We assume that C is traversed counterclockwise.

Proof. The proof is by induction. We assume that f has $n-1$ derivatives and that

$$f^{(n-1)}(z) = \frac{(n-1)!}{2\pi i}\int_C \frac{f(\xi)}{(\xi - z)^n}d\xi.$$

We then consider the derivative quotient. Namely

$$\frac{f^{(n-1)}(z+h) - f^{(n-1)}(z)}{h} = \\ = \frac{(n-1)!}{2\pi i}\int_C f(\xi)\frac{1}{h}\left[\frac{1}{(\xi - z - h)^n} - \frac{1}{(\xi - z)^n}\right]d\xi. \tag{5.34}$$

The key is to show that

$$\lim_{h\to 0}\frac{1}{h}\left[\frac{1}{(\xi - z - h)^n} - \frac{1}{(\xi - z)^n}\right] = \frac{n}{(\xi - z)^{n+1}} - (*)$$

as $h \to 0$. This limit is simply

$$\frac{d}{dz}\frac{1}{(\xi - z)^{n-1}} = \frac{n}{(\xi - z)^{n+1}}.$$

However let us calculate the limit directly. We recall the factorisation

$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1}).$$

We take $a = \dfrac{1}{(\xi - z - h)}$ and $b = \dfrac{1}{(\xi - z)}$. We see that

$$a - b = \frac{h}{(\xi - z - h)(\xi - z)}.$$

Which gives the expression

$$\frac{1}{h}\left[\frac{1}{(\xi - z - h)^n} - \frac{1}{(\xi - z)^n}\right] \\ = \frac{1}{h}\frac{h}{(\xi - z - h)(\xi - z)}(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1}) \\ = \frac{1}{(\xi - z - h)(\xi - z)}(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1}).$$

The limit as $h \to 0$ of (*) is now easy to calculate.

$$\lim_{h\to 0}\left(\frac{1}{(\xi - z - h)}\right)^k = \frac{1}{(\xi - z)^k},$$

for all $k = 0, 1, 2, ...$ and

$$\frac{1}{(\xi - z - h)(\xi - z)} \to \frac{1}{(\xi - z)^2}.$$

So that

$$\frac{1}{(\xi - z - h)(\xi - z)}(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1})$$
$$\to \frac{1}{(\xi - z)^{n+1}} + \cdots + \frac{1}{(\xi - z)^{n+1}}.$$

In other words

$$\frac{1}{h}\left[\frac{1}{(\xi - z - h)^n} - \frac{1}{(\xi - z)^n}\right] \to \frac{n}{(\xi - z)^{n+1}},$$

as claimed. Thus

$$\lim_{h\to 0}\frac{f^{(n-1)}(z+h) - f^{(n-1)}(z)}{h} = f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(\xi)}{(\xi - z)^{n+1}}d\xi.$$

Since the result holds in the $n = 1$ case, it holds for all n by induction. □

As our first application of this result we will prove that every differentiable complex function is equal to its Taylor series.

Theorem 5.11. *Let f be a differentiable function in an open set D. If S is a disc centered at z_0 with closure contained in D then f has a power series expansion in S centered at z_0. That is*

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n,$$

for all $z \in S$ with the coefficients of $(z - z_0)^n$ given by

$$a_n = \frac{1}{n!}f^{(n)}(z_0).$$

Proof. We begin with the Cauchy integral formula. Let C be the boundary of the disc S. Then

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\xi)}{\xi - z}d\xi.$$

Observe that

$$\frac{1}{\xi - z} = \frac{1}{\xi - z_0 - (z - z_0)} = \frac{1}{\xi - z_0}\frac{1}{1 - \left(\frac{z - z_0}{\xi - z_0}\right)}.$$

We now expand in a geometric series. Since z and z_0 are in the interior of the disc and ξ is on the boundary of the disc, is clear that $\left|\frac{z-z_0}{\xi - z_0}\right| < 1$. Therefore, since for $|r| < 1$ we have $1/(1-r) = 1 + r + r^2 + \cdots$ we can write

$$\frac{1}{1 - \left(\frac{z - z_0}{\xi - z_0}\right)} = 1 + \left(\frac{z - z_0}{\xi - z_0}\right) + \left(\frac{z - z_0}{\xi - z_0}\right)^2 + \cdots = \sum_{n=0}^{\infty}\left(\frac{z - z_0}{\xi - z_0}\right)^n.$$

This gives the expression for f

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_C \frac{f(\xi)}{\xi - z} d\xi \\ &= \frac{1}{2\pi i}\int_C \frac{f(\xi)}{\xi - z_0}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{\xi - z_0}\right)^n d\xi \\ &= \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_C \frac{f(\xi)}{(\xi - z_0)^{n+1}}\right)(z-z_0)^n d\xi \\ &= \sum_{n=0}^{\infty}\frac{1}{n!}f^{(n)}(z_0)(z-z_0)^n, \end{aligned}$$

by the Cauchy integral formula for $f^{(n)}(z_0)$. We can swap the integral and sum since the geometric series converges uniformly. □

Thus a complex function which is differentiable once is automatically differentiable infinitely often and is equal to its Taylor series expansion! This appears to be miraculous. It justifies using of the term 'analytic' to describe a function of a complex variable which has one derivative, because such a function *is* analytic in the same sense of the word that we gave for real valued functions. From this point on we will use the term 'analytic function' more freely, particularly in cases where we the existence of a Taylor series expansion is important.

To restate a point made previously, a complex valued function is differentiable if $\lim_{h\to 0}\frac{f(z+h)-f(z)}{h}$, exists. Now however, the origin can be approached from infinitely many directions. Thus it is *harder* for a function of a complex variable to be differentiable than for a function of a real variable. It turns out that the only functions of a complex variable which are differentiable even once, are precisely those given by a power series expansion. If a function of a complex variable is not given by a power series expansion, it cannot be differentiable. That is what we have shown. So our apparent miracle is not really all that surprising. Because differentiability is such a restrictive property for functions of a complex variable, those functions which are differentiable have much nicer properties than is the case for differentiable real valued functions.

Another useful corollary of the Cauchy formula is the following. We will use it in our proof of Liouville's Theorem. We remark that the closure of a region D is the union of D and its boundary. So if for example $D = \{z \in \mathbb{C} : |z - z_0| < R\}$, the closure of D is

$$\overline{D} = \{z \in \mathbb{C} : |z - z_0| \leq R\}.$$

Corollary 5.12 (The Cauchy inequalities)**.** *Suppose that f is differentiable in an open set that contains the closure of an open disc D of radius R centered at z_0. Denote the boundary of D by C. Then $f^{(n)}$*

satisfies the inequality

$$|f^{(n)}(z_0)| \leq \frac{n!||f||_C}{R^n}, \tag{5.35}$$

in which $||f||_C = \sup_{z\in C}|f(z)|$ *is the supremum of* f *on the circle* C.

Proof. We parameterise the circle C by $C(t) = z_0 + Re^{it}$ for $t \in [0, 2\pi]$. Then by the Cauchy integral formula for the nth derivative

$$\begin{aligned}
|f^{(n)}(z)| &= \left|\frac{n!}{2\pi i}\int_C \frac{f(\xi)}{(\xi - z)^{n+1}} d\xi\right| \\
&\leq \frac{n!}{2\pi}\int_C \left|\frac{f(\xi)}{(\xi - z)^{n+1}}\right| d\xi \\
&= \frac{n!}{2\pi}\int_0^{2\pi} \left|\frac{f(z_0 + Re^{it})}{R^{n+1}} Rie^{it}\right| dt \\
&= \frac{n!}{2\pi R^n}\int_0^{2\pi} \left|f(z_0 + Re^{it})\right| dt \\
&\leq \frac{n!}{2\pi R^n}\int_0^{2\pi} ||f||_C dt = \frac{n!||f||_C}{R^n}.
\end{aligned}$$

□

This is a technical result which we use to prove another very important property of differentiable functions. We recall that function complex function $f(z)$ which is differentiable for all $z \in \mathbb{C}$ is said to be *entire*.

One of the differences between infinitely differentiable functions on the real line and entire functions on $\mathbb{C}$ is given by Liouville's Theorem.

Theorem 5.13 (Liouville's Theorem). *If* f *is an entire bounded function then it is constant.*

Proof. The proof of this is a consequence of the Cauchy inequalities. We take $n = 1$. Suppose that f is bounded. Then there is a real $M > 0$ such that $|f(z)| \leq M$ for all $z \in \mathbb{C}$. Let z_0 be the centre of a circle of radius R. By the Cauchy inequality with $n = 1$ we have

$$|f'(z_0)| \leq \frac{M}{R}. \tag{5.36}$$

This holds for any R and any z_0. Letting $R \to \infty$ we get $f'(z_0) = 0$. Since this holds for all z_0 we conclude that f must be a constant, since the only functions whose derivative is everywhere zero are the constant functions. □

This has some interesting applications. One is to answer an important question about polynomials. This question goes back centuries. Consider a polynomial $p_n(x) = a_0 + a_1x + \cdots a_nx^n$ where $a_1, ..., a_n$ are possibly complex numbers and a_n is not zero. How many solutions does the equation $p_n(x) = 0$ have? The answer is n and this was first proved by Argand in 1806, though Gauss had given an incomplete proof in 1799. (The gap in Gauss' proof is extremely subtle and was not noticed at the time). Euler had failed to prove it and Leibniz and Nicholas Bernoulli thought it was false. Since Argand, many different proofs have been presented. The proof we give here is quite simple.

Theorem 5.14 (The fundamental theorem of algebra)**.** *Every polynomial of degree n has exactly n complex roots.*

Proof. We suppose that a polynomial $P_n(z)$ is a polynomial of degree n with no zeroes. Since $P_n(z)$ is entire, then $1/P_n(z)$ must be an entire function as well, since $P_n(z)$ is never zero. But $|1/P_n(z)| \to 0$ as $|z| \to \infty$. Thus $1/P_n(z)$ is a bounded entire function. But $1/P_n(z)$ is not a constant, so this contradicts Liouville's Theorem. Thus $P_n(z)$ must have at least one zero, z_1. Now define $P_{n-1}(z) = P_n(z)/(z - z_1)$. This is a polynomial of degree $n-1$. Now repeat the previous argument with $P_n(z)$ replaced by $P_{n-1}(z)$. We see that $P_{n-1}(z)$ must have at least one zero z_2. Repeating the argument n times proves the result. □

It is worth noting that the roots of a polynomial with real coefficients have the following property. Suppose $p(z) = a_n z^n + \cdots + a_0$ and $a_0, a_1, ...a_n$ are real numbers. If z_1 is a root of the polynomial, then so is $\bar{z_1}$, the complex conjugate of z_1. This is because of the easily verifiable fact that for every complex number z, $(\bar{z})^n = \bar{z^n}$. So that

$$p(\bar{z_1}) = \overline{p(z_1)} = 0.$$

Thus the complex roots come in pairs. This means that a real polynomial of odd degree must have at least one real root, a fact which is obvious anyway.

Liouville's Theorem means that certain familiar functions have quite different properties when we allow complex arguments. For example, $|\cos x| \leq 1$ for real x, but this is no longer true if we let x be complex number. In fact $\cos z = w$ always has a solution for all $w \in \mathbb{C}$. The same is true for $\sin z$. Yet it is still true that $\cos^2 z + \sin^2 z = 1$.

5.4. **Analytic Continuation.** Complex differentiable functions have more remarkable properties. In fact so strong is the notion of differentiability that it gives rise to the concept of *analytic continuation.* Specifically if two functions f and g agree on a small set, where they are differentiable, then they must be the same function. This is the content of the next two results. Since this section relies on the existence of a Taylor expansion, we will use the term analytic here.

Theorem 5.15. *Suppose that f is analytic in a region Ω and that f vanishes on a sequence of distinct points with a limit point contained in Ω. Then f is identically 0.*

Proof. The key is to use the fact that f has a Taylor series expansion. We let the sequence on which f vanishes be $\{\xi_k\}_{k=1}^{\infty}$ and suppose that the limit point of the sequence is z_0. Next pick a disc D centered at $z = z_0$ and let

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

Since f is non zero, there is a smallest value of m for which $a_m \neq 0$. We thus write

$$\begin{aligned} f(z) &= \sum_{n=m}^{\infty} a_n (z - z_0)^n \\ &= a_m (z - z_0)^m \sum_{n=0}^{\infty} \frac{a_{n+m}}{a_m} (z - z_0)^n \\ &= a_m (z - z_0)^m \left(1 + g(z - z_0)\right). \end{aligned}$$

The function $g(z - z_0) \to 0$ as $z \to z_0$.

Now we have assumed that $f(\xi_k) = 0$ for all $k = 1, 2, 3, ...$ However $a_m(z - z_0)^m$ has exactly one root of order m. Namely $z = z_0$. So if $a_m(\xi_k - z_0)^m \neq 0$ for all k then $a_m = 0$. Further $1 + g(\xi - z_0)^n \neq 0$ for all k. But this means that $f = 0$. The remainder of the proof is to extend this to the whole of Ω. We let U be the set of points in Ω for which $f(z) = 0$. Then define V to be the complement of U. Clearly U is nonempty since it contains z_0. U is also closed. Since $\Omega = U \cup V$ and Ω is connected, it follows that either U is empty or V is empty. But U is not empty. So V must be. Thus $\Omega = U$. In other words, f is zero on the whole of Ω. □

The corollary of this result which we are interested in is the following.

Corollary 5.16. *Suppose that f and g are analytic in a region Ω and $f(z) = g(z)$ for all z in some non-empty subset of Ω (or for a distinct sequence of points with limit in Ω). Then $f(z) = g(z)$ throughout Ω.*

Proof. We apply the previous theorem to the function $h = f - g$. □

Analytic continuation is very useful and it has surprising consequences. The idea is to take an analytic function defined on some domain and extend it in a natural way to a larger domain. The point of this theorem is that the analytic continuation is unique under certain reasonable conditions. The analytic continuation of a function can reveal information that the original version of the function does not. This is best illustrated by the Riemann zeta function, which we will discuss shortly.

Consider as a simple example the two functions defined by

$$f(z) = 1 - z^2 + z^4 - z^6 + \cdots, g(z) = \frac{1}{1 + z^2}.$$

It is clear that for $|z| < 1$

$$f(z) = \frac{1}{1+z^2}.$$

Hence for all z in the interior of the unit disc, f and g are identical. However g is defined for all $z \in \mathbb{C}$ except $z = \pm i$. We may thus view g as an analytic continuation of f to the whole of the complex plane, except $z = \pm i$.

There exist functions which cannot be analytically continued. A classic example of this is the function defined by the power series

$$f(z) = \sum_{n=0}^{\infty} z^{n!}.$$

This function is defined on the unit disc where the series converges absolutely. However it cannot be extended to any analytic function defined outside the disc. The reason for this is that the boundary of the disc is a natural barrier to the continuation of the function. If $n \geq q$ then for any $z_{pq} = e^{i\pi p/q}$, which is a point on the boundary of the disc, we have $z_{pq}^{n!} = 1$. From this it is not hard to show that for such a z_{pq}, $f(z) \to \infty$ as $z \to z_{pq}$.

What is important here is that the points of the form $e^{i\pi p/q}$ are dense on the boundary of the disc. What this means is that any point on the boundary can be approximated arbitrarily closely by a point of the form z_{pq}. We assume that there is an analytic continuation of f which we denote by F. The function F is to be defined on some region Ω which overlaps in some part with the unit disc on which f is defined. Since the points of the form z_{pq} are dense on the boundary of the disc, no matter how Ω is chosen it must contain at least one point of the form z_{pq}. Call this point z_0. F is assumed to be analytic in Ω and hence continuous. $F(z_0)$ must therefore be finite. Now there must be some set of points around z_0 on which F and f agree. So we can find an r with $0 < r < 1$ for which $F(rz_0) = f(rz_0)$. By continuity $F(rz_0) \to F(z_0)$ as $r \to 1$. But $F(rz_0) = f(rz_0)$ for all r so $F(rz_0) \to \infty$ as $r \to 1$. Hence $F(z_0) = \infty$. Which is a contradiction. So no such function F can exist and f does not have an analytic continuation.

Exercise. Show that if we define $f(z) = \sum_{n=0}^{\infty} z^{2^n}$ for $|z| < 1$ then f has no analytic continuation past the unit disc.

We will conclude this section with two theorems on extending functions. We will use the following in both theorems.

Definition 5.17. Let Ω^+ be a region in the complex plane with the property that for all $z \in \Omega^+$, $\Re(z) > 0$. That is, every element in Ω^+ sits above the real axis or on the real axis. Assume further that there is an interval I on the real axis which forms a lower boundary for Ω^+. Then Ω^- is the reflection of Ω^+ in the real axis. That is $\Omega^- = \{z \in \mathbb{C} \mid \bar{z} \in \Omega^+\}$. We set $\Omega = \Omega^+ \cup I \cup \Omega^-$.

The proof of the next theorem may be found in Stein and Shakarchi [3]. Our formulation is taken from there.

Theorem 5.18 (Symmetry principle)**.** *If f^+ and f^- are analytic functions on Ω^+ and Ω^- respectively, that extend continuously to I and $f^+(x) = f^-(x)$ for all $x \in I$, then the function defined on Ω by*

$$f(z) = \begin{cases} f^+(z) & if\ z \in \Omega^+ \\ f^\pm(z) & if\ z \in I \\ f^-(z) & if\ z \in \Omega^-, \end{cases}$$

is analytic in Ω.

The main application of this result is the famous Schwarz reflection principle. Again the formulation is taken from Stein and Shakarchi.

Theorem 5.19 (Schwarz reflection principle)**.** *Suppose that f is an analytic function in Ω^+ that extends continuously to I and such that f is real valued on I. Then there exists a function F analytic in all of Ω such that $F = f$ on Ω^+.*

Proof. The proof is really an application of the symmetry principle, which depends on actually constructing F by an obvious method. We wish to define a function which is extends f to values of z below the imaginary axis. So we define it to be the complex conjugate of f. That is, for $z \in \Omega^+$ we set

$$F(z) = \overline{f(\bar{z})}.$$

To understand this construction, recall that elements of Ω^- are the complex conjugate of elements of Ω^+. So that if $z \in \Omega^-$ then $\bar{z} \in \Omega^+$. Thus for $z \in \Omega^-$, it is clear that $f(\bar{z})$ exists because $\bar{z}$ is above the axis. We then take the complex conjugate of this so that we don't obtain the same values for F twice. ($F(z) = f(z)$ for z above the real axis. So if we just set $F(z) = f(\bar{z})$ for z below the axis, we would get the same values twice. Taking the conjugate reflects the values about the real axis).

We now have to show that F is analytic. Take two points $z, z_0 \in \Omega^-$. Then $\bar{z}, \bar{z_0} \in \Omega^+$. We know that f is analytic so it has a power series expansion which we write

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

Thus

$$f(\bar{z}) = \sum_{n=0}^{\infty} a_n (\bar{z} - \bar{z_0})^n.$$

Which means that

$$F(z) = \overline{f(\bar{z})} = \overline{\sum_{n=0}^{\infty} a_n(\bar{z} - \bar{z}_0)^n} \tag{5.37}$$

$$= \sum_{n=0}^{\infty} \overline{a_n}(z - z_0)^n. \tag{5.38}$$

Since F has a power series expansion in Ω^- it is analytic in Ω^-. Finally we note that since f is real valued on I, then F extends continuously onto I. Application of the symmetry principle tells us that F is analytic on the whole of Ω and this completes the proof. □

5.4.1. *The Riemann Zeta Function.* One of the most important examples of analytic continuation arises from the Riemann Zeta function, which is usually written as

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}.$$

This series is easily seen to converge for all $z > 1$ when z is real.

Euler studied this function and was able to relate it to the prime numbers in the Euler product formula, which is one of the most famous formulas in mathematics. Euler was a master at the manipulation of infinite series and many of his proofs rely upon such manipulations. This result uses one of the best examples of this style of proof.

Theorem 5.20 (Euler)**.** *Let $z > 1$. Then*

$$\begin{aligned}\zeta(z) &= \prod_{p \text{ prime}} \left(1 - \frac{1}{p^z}\right)^{-1} \\ &= \frac{1}{\left(1 - \frac{1}{2^z}\right)} \frac{1}{\left(1 - \frac{1}{3^z}\right)} \frac{1}{\left(1 - \frac{1}{5^z}\right)} \frac{1}{\left(1 - \frac{1}{7^z}\right)} \frac{1}{\left(1 - \frac{1}{11^z}\right)} \cdots .\end{aligned}$$

Proof. This uses the Fundamental Theorem of Arithmetic: Every integer is either prime or can be written as a product of primes which is unique up to order. We write each term $\frac{1}{\left(1 - \frac{1}{p^z}\right)}$ as the sum of a geometric series and multiply out the terms. By the Fundamental Theorem of Arithmetic, for every n

$$\frac{1}{n^z} = \frac{1}{p_1^z \cdots p_k^z},$$

for some set of primes $p_1, ..., p_k$. Since the product is over all primes we can write

$$\prod_{p \text{ prime}} \left(1 - \frac{1}{p^z}\right)^{-1} = \left(1 + \frac{1}{2^z} + \frac{1}{2^{2z}} + \cdots\right)\left(1 + \frac{1}{3^z} + \frac{1}{3^{2z}} + \cdots\right)$$
$$\times \left(1 + \frac{1}{5^z} + \frac{1}{5^{2z}} + \cdots\right)\left(1 + \frac{1}{7^z} + \frac{1}{7^{2z}} + \cdots\right)\cdots$$
$$= 1 + \frac{1}{2^z} + \frac{1}{3^z} + \frac{1}{2^{2z}} + \frac{1}{5^z} + \frac{1}{2^z 3^z} + \frac{1}{7^z} + \cdots$$
$$= 1 + \frac{1}{2^z} + \frac{1}{3^z} + \frac{1}{4^z} + \frac{1}{5^z} + \frac{1}{6^z} + \frac{1}{7^z} + \cdots$$
$$= \zeta(z).$$

□

Euler's product formula came to provide a link between analysis and the theory of prime numbers. The distribution of the primes seems almost random. Although Euclid's Elements contain a proof that there are infinitely many of them, little could actually be said about the primes for the next two thousand years. Gauss at the age of 14 made a conjecture about the actual number of primes less than a given number. Legendre made the same conjecture a little later and actually published it in 1792. If $\pi(x)$ is the number of primes less than x, the conjecture was that

$$\lim_{x\to\infty} \frac{\pi(x)\ln x}{x} = 1. \tag{5.39}$$

In other words, the number of primes less than x is roughly $x/\ln x$. Gauss later improved on this by arguing that $\pi(x)$ is actually better estimated by $\int_2^x \frac{dt}{\ln t}$. That is Gauss claimed

$$\lim_{x\to\infty} \frac{\pi(x)}{\int_2^x \frac{dt}{\ln t}} = 1. \tag{5.40}$$

The Russian mathematician Chebyshev made an enormous contribution to the problem, using only algebraic methods. He showed that the result is at least "nearly" true. Meaning that the limit in (5.39) is at least close to 1. The proof that equation (5.39) is true took over a hundred years from Legendre's initial paper. It is now called the Prime Number Theorem and was proved independently in 1898 by Jacques Hadamard and Charles de la Vallée Poussin. It turns out that both (5.39) and (5.40) are true, but Gauss was right and the second gives a much better estimate than the first. The first arises from the second using integration by parts:

$$\int_2^x \frac{dt}{\ln t} = \frac{x}{\ln x} - \frac{2}{\ln 2} + \int_2^x \frac{1}{(\ln t)^2}dt.$$

There are 168 primes less than 1000 and $\int_2^{1000} \frac{dt}{\ln t} = 176.56$. This is not too bad. There are 9592 primes less than 100000 and $\int_2^{100000} \frac{dt}{\ln t} = 9628.76$. Now the absolute error of this estimate is larger than for $x = 100$, but the relative error is smaller: $|168 - 176.56| = 8.56$ and $|9592 - 9628.76| = 36.76$. However $169/176.56 = .952$ and $9592/9628.76 = .996$.

As x grows, the relative error gets smaller, but the absolute error increases. That is $\left|\pi(x) - \int_2^x \frac{dt}{\ln t}\right|$ increases, but $\pi(x)/\int_2^x \frac{dt}{\ln t}$ gets closer and closer to 1. The error in the approximation is believed to grow as $\sqrt{x}\ln x$ but this is unproven. In fact this can be shown to be equivalent to the *Riemann Hypothesis*.

Riemann founded the branch of mathematics which lead to the proof of the Prime Number Theorem in an 8 page paper published in 1859. He used complex analysis to study the Zeta function. His first step was to establish an analytic continuation of the function. As Euler had already done, he turned the sum into an integral.

Define the Gamma function

$$\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}dt, \tag{5.41}$$

which was first introduced by Euler. This is itself an analytic continuation of the factorial, since $\Gamma(n+1) = n!$. Now let $t = nx$ to get

$$\Gamma(z) = \int_0^\infty n^z x^{z-1}e^{-nx}dx. \tag{5.42}$$

This can be rearranged to give

$$\frac{1}{n^z} = \frac{1}{\Gamma(z)}\int_0^\infty x^{z-1}e^{-nx}dx, \tag{5.43}$$

from which we have

$$\begin{aligned}
\zeta(z) &= \sum_{n=1}^\infty \frac{1}{\Gamma(z)}\int_0^\infty x^{z-1}e^{-nx}dx \\
&= \frac{1}{\Gamma(z)}\int_0^\infty x^{z-1}\sum_{n=1}^\infty (e^{-x})^n dx \\
&= \frac{1}{\Gamma(z)}\int_0^\infty x^{z-1}\frac{e^{-x}}{1-e{-x}}dx \\
&= \frac{1}{\Gamma(z)}\int_0^\infty \frac{x^{z-1}}{e^x - 1}dx.
\end{aligned}$$

Riemann then went much further than Euler had and turned this integral for $\zeta(z)$ into a contour integral. His contour is rather odd. It starts at $+\infty$, moves along at a height ϵ above the positive real axis, circles zero and moves back to $+\infty$ at a distance ϵ beneath the real axis. This contour we call C_ϵ. It can also be defined as coming from $-\infty$, circling the origin and moving back to $-\infty$ again. Riemann proved that

$$2\sin(\pi z)\Gamma(z)\zeta(z) = i\lim_{\epsilon\to 0}\oint_{C_\epsilon}\frac{(-x)^{z-1}}{e^x-1}dx.$$

Starting with this, he was able to establish that ζ is an analytic function with one singularity at the point $z=1$ (in fact the singularity is a pole or order one. Poles will be discussed later), and that it satisfies a remarkable formula. He proved that

$$\zeta(z) = 2^z\pi^{z-1}\Gamma(1-z)\sin\left(\frac{\pi z}{2}\right)\zeta(1-z). \tag{5.44}$$

This is called the functional equation for the Riemann zeta function. Euler had conjectured that something like this was true. It is clear from the functional equation that $\zeta(-2n)=0$ for all $n=0,1,2,...$, because the term $\sin\left(\frac{\pi z}{2}\right)$ is zero at these points and $\Gamma(1+2n)=(2n)!$. These are called the trivial zeroes. At the positive even integers this argument fails because of the $\Gamma(1-2n)$ term. Specifically

$$\lim_{z\to 2n}\sin\left(\frac{\pi z}{2}\right)\Gamma(1-z)\neq 0.$$

To see this we use a remarkable property of the Gamma function, known as the reflection formula which is due to Euler.

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}.$$

So

$$\sin\left(\frac{\pi z}{2}\right)\Gamma(1-z) = \frac{\pi\sin\left(\frac{\pi z}{2}\right)}{\Gamma(z)\sin(\pi z)}.$$

Since this limit is of the form $0/0$ we use L'Hôpital's rule to see that

$$\begin{aligned}\lim_{z\to 2n}\sin\left(\frac{\pi z}{2}\right)\Gamma(1-z) &= \lim_{z\to 2n}\frac{\pi\sin\left(\frac{\pi z}{2}\right)}{\Gamma(z)\sin(\pi z)}\\ &= \lim_{z\to 2n}\frac{\pi^2\cos\left(\frac{\pi z}{2}\right)}{2\pi\Gamma(z)\cos(\pi z)+\Gamma'(z)\sin(\pi z)}\\ &= \frac{(-1)^n\pi}{2\Gamma(2n)}.\end{aligned}$$

So $\zeta(2n)$ is non-zero. Riemann conjectured that there should be other zeroes in the complex plane and found the first examples. The first non trivial zeroes are approximately,

$$\frac{1}{2}\pm 14.1347i, \frac{1}{2}\pm 21.022i, \frac{1}{2}\pm 25.011i, ...$$

Notice that the real part of each of these is $1/2$. Riemann believed that every one of the so called non-trivial zero had real part equal to $\frac{1}{2}$. This became known as the Riemann Hypothesis and it remains unproven. It is considered to be one of the most important unsolved problems in mathematics. If it is true, it has profound consequences, but that is well beyond the scope of our discussion.

Using Riemann's ideas, Hadamard and de la Vallée Poussin were able to prove the Prime Number Theorem. It turns out to be enough to show that $\zeta(z)$ has no zeroes with real part equal to 1, but that requires a considerable amount of work to prove. The point we want to emphasise is that complex variable theory now plays a fundamental role in the study of prime numbers.

5.5. **The Converse of Cauchy's Theorem.** The converse of Cauchy's Theorem is true. This is known as Morera's theorem.

Theorem 5.21 (Morera's Theorem)**.** *Suppose that f is a complex integrable function such that $\int_\gamma f = 0$ for all simple, closed contours γ contained in a simply connected domain D. Then f is analytic in D.*

Proof. This is a consequence of Theorem 4.6. This result tells us that if $\int_\gamma f = 0$ for all contours γ then f has an antiderivative F. That is $F' = f$. However we now know that if F is differentiable, then it is infinitely differentiable. This means that $F'' = \frac{d}{dz}F' = \frac{d}{dz}f = f'$ exists. So f is analytic. $\square$

This result contains another quite remarkable fact about complex differentiable functions. If a function f is not differentiable, it cannot have an antiderivative! This is not true in the real case. In the real case, every continuous function g on a closed interval $[a, b]$ has an antiderivative defined by

$$G(x) = \int_a^x g(t)dt.$$

This is the content of the fundamental theorem of calculus.

Notice that if we pick g so that it is everywhere continuous but nowhere differentiable, then G' exists and in fact $G' = g$, but G'' does not exist.

In the world of differentiable functions of a complex variable, this behaviour is impossible. If F' exists, then $F^{(n)}$ must exist for all n. So if F is an antiderivative for f, then $F' = f$ exists, which implies that $F'' = f'$ exists. So if f has an antiderivative, then f must itself be differentiable.

5.5.1. *Sequences of Analytic Functions.* Another way in which real valued functions differ from complex valued functions is in the behaviour of limits of sequence. In the theory of real variables, the behaviour of sequences of functions is of great importance. There is considerable subtlety involved in the subject, because limits of sequences of functions don't necessarily have the properties that we would wish. For example, suppose that $\{f_n\}$ is a sequence of real, differentiable functions that converge uniformly to the function f. It does not follow that f is differentiable. The limit must be continuous, uniform convergence guarantees that, but there is no reason why it has to be differentiable. For example, if $f_n(x) = \sqrt{x^2 + 1/n^2}$, then $f_n(x) \to |x|$ uniformly, but the limit function is not differentiable.

We do however have the following result which also holds for complex valued functions. We will use it in the next theorem.

Theorem 5.22. *Let* $\{f_n\}_{n=1}^{\infty}$ *be a sequence of functions which converges uniformly to a function* f *on some domain* D*. Then*

$$\int_D f_n \to \int_D f.$$

The proof of this may be found in any standard real analysis text such as Stein and Shakarchi's book on real analysis [4].

With analytic functions of a complex variable, this problem does not exist. If a sequence of analytic functions $\{f_n\}_{n=1}^{\infty}$ converges uniformly then the limit function has to be analytic. We have the following result.

Theorem 5.23. *Suppose that* $\{f_n\}_{n=1}^{\infty}$ *is a sequence of analytic functions that converges uniformly on every compact subset of* Ω *to a function* f*. Then* f *is analytic in* Ω*.*

Proof. This is an application of Morera's Theorem. Suppose that D is a disc contained in the closure of Ω and that T is any contour contained in D. Observe that D is compact and that we can cover Ω with an arbitrary collection of discs. By Cauchy's Theorem $\int_T f_n = 0$ for all n. Since $f_n \to f$ uniformly in D, then f is continuous and by Theorem 5.22

$$\int_T f_n \to \int_T f.$$

Since $\int_T f_n = 0$ for all n, it follows that $\int_T f_n \to 0$ and so $\int_T f = 0$. Since this is true for any T in D, it follows from Morera's theorem that f is analytic. Since we can perform this for any disc contained in Ω, it follows that f is analytic in Ω. ☐

More than this is true. We can also say the following.

Theorem 5.24. *Under the hypothesis of Theorem 5.23, the sequence of derivatives* $\{f_n'\}_{n=1}^{\infty}$ *converges uniformly to* f' *on every compact subset of* Ω*.*

Proof. See Stein and Shakarchi, [3]. ☐

We conclude this section with a discussion of the behaviour of integrals. This next result is extremely useful. The proof is not difficult and may also be found in [3].

Theorem 5.25. *Suppose that* $\Omega \subseteq \mathbb{C}$ *is an open set and* $F(z,s)$ *is a function defined on the set* $\Omega \times [0,1]$*. Suppose that* $F(z,s)$ *is analytic in* z *for each fixed value of* s *and that* F *is continuous on* $\Omega \times [0,1]$*. Then the function defined by*

$$f(z) = \int_0^1 F(z,s)ds$$

is analytic on Ω*.*

There is nothing particularly special about the interval $[0, 1]$. We can use any compact interval $[a, b]$, by a simple change of variables. As an example, consider the function $F(z) = g(s)e^{-2i\pi zs}$, where g is a real valued, continuous function. Then F satisfies the conditions of the theorem. So

$$f(z) = \int_0^1 g(s)e^{-2i\pi zs}ds$$

is an analytic function of z. But this is simply the Fourier coefficient of g. Thus the Fourier coefficients of a continuous function g are analytic.

5.6. **The Cauchy Integral Formula and the Poisson Kernel.** We show in 35231 that a solution of the Laplace equation on the disc D_R centered at 0 of radius R, with boundary value g is given by

$$u(r, \theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{g(\varphi)(R^2 - r^2)}{R^2 - 2rR\cos(\theta - \varphi) + r^2}d\varphi.$$

This expression is easily seen to be a consequence of the Cauchy integral formula. We know that for f differentiable,

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\xi)}{\xi - z}d\xi.$$

We will write the real and imaginary parts of f in terms of polar coordinates, r, θ.(i.e $f(z) = u(r, \theta) + iv(r, \theta)$) and set $\xi = Re^{i\varphi}, z = re^{i\theta}$ for $\varphi, \theta \in [0, 2\pi]$. Now observe that the point $R^2/\bar{z}$ lies outside the circle C and so by the Cauchy integral formula

$$\frac{1}{2\pi i}\int_C \frac{f(\xi)}{\xi - R^2/\bar{z}}d\xi = 0. \tag{5.45}$$

So we can write

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_C f(\xi)\left(\frac{1}{\xi - z} - \frac{1}{\xi - R^2/\bar{z}}\right)d\xi \\ &= \frac{1}{2\pi i}\int_C f(\xi)\frac{z - R^2/\bar{z}}{(\xi - z)(w - R^2/\bar{z})}d\xi. \end{aligned}$$

Using $\xi = Re^{i\varphi}, z = re^{i\theta}$ we get

$$\begin{aligned} f(re^{i\theta}) &= \frac{1}{2\pi i}\int_0^{2\pi} f(Re^{i\varphi})\frac{[re^{i\theta} - (R^2/r)e^{i\theta}]iRe^{i\varphi}d\varphi}{[Re^{i\varphi} - re^{i\theta}][Re^{i\varphi} - (R^2/r)e^{i\theta}]} \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(Re^{i\varphi})\frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - \varphi) + r^2}d\varphi, \end{aligned}$$

after some rather tedious messing about with Euler's formula for e^{ix}.

Thus

$$u(r, \theta) + iv(r, \theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(u(R, \varphi) + iv(R, \varphi))(R^2 - r^2)}{R^2 - 2Rr\cos(\theta - \varphi) + r^2}d\varphi,$$

If R is fixed, we can consider $u(R,\varphi)$ and $v(R,\varphi)$ as functions of φ only. Equating the real and imaginary parts we obtain

$$u(r,\theta)=\frac{1}{2\pi}\int_0^{2\pi}\frac{\tilde{u}(\varphi)(R^2-r^2)}{R^2-2rR\cos(\theta-\varphi)+r^2}d\varphi \tag{5.46}$$

and

$$v(r,\theta)=\frac{1}{2\pi}\int_0^{2\pi}\frac{\tilde{v}(\varphi)(R^2-r^2)}{R^2-2rR\cos(\theta-\varphi)+r^2}d\varphi, \tag{5.47}$$

where $\tilde{u}(\varphi)=u(R,\varphi)$ and $\tilde{v}(\varphi)=v(R,\varphi)$.

We know that the real and imaginary parts of f are harmonic. Thus (5.47) and (5.46) are harmonic. Hence the Poisson kernel solution of the Laplace equation is nothing more than a restatement of the Cauchy integral formula. There is a great deal more that can be said about the connection between Laplace's equation and analytic functions. This is a subject known as *potential theory*.

6. Residues and the Evaluation of Integrals

6.1. **Poles.** We have seen that differentiable functions of a complex variable are analytic in the sense that are equal to their Taylor expansion in the region where the derivative exists. We are now going to exploit this fact.

One of the most important applications of Cauchy's integral formula is the so called *residue theorem.* We will present this important result in this section together with some applications. The idea behind the residue theorem is the general principle in complex analysis that an analytic function is characterised by its singularities and its zeroes.

For an analytic function there are three types of singularities possible.

(1) Removable singularities,

(2) Poles,

(3) Essential singularities.

Removable singularities are basically harmless. The idea is that the singularity is only an apparent singularity because we have not described the function appropriately. Suppose that f is not defined at z_0 but $\lim_{z\to z_0} f(z) = L$ exists and is finite. Then z_0 is a removable singularity, because we can redefine f to equal L at z_0.

For example, suppose

$$f(z) = \frac{\sin z}{z}.$$

Then f is not defined at $z = 0$, but $\lim_{z\to 0} f(z) = 1$. So we can define

$$F(z) = \begin{cases} \frac{\sin z}{z} & z \neq 0 \\ 1 & z = 0. \end{cases}$$

So if we take this as our definition, the apparent singularity at zero vanishes and we have a continuous function.

Poles cannot be removed. However they are enormously useful. Suppose that f has a singularity at z_0, in the sense that $|\lim_{z\to z_0} f(z)| = \infty$, but that there exists a disc D around z_0 such that for all $z \in D$, with $z \neq z_0$, f is finite and analytic. Then z_0 is an *isolated singularity.*

Definition 6.1. Suppose that f is an analytic function with an isolated singularity at z_0. We say that z_0 is a pole of order if the function defined to be $1/f$ in a region D containing z_0 and equal to zero at z_0, is analytic in D. A singularity which is neither removable, nor a pole, is called an essential singularity.

An important property of a pole is its order. We will define this shortly.

For the next result we require the following property of analytic functions.

Theorem 6.2. *Suppose that f is an analytic function in some connected region Ω with an isolated zero at z_0. Then there is a neighbourhood $D \subseteq \Omega$ containing z_0 and a nowhere vanishing function g and a unique positive integer n such that*

$$f(z) = (z - z_0)^n g(z)$$

for all $z \in D$.

Proof. Since f is analytic it has a Taylor series expansion. We write

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k.$$

There is a smallest integer n such that $a_n \neq 0$. So we are able to write

$$\begin{aligned} f(z) &= \sum_{k=n}^{\infty} a_k (z - z_0)^k = (z - z_0)^n \left(a_n + \sum_{k=n+1}^{\infty} a_k (z - z_0)^{k-n} \right) \\ &= (z - z_0)^n g(z), \end{aligned} \tag{6.1}$$

where $g(z) = a_n + \sum_{k=n+1}^{\infty} a_k (z - z_0)^{k-n}$. Clearly $g(z_0) \neq 0$, since $a_n \neq 0$ and the terms in the power series vanish at 0. By continuity it is clear that for z sufficiently close to z_0, g will not vanish.

Now we want to show that the value of n is unique. Suppose that we had two such integers n, m with $n \neq m$. We could then write

$$f(z) = (z - z_0)^n g(z) = (z - z_0)^m h(z).$$

Suppose without loss of generality that $n > m$. Then we write

$$g(z) = (z - z_0)^{n-m} h(z).$$

But this implies that $g(z_0) = 0$ which is a contradiction. So the value of n is unique. $\square$

Theorem 6.3. *Suppose that f has a pole at $z_0 \in \Omega$. Then there is a positive integer n and neighbourhood D containing z_0 and an analytic function h such that for all $z \in D$ with $z \neq z_0$*

$$f(z) = (z - z_0)^{-n} h(z).$$

We say that n is the order of the pole.

Proof. Since z_0 is a pole, then $1/f$ is analytic with a zero at z_0. So there exists a non-vanishing analytic function $g(z)$ and an integer n such that $1/f(z) = (z - z_0)^n g(z)$. Now g is nonvanishing, so $h = 1/g$ is analytic. Thus

$$f(z) = (z - z_0)^{-n} h(z),$$

as claimed. $\square$

Remark 6.4. A pole of order 1 is said to be a simple pole.

6.2. **Laurent Series.** Functions which possess poles have a natural expansion which is analogous to a Taylor series and may indeed be seen as a generalisation of the classical Taylor series.

Theorem 6.5. *Suppose that f is an analytic function with a pole of order n at z_0. Then there exist numbers $\{a_k\}_{k=-n}^{\infty}$ such that*

$$f(z) = \frac{a_{-n}}{(z-z_0)^n} + \frac{a_{-n+1}}{(z-z_0)^{n-1}} + \cdots + \frac{a_{-1}}{z-z_0} + \sum_{k=0}^{\infty} a_k (z-z_0)^k. \tag{6.2}$$

Such an expansion is known as a Laurent series for f.

Proof. The proof is a corollary of our previous result. We can write

$$f(z) = (z-z_0)^{-n} h(z),$$

where, since h is analytic, we have $h(z) = \sum_{k=0}^{\infty} A_k (z-z_0)^k$. Hence

$$\begin{aligned} f(z) &= \frac{A_0}{(z-z_0)^n} + \frac{A_1}{(z-z_0)^{n-1}} + \cdots + \frac{A_{n-1}}{z-z_0} + \sum_{k=n}^{\infty} A_k (z-z_0)^{k-n} \\ &= \frac{a_{-n}}{(z-z_0)^n} + \frac{a_{-n+1}}{(z-z_0)^{n-1}} + \cdots + \frac{a_{-1}}{z-z_0} + \sum_{k=0}^{\infty} a_k (z-z_0)^k, \end{aligned}$$

where $a_{-n} = A_0, a_{-n+1} = A_1$ etc. □

Definition 6.6. The terms

$$\frac{a_{-n}}{(z-z_0)^n} + \frac{a_{-n+1}}{(z-z_0)^{n-1}} + \cdots + \frac{a_{-1}}{z-z_0},$$

are known as the principle part of the Laurent expansion. The number a_{-1} is known as the residue of f at z_0. We write $\mathrm{Res}(f(z), z_0) = a_{-1}$.

Let us consider some examples. If we know the Taylor series expansion for a function, computing the Laurent expansion is generally very easy.

Example 6.1. Let $f(z) = \dfrac{e^z}{z^3}$. Then we know that

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} + \cdots .$$

Hence

$$\frac{e^z}{z^3} = \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{2z} + \frac{1}{3!} + \frac{z}{4!} + \frac{z^2}{5!} + \cdots .$$

We see that $z = 0$ is a pole of order 3 and the residue at the pole is 1/2.

Example 6.2. The function $f(z) = \dfrac{\cos z}{z}$ has a Laurent series

$$\begin{aligned}\frac{\cos z}{z} &= \frac{1}{z}\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots\right)\\ &= \frac{1}{z} - \frac{z}{2!} + \frac{z^3}{4!} - \cdots.\end{aligned}$$

So $z = 0$ is a pole of order 1 and the reside is 1.

Example 6.3. The function $f(z) = \dfrac{\cos z}{z^2}$ has a Laurent series

$$\begin{aligned}\frac{\cos z}{z} &= \frac{1}{z^2}\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots\right)\\ &= \frac{1}{z^2} - \frac{1}{2!} + \frac{z^2}{4!} - \cdots.\end{aligned}$$

So $z = 0$ is a pole of order 2 and the reside is 0.

Example 6.4. We find the Laurent expansion for $f(z) = \dfrac{\sin z}{z^4}$. We use

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots.$$

Then

$$\frac{\sin z}{z^4} = \frac{1}{z^3} - \frac{1}{6z} + \frac{z}{5!} - \frac{z^3}{7!} + \cdots.$$

The pole is again of order 3 and the pole is $-1/6$.

Example 6.5. Let $f(z) = \dfrac{\cos z \sin z}{z^4}$. We can do this two ways. First we multiply the series for $\cos z$ and $\sin z$ together. Now

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots$$

We multiply the series for $\cos z$ and $\sin z$ term by term to get

$$\begin{aligned}\cos z \sin z &= \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots\right)\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\right)\\ &= z - \frac{2z^3}{3} + \frac{2z^5}{15} - \frac{4z^7}{315} + \cdots\end{aligned}$$

And we have

$$f(z) = \frac{1}{z^3} - \frac{2}{3z} + \frac{2z}{15} - \frac{4z^3}{315} + \cdots.$$

We have another pole of order 3 and the residue is $-2/3$. An easier way is to just note that $\sin(2z) = 2\sin z \cos z$ so that

$$\cos z \sin z = \frac{1}{2}\left(2z - \frac{(2z)^3}{3!} + \frac{(2z)^5}{5!} + \cdots\right),$$

and the result follows.

The most basic series is the geometric series. This is

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots = \sum_{n=0}^{\infty} z^n.$$

This series is convergent for $|z| < 1$. What about for $|z| > 1$? Can we express this function as a series? Let us observe that

$$\begin{aligned}\frac{1}{1-z} &= \frac{1}{z(\frac{1}{z}-1)} \\ &= -\frac{1}{z}(1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots \\ &= -\frac{1}{z^2} - \frac{1}{z^3} - \frac{1}{z^4} - \frac{1}{z^5} - \cdots\end{aligned}$$

What we have done is expanded $\frac{1}{1-\frac{1}{z}}$ in powers of $1/z$, which will converge when $1/|z| < 1$, or $|z| > 1$. So we have an *infinite* Laurent expansion of the function, valid for $|z| > 1$.

Many series can be derived from the geometric series. For example, suppose we take the identity

$$\tan^{-1}(z) = \int_0^z \frac{dt}{1+t^2}.$$

Setting $r = -t^2$ we have

$$\frac{1}{1+t^2} = 1 - t^2 + t^4 - t^6 + t^8 - \cdots$$

and integrating term by term we get

$$\begin{aligned}\tan^{-1}(z) &= \int_0^z \frac{dt}{1+t^2} \\ &= \int_0^z (1 - t^2 + t^4 - t^6 + t^8 - \cdots)dt \\ &= z - \frac{z^3}{3} + \frac{z^5}{5} - \frac{z^7}{7} + \cdots\end{aligned}$$

and this will be valid again for $|z| < 1$. We can also observe that

$$\begin{aligned}\frac{1}{1+t^2} &= \frac{1}{t^2(1+\frac{1}{t^2})} \\ &= \frac{1}{t^2}(1 - \frac{1}{t^2} + \frac{1}{t^4} - \frac{1}{t^6} + \cdots) \\ &= \frac{1}{t^2} - \frac{1}{t^4} + \frac{1}{t^6} - \frac{1}{t^8} + \cdots\end{aligned}$$

Now for $|z| > 1$ we have

$$\int_z^{\infty} \frac{dt}{1+t^2} = \frac{\pi}{2} - \tan^{-1} z. \tag{6.3}$$

So for $|z| > 1$

$$\begin{aligned}\frac{\pi}{2} - \tan^{-1} z &= \int_z^\infty \left(\frac{1}{t^2} - \frac{1}{t^4} + \frac{1}{t^6} - \frac{1}{t^8} + \cdots\right) dt \\ &= \frac{1}{z} - \frac{1}{3z^3} + \frac{1}{5z^5} - \frac{1}{7z^7} + \cdots\end{aligned}$$

Partial fractions are also useful sometimes. For example

$$\begin{aligned}\frac{z}{z^2-1} &= \frac{1}{2(z+1)} + \frac{1}{2(z-1)} \\ &= \frac{1}{2}\left((1 - z + z^2 - z^3 + \cdots) - (1 + z + z^2 + z^3 + \cdots)\right) \\ &= -\frac{1}{2}(z + z^3 + z^5 + z^7 + \cdots)\end{aligned}$$

The residue of f at a pole is a crucial number. Residues are essential in the calculation of many integrals. Therefore it is important to be able to calculate residues effectively. There is, fortunately, a straightforward method of doing this.

Theorem 6.7. *Suppose that f has a pole at z_0. If z_0 is a simple pole then*

$$\operatorname{Res}(f, z_0) = \lim_{z\to z_0} (z - z_0) f(z). \tag{6.4}$$

If z_0 is a pole of degree $n \neq 1$ then

$$\operatorname{Res}(f, z_0) = \lim_{z\to z_0} \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} \left((z - z_0)^n f(z)\right). \tag{6.5}$$

Proof. This is a simple exercise that we leave to the reader. It follows easily from the definition of the residue and the Laurent expansion for f. □

We often do not need to know the Laurent expansion for a function to identify the poles and their orders. If we have a function

$$f(z) = \frac{h(z)}{P(z)}, \ z \in D \subseteq \mathbb{C},$$

where P is a polynomial and h is analytic in D, then $z_0 \in D$ will be a pole of f if $P(z_0) = 0$ and $h(z_0) \neq 0$. The order of the pole will be the multiplicity of the root. Essentially, if h is analytic in D and, $z_1, ..., z_m \in D$, then

$$f(z) = \frac{h(z)}{(z - z_1)^{n_1}(z - z_2)^{n_2} \cdots (z - z_m)^{n_m}},$$

has a pole of order n_1 at z_1, a pole or order n_2 at z_2 etc.

Example 6.6. Consider the function

$$\frac{e^{iz}}{z^2+1} = \frac{e^{iz}}{(z+i)(z-i)}.$$

Since the numerator is analytic everywhere, the poles are the roots of $z^2 + 1 = 0$, which are clearly $z = i, -i$. The order of each pole is 1. That is, they are simple poles. The residue at $z = i$ is then

$$\operatorname{Res}\left(\frac{e^{iz}}{z^2+1}, i\right) = \lim_{z \to i}\left((z-i)\frac{e^{iz}}{(z+i)(z-i)}\right) = \frac{e^{-1}}{2i}.$$

At $z = -i$, the residue is

$$\operatorname{Res}\left(\frac{e^{iz}}{z^2+1}, -i\right) = \lim_{z \to -i}\left((z+i)\frac{e^{iz}}{(z+i)(z-i)}\right) = \frac{e^{1}}{-2i}.$$

Example 6.7. Consider the function

$$\frac{e^{iz}}{(z^2+1)^2} = \frac{e^{iz}}{(z+i)^2(z-i)^2}.$$

As in the previous example, since the numerator is analytic everywhere, the poles are the singularities of the function, which are the roots of the $(z^2 + 1)^2 = 0$. These are clearly $z = i, -i$. The order of each pole in this case is 2.

Since $z = i$ is a pole of order 2

$$\begin{aligned}
\operatorname{Res}\left(\frac{e^{iz}}{(z^2+1)^2}, i\right) &= \lim_{z \to i}\frac{d}{dz}\left((z-i)^2\frac{e^{iz}}{(z^2+1)^2}\right) \\
&= \lim_{z \to i}\frac{d}{dz}\left((z-i)^2\frac{e^{iz}}{(z+i)^2(z-i)^2}\right) \\
&= \lim_{z \to i}\frac{d}{dz}\left(\frac{e^{iz}}{(z+i)^2}\right) \\
&= \lim_{z \to i}\left(\frac{ie^{iz}}{(z+i)^2} - \frac{2e^{iz}}{(z+i)^3}\right) \\
&= -\frac{i}{2e}.
\end{aligned}$$

Similarly

$$\begin{aligned}
\operatorname{Res}\left(\frac{e^{iz}}{(z^2+1)^2}, -i\right) &= \lim_{z \to -i}\frac{d}{dz}\left((z+i)^2\frac{e^{iz}}{(z^2+1)^2}\right) \\
&= \lim_{z \to i}\frac{d}{dz}\left((z+i)^2\frac{e^{iz}}{(z+i)^2(z-i)^2}\right) \\
&= \lim_{z \to -i}\frac{d}{dz}\left(\frac{e^{iz}}{(z-i)^2}\right) \\
&= \lim_{z \to -i}\frac{ie^{iz}}{(z-i)^2} - \frac{2e^{iz}}{(z-i)^3} \\
&= 0.
\end{aligned}$$

Example 6.8. The function $f(z) = \dfrac{\sin z}{z-1}$ has a simple pole at $z = 1$. We have

$$\operatorname{Res}\left(\frac{\sin z}{z-1}, 1\right) = \lim_{z\to 1}(z-1)\frac{\sin z}{z-1} = \sin 1.$$

Example 6.9. A harder example is $f(z) = \tan z = \dfrac{\sin z}{\cos z}$. The singularities of this function are at $z_n = (2n+1)\frac{\pi}{2}$, $n = 0, \pm 1, \pm 2, \ldots$. These are all simple poles. To see that $z = \pi/2$ is a simple pole, we can compute the Laurent series around that point. We can expand $\sin z$ and $\cos z$ as Taylor series around $z = \pi/2$. We use the formula

$$f(z) = f(z_0) + f'(z_0)(z-z_0) + \frac{1}{2!}(z-z_0)^2 + \frac{1}{3!}(z-z_0)^3 + \cdots.$$

This gives

$$\sin z = 1 - \frac{1}{2}\left(z-\frac{\pi}{2}\right)^2 + \frac{1}{4!}\left(z-\frac{\pi}{2}\right)^4 - \frac{1}{6!}\left(z-\frac{\pi}{2}\right)^6 + \cdots,$$
$$\cos z = -\left(z-\frac{\pi}{2}\right) + \frac{1}{3!}\left(z-\frac{\pi}{2}\right)^3 - \frac{1}{5!}\left(z-\frac{\pi}{2}\right)^5 + \cdots.$$

Notice that this is quite different from the familiar expansion around $z_0 = 0$. Now we are expanding $\sin z$ in even powers and $\cos z$ in odd powers. We therefore have

$$\tan z = \frac{1 - \frac{1}{2}\left(z-\frac{\pi}{2}\right)^2 + \frac{1}{4!}\left(z-\frac{\pi}{2}\right)^4 - \frac{1}{6!}\left(z-\frac{\pi}{2}\right)^6 + \cdots}{-\left(z-\frac{\pi}{2}\right) + \frac{1}{3!}\left(z-\frac{\pi}{2}\right)^3 - \frac{1}{5!}\left(z-\frac{\pi}{2}\right)^5 + \cdots}.$$

Getting a series expansion for $\tan z$ in powers of $z-\frac{\pi}{2}$ is just a laborious exercise. Let us extract the first term of the series. We write

$$\begin{aligned}
&\frac{1}{-\left(z-\frac{\pi}{2}\right) + 1/6\left(z-\frac{\pi}{2}\right)^3 - 1/120\left(z-\frac{\pi}{2}\right)^5 + \cdots} \\
&\quad = \frac{1}{-\left(z-\frac{\pi}{2}\right)}\frac{1}{1 - \frac{1}{3!}\left(z-\frac{\pi}{2}\right)^2 + \frac{1}{5!}\left(z-\frac{\pi}{2}\right)^4 + \cdots} \\
&\quad = -\frac{1}{\left(z-\frac{\pi}{2}\right)}\left(1 + h(z) + h(z)^2 + \cdots\right) \\
&\quad = -\frac{1}{z-\frac{\pi}{2}} - \frac{h(z)}{z-\frac{\pi}{2}} - \frac{h(z)^2}{z-\frac{\pi}{2}} - \cdots,
\end{aligned}$$

where $h(z) = \frac{1}{3!}\left(z-\frac{\pi}{2}\right)^2 - \frac{1}{5!}\left(z-\frac{\pi}{2}\right)^4 + \cdots$. So the first term of the Laurent series is $\dfrac{-1}{z-\frac{\pi}{2}}$. Continuing this way we can slowly extract the series

$$\tan z = -\frac{1}{z-\frac{\pi}{2}} + \frac{1}{3}\left(z-\frac{\pi}{2}\right) + \frac{1}{45}\left(z-\frac{\pi}{2}\right)^3 + \frac{2}{945}\left(z-\frac{\pi}{2}\right)^5 + \cdots$$

We immediately see that the residue at $z = \pi/2$ is -1. The residues at the other poles have the same value. An easier way to compute this Laurent expansion is with Mathematica. The command

$$\text{Series}[f[z], \{z, z_0, n\}],$$

will return the first n terms of the series for f about the point z_0. Notice also that

$$\begin{aligned}
\lim_{z\to\pi/2}(z - \frac{\pi}{2}) \tan z &= \lim_{z\to\pi/2} \frac{(z - \frac{\pi}{2}) \sin z}{\cos z} \\
&= \lim_{z\to\pi/2} \frac{\sin z + (z - \frac{\pi}{2}) \cos z}{-\sin z} \\
&= -1.
\end{aligned}$$

Which is the value of the residue obtained from the Laurent series.

Notice also that

$$\begin{aligned}
\sec^2 z &= \frac{d}{dz} \tan z \\
&= \frac{1}{\left(z - \frac{\pi}{2}\right)^2} + \frac{1}{3} + \frac{1}{15}\left(z - \frac{\pi}{2}\right)^2 + \frac{2}{189}\left(z - \frac{\pi}{2}\right)^4 + \cdots,
\end{aligned}$$

so that $\sec^2 z$ has a pole or order 2 at $z = \pi/2$ and the residue at this pole is zero.

Let us now observe something important about the residue of f at a pole. Let

$$P(z) = \frac{a_{-n}}{(z - z_0)^n} + \frac{a_{-n+1}}{(z - z_0)^{n-1}} + \cdots + \frac{a_{-1}}{z - z_0},$$

and let h denote the analytic part of f. That is, $f(z) = P(z) + h(z)$. Now let C_R be a circle or radius R which is centered at z_0 but does not contain any other singularities of the function. We wish to calculate the value of $\int_{C_R} f(z)dz$.

First we note that since the integral is linear and h is analytic. Therefore $\int_{C_R} h(z)dz = 0$ by Cauchy's Theorem. So we have

$$\begin{aligned}
\int_{C_R} f(z)dz = \int_{C_R} (P(z) + h(z))dz &= \int_{C_R} P(z)dz + \int_{C_R} h(z)dz \\
&= \int_{C_R} P(z)dz.
\end{aligned}$$

Next we calculate the term involving a_{-1}. We have seen this done before, but for convenience we repeat the computation here. Set $z(t) = z_0 + Re^{it}, t \in [0, 2\pi]$. Then by definition of a contour integral we get

$$\int_{C_R} \frac{a_{-1}}{z - z_0}dz = a_{-1} \int_{C_R} \frac{dz}{z - z_0}dz = a_{-1} \int_0^{2\pi} \frac{iRe^{it}}{Re^{it}}dt = 2\pi i a_{-1}.$$

Now we consider the terms involving $a_{-k}, k \neq 1$. With the same parametrisation as before for z we get

$$\begin{aligned}
\int_{C_R} \frac{a_{-k}}{(z-z_0)^k} dz &= a_{-k} \int_{C_R} \frac{dz}{(z-z_0)^k} dz \\
&= a_{-k} \int_0^{2\pi} \frac{iRe^{it}}{(Re^{it})^k} dt \\
&= ia_{-k} R^{k-1} \int_0^{2\pi} e^{(k-1)it} dt \\
&= a_{-k} R^{k-1} \left[\frac{1}{k-1} e^{i(k-1)t} \right]_0^{2\pi} = 0. \qquad (6.6)
\end{aligned}$$

So the only contribution to the integral comes from the residue at the pole. We have thus proved the following remarkable theorem.

6.3. **The Residue Theorem.**

Theorem 6.8. *Suppose that f is analytic function on a domain D with a single pole of order n at $z_0 \in D$. Let C_R be a circle of radius R centered at z_0. Then*

$$\int_{C_R} f(z) dz = 2\pi i \operatorname{Res}(f(z), z_0). \qquad (6.7)$$

This result can be easily generalised. A proof is in Stewart and Tall [5].

Theorem 6.9 (The residue theorem)**.** *Suppose that f is analytic in a region Ω except at poles $z_1, ..., z_N$. Suppose that γ is a closed, simple contour, traversed counterclockwise which does not pass through any pole, but contains all the poles $z_1, ..., z_N$ in its interior. Then*

$$\int_\gamma f(z) dz = 2\pi i \sum_{k=1}^{N} \operatorname{Res}(f(z), z_k). \qquad (6.8)$$

Equation (6.8) is known as the residue formula.

Example 6.10. Let C_R be a circle of radius $R > 0$ centered at zero. Then

$$\int_{C_R} \frac{e^z}{z^3} dz = 2\pi i \times \frac{1}{2} = \pi i.$$

We computed the residue previously when we obtained the Laurent series, see Example 6.1.

The only things that matter when doing an integral around a closed contour are the values of the residues at the poles inside the contour. If there is a pole outside the contour, it makes no contribution to the integral.

Example 6.11. Let C_1 be a circle of radius 1 centered at zero. Then

$$\int_{C_1} \frac{e^z}{(z+4)^3} dz = 0$$

since the pole at $z = -4$ is not inside the contour.

6.4. **The Evaluation of Trigonometric Integrals.** We can use the residue theorem to evaluate a huge variety of real integrals. Let us begin by evaluating some trigonometric integrals. The basic idea is to use the fact that if $z = e^{it}$ then $\cos t = 1/2(z + 1/z)$ and $\sin t = 1/2i(z - 1/z)$. Further $dt = dz/iz$. In this way many trigonometric integrals can be reduced to contour integrals and evaluated by residues. The point is that setting $z = e^{it}$ maps the interval $[0, 2\pi]$ onto the unit circle centered at the origin.

Example 6.1. Evaluate the integral $\int_0^{2\pi} (\cos^3 t + \sin^2 t) dt$.

Setting $z = e^{it}$ we map the interval $[0, 2\pi]$ onto the unit circle. That is, we can express the integral as

$$\int_0^{2\pi} (\cos^3 t + \sin^2 t) dt = \int_C \left[\left(\frac{1}{2}(z + 1/z) \right)^3 + \left(\frac{1}{2i}(z - \frac{1}{z}) \right)^2 \right] \frac{dz}{iz}$$

$$= \int_C \left[\frac{1}{8i} z^2 - \frac{1}{4i} z - 3i + \frac{1}{2iz} + \frac{3}{iz^2} - \frac{1}{4iz^3} + \frac{1}{iz^4} \right] dz,$$

where C denotes the unit circle. Let

$$f(z) = \frac{1}{8i} z^2 - \frac{1}{4i} z - 3i + \frac{1}{2iz} + \frac{3}{iz^2} - \frac{1}{4iz^3} + \frac{1}{iz^4}.$$

Then f has one pole. It is at $z = 0$. The residue can be simply read off from the expansion. It is by definition the coefficient of $1/z$ which in this instance is $1/2i$.

Thus

$$\int_0^{2\pi} (\cos^3 t + \sin^2 t) dt = \int_C f(z) dz = 2\pi i Res(f, 0) = 2\pi i \times 1/2i = \pi.$$

Most integrals of the form $\int_0^{2\pi} Q(\cos t, \sin t) dt$ can, in principle at least, be done this way.

Exercise 6.1. Evaluate the integral $\int_0^{2\pi} (\cos^4 t + \sin^4 t) dt$ using residues.

Example 6.2. Calculate the integral $\displaystyle\int_0^{2\pi} \frac{dt}{a + b\cos t}$ for $a > |b|$.

Solution. As before, we make the substitution $z = e^{it}$. Then

$$\frac{1}{a + b\cos t} = \frac{1}{a + \frac{b}{2}(z + 1/z)}$$

$$= \frac{z}{\frac{b}{2} z^2 + az + \frac{b}{2}} = \frac{2z/b}{z^2 + 2az/b + 1}.$$

Our integral therefore becomes

$$\begin{aligned}\int_0^{2\pi} \frac{dt}{a+b\cos t} &= \int_C \frac{2z/b}{z^2+2az/b+1}\frac{dz}{iz} \\ &= \frac{2}{ib}\int_C \frac{1}{z^2+2az/b+1}dz. \qquad (6.9)\end{aligned}$$

The function $f(z) = \dfrac{1}{z^2+2az/b+1}$ has poles at the roots of

$$z^2+2az/b+1=0.$$

By the quadratic formula

$$z = \frac{-a \pm \sqrt{a^2-b^2}}{b}.$$

Since $a > |b|$ the roots are real. For the calculation of the integral we are only concerned with the pole inside the unit circle. Remember, only the poles inside the contour make a contribution to the integral. The poles which lie outside the contour are ignored. The question is which root, $z_1 = -(a+\sqrt{a^2-b^2})/b$ or $z_2 = -(a-\sqrt{a^2-b^2})/b$, is inside the unit circle. Since $a > |b|$, it is clear that

$$|z_1| = \left|(a+\sqrt{a^2-b^2})/b\right| \geq |a/b| > 1,$$

so z_1 is not in the unit circle. The other pole is in the unit circle. We know that $a - \sqrt{a^2-b^2} > 0$. The statement $|z_2| < 1$ implies that

$$a - \sqrt{a^2-b^2} < |b| \implies a^2 - 2a|b| + b^2 < a^2 - b^2, \qquad (6.10)$$

which in turn implies that $a > |b|$ which is true. Thus z_2 is in the unit circle and we need only calculate the residue at z_2. We have

$$\begin{aligned}\operatorname{Res}(f(z), z_2) &= \lim_{z\to z_2}(z-z_2)f(z) \\ &= \lim_{z\to-(a-\sqrt{a^2-b^2})/b} \frac{(z+(a-\sqrt{a^2-b^2})/b)}{(z+(a+\sqrt{a^2-b^2})/b)(z+(a-\sqrt{a^2-b^2})/b)} \\ &= \frac{b}{2\sqrt{a^2-b^2}}. \qquad (6.11)\end{aligned}$$

So by the residue theorem we have

$$\begin{aligned}\int_0^{2\pi} \frac{dt}{a+b\cos t} &= \frac{2}{ib}\int_C f(z)dz = \frac{2}{ib}2\pi i \operatorname{Res}(f(z), z_2) \\ &= \frac{2\pi}{\sqrt{a^2-b^2}}. \qquad (6.12)\end{aligned}$$

Example 6.3. Evaluate the integral $\displaystyle\int_0^{\pi} \frac{dt}{1+b\cos^2 t}, \quad b \geq 1.$

Solution. First approach. We observe that

$$\cos(t-\pi) = \cos t \cos \pi - \sin \pi \sin t = -\cos t.$$

Thus $\cos^2 t = \cos^2(t-\pi)$. Now we use elementary properties of integrals to write

$$\begin{aligned}\int_0^{2\pi} \frac{dt}{1+b\cos^2 t} &= \int_0^{\pi} \frac{dt}{1+b\cos^2 t} + \int_{\pi}^{2\pi} \frac{dt}{1+b\cos^2 t} \\ &= \int_0^{\pi} \frac{dt}{1+b\cos^2 t} + \int_0^{\pi} \frac{du}{1+b\cos^2(u-\pi)} \\ &= \int_0^{\pi} \frac{dt}{1+b\cos^2 t} + \int_0^{\pi} \frac{du}{1+b\cos^2 u} \\ &= 2\int_0^{\pi} \frac{dt}{1+b\cos^2 t}.\end{aligned}$$

So

$$\int_0^{\pi} \frac{dt}{1+b\cos^2 t} = \frac{1}{2}\int_0^{2\pi} \frac{dt}{1+b\cos^2 t}.$$

As in the preceding two examples we now set $z = e^{it}$ to map the interval $[0, 2\pi]$ onto the unit circle C. This gives

$$\begin{aligned}\int_0^{\pi} \frac{dt}{1+b\cos^2 t} &= \frac{1}{2}\int_C \frac{1}{1+\frac{b}{4}(z+1/z)^2}\frac{dz}{iz} \\ &= \frac{2}{ib}\int_C \frac{zdz}{z^4+(2+4/b)z^2+1}.\end{aligned}$$

This integral can be evaluated by the residue theorem, though it is not especially pleasant to have to compute the roots of the quartic here. Let us try another method.

Second approach. Notice that $\cos^2 t = \frac{1}{2}(1+\cos 2t)$. So

$$\int_0^{\pi} \frac{dt}{1+b\cos^2 t} = \int_0^{\pi} \frac{dt}{1+b/2(1+\cos 2t)}.$$

Now setting $z = e^{2it}$ maps the interval $[0, \pi]$ onto the unit circle, and $\cos 2t = \frac{1}{2}(z+1/z)$ and $dt = dz/2iz$. Making the change of variables in the integral produces

$$\begin{aligned}\int_0^{\pi} \frac{dt}{1+b/2(1+\cos 2t)} &= \int_C \frac{1}{1+b/2+b/4(z+1/z)}\frac{dz}{2iz} \\ &= \frac{2}{ib}\int_C \frac{dz}{z^2+(2+4/b)z+1} = \frac{2}{ib}\int_C f(z)dz.\end{aligned}$$

The roots of $z^2+(2+4/b)z+1=0$ are

$$z_1 = \frac{-2-b-2\sqrt{1+b}}{b}, \quad z_2 = \frac{-2-b+2\sqrt{1+b}}{b}.$$

It is clear that $|z_1| > 1$ since

$$\left|\frac{-2-b-2\sqrt{1+b}}{b}\right| = \left|\frac{2+b+2\sqrt{1+b}}{b}\right| > 1.$$

The fact z_2 is inside the unit circle follows from the fact that z_2 is a decreasing function of b with

$$\lim_{b\to\infty} z_2 = \lim_{b\to\infty}\left(2\frac{\sqrt{1+b}-1}{b} - 1\right) = -1.$$

It is also clear that if $b = 1$ then $z_2 \approx -0.171$. Therefore $-0.17 < z_2 < 1$ so that z_2 is inside the unit circle.

We must calculate the residue of $f(z) = \frac{1}{z^2+(2+4/b)z+1}$ at z_2.

$$\begin{aligned}\operatorname{Res}(f(z), z_2) &= \lim_{z\to z_2}(z - z_2)f(z)\\ &= \lim_{z\to\frac{-2-b+2\sqrt{1+b}}{b}} \frac{z - \frac{-2-b+2\sqrt{1+b}}{b}}{(z - (\frac{-2-b-2\sqrt{1+b}}{b}))(z - (\frac{-2-b+2\sqrt{1+b}}{b}))}\\ &= \frac{b}{4\sqrt{1+b}}.\end{aligned}$$

Therefore

$$\begin{aligned}\int_0^\pi \frac{dt}{1+b\cos^2 t} &= \frac{2}{ib}2\pi i\operatorname{Res}(f, z_2) = \frac{4\pi}{b}\frac{b}{4\sqrt{1+b}}\\ &= \frac{\pi}{\sqrt{1+b}}.\end{aligned}$$

This result is actually true for all $b > -1$. It is a useful exercise to prove this.

Exercise 6.2. Use contour integration to show that

$$\int_0^\pi \frac{dt}{1+b\sin^2 t} = \frac{\pi}{\sqrt{1+b}},\ \ b > -1.$$

6.5. **Integration over the real line.** Next we consider how to evaluate integrals of the form $\int_{-\infty}^\infty f(x)dx$ by the residue theorem. We recall from earlier in the notes some important facts. By definition, if f is a continuous function then

$$\int_{-\infty}^\infty f(x)dx = \lim_{T\to\infty}\int_{-T}^\infty f(x)dx + \lim_{K\to\infty}\int_0^K f(x)dx. \tag{6.13}$$

If the limit (6.13) exists, then it is true that

$$\int_{-\infty}^\infty f(x)dx = \text{P.V.}\int_{-\infty}^\infty f(x)dx, \tag{6.14}$$

where the *Cauchy principal value* is defined by

$$\text{PV}\int_{-\infty}^\infty f(x)dx = \lim_{R\to\infty}\int_{-R}^R f(x)dx.$$

It must be stressed that the Cauchy principal value can exist, even when (6.13) does not exist. For example let f be odd. Then

$$\int_{-R}^{R} f(x)dx = \int_{-R}^{0} f(x)dx + \int_{0}^{R} f(x)dx,$$

and putting $x = -t$ in the first integral gives

$$\int_{-R}^{0} f(x)dx = -\int_{0}^{R} f(t)dt \tag{6.15}$$

so that

$$\begin{aligned}\int_{-R}^{R} f(x)dx &= \int_{-R}^{0} f(x)dx + \int_{0}^{R} f(x)dx \\ &= -\int_{0}^{R} f(x)dx + \int_{0}^{R} f(x)dx \\ &= 0.\end{aligned}$$

Thus for f odd,

$$\begin{aligned}\text{P.V.} \int_{-\infty}^{\infty} f(x)dx &= \lim_{R\to\infty} \int_{-R}^{R} f(x)dx \\ &= \lim_{R\to\infty} 0 \\ &= 0.\end{aligned}$$

Yet if we take $f(x) = \sin x$ which is odd, the integral

$$\int_{-\infty}^{\infty} \sin x dx = \lim_{T\to\infty} \int_{-T}^{\infty} \sin x dx + \lim_{K\to\infty} \int_{0}^{K} \sin x dx$$

does not exist. Thus $\int_{-\infty}^{\infty} \sin x dx$ does not exist, but

$$\text{P.V.} \int_{-\infty}^{\infty} \sin x dx = 0.$$

What we will do is derive a method for computing the Cauchy principal value of an integral. If the integral converges, then this principal value will be the actual value of the integral.

First we present a simple example, then we will prove a result which will allow many more such integrals to be evaluated with less work.

Example 6.4. Use contour integration to show that

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi.$$

Solution. This integral can be done by elementary means, since

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{dx}{1+x^2} &= \lim_{R\to\infty}\int_{-R}^{0} \frac{dx}{1+x^2} + \lim_{R\to\infty}\int_{0}^{R} \frac{dx}{1+x^2} \\ &= \lim_{R\to\infty}[\tan^{-1} x]_{-R}^{0} + \lim_{T\to\infty}[\tan^{-1} x]_{0}^{T} \\ &= \frac{\pi}{2} + \frac{\pi}{2} = \pi.\end{aligned}$$

We however will use a contour integral to illustrate a very powerful

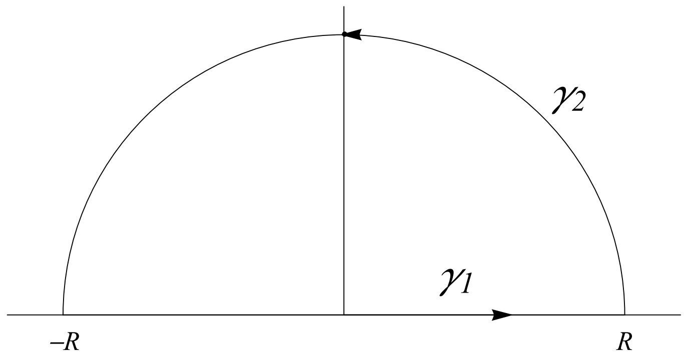

FIGURE 3. A Semicircular Contour.

method. We integrate $f(z) = \dfrac{1}{1+z^2}$ along the semicircular contour $\gamma = \gamma_1(x)+\gamma_2(x)$ where $\gamma_1(x) = x$, $-R \le x \le R$ and $\gamma_2(x) = Re^{ix}$, $0 \le x \le \pi$. See Figure 3.

The function f has two simple poles at $\pm i$ since

$$f(z) = \frac{1}{(z+i)(z-i)}.$$

The only pole inside the contour is at $z = i$. We easily see that

$$\mathrm{Res}(f(z), i) = \lim_{z\to i}(z-i)f(z) = \lim_{z\to i}\frac{z-i}{(z+i)(z-i)} = \frac{1}{2i}.$$

Now the residue theorem says that

$$\int_\gamma f(z)dz = 2\pi i \mathrm{Res}(f(z), i) = \pi,$$

since $z = i$ is the only pole inside γ. (Remember, only the poles inside the curve contribute to the integral).

By the definition of a contour integral we have

$$\int_\gamma f(z)dz = \int_{-R}^{R}\frac{dx}{1+x^2} + \int_0^\pi \frac{Rie^{ix}}{R^2e^{2ix}+1}dx.$$

We see that $R^2e^{2ix}+1 = R^2(\cos(2x)+i\sin(2x))+1$. Since $\cos(2x) \geq -1$ we have

$$\begin{aligned} |R^2e^{2ix}+1| &= \sqrt{(R^2\cos(2x)+1)^2 + R^4\sin^2(2x)} \\ &= \sqrt{R^4+2R^2\cos(2x)+1} \\ &\geq \sqrt{R^4-2R^2+1} \\ &= \sqrt{(R^2-1)^2}, \end{aligned}$$

so that for $R > 1$, $|R^2e^{2ix}+1| \geq R^2-1$. From this we have the estimate

$$\left|\frac{Rie^{ix}}{R^2e^{2ix}+1}\right| \leq \frac{R}{R^2-1}.$$

Therefore

$$\left|\int_0^\pi \frac{Rie^{ix}}{R^2e^{2ix}+1}dx\right| \leq \int_0^\pi \frac{R}{R^2-1}dx = \frac{\pi R}{R^2-1} \to 0$$

as $R \to \infty$.

We conclude that

$$\begin{aligned} \lim_{R\to\infty}\int_\gamma f(z)dz &= \lim_{R\to\infty}\int_{-R}^R \frac{dx}{1+x^2} + \lim_{R\to\infty}\int_0^\pi \frac{Rie^{ix}}{R^2e^{2ix}+1}dx \\ &= \lim_{R\to\infty}\int_{-R}^R \frac{dx}{1+x^2} = \pi. \end{aligned}$$

What does this mean? What we have shown is that

$$\text{P.V}\int_{-\infty}^\infty \frac{dx}{1+x^2} = \pi.$$

However, if the integral $\int_{-\infty}^\infty f(x)dx$ exists, it is equal to its principle value. The integral we want exists, and so we can conclude that

$$\int_{-\infty}^\infty \frac{dx}{1+x^2} = \pi.$$

This might seem rather a lot of work to do an integral that can be done by elementary means. However, we can also prove the following.

Proposition 6.10. *Suppose that f is analytic in the upper half plane $\Im(z) \geq 0$ except at finitely many poles, none of which lie on the real axis. Suppose further that there is a constant $A > 0$ such that for large enough R, $|f(z)| \leq A/R^k, k \geq 2$ on the semicircle $z = Re^{it},\ 0 \leq t \leq \pi$. Then*

$$\int_{-\infty}^\infty f(x)dx = \pi i\Sigma \tag{6.16}$$

in which Σ denotes the sum of the residues in the upper half plane.

Proof. The residue theorem tells us that the integral around the closed contour γ of the previous example satisfies $\int_\gamma f(z)dz = 2\pi i\Sigma$. The condition on the growth of f guarantees that

$$\left|\int_0^\pi f(Re^{it})Rie^{it}dt\right| \leq \frac{2\pi A}{R^{k-1}} \to 0$$

as $R \to \infty$, since $k > 1$. Now since

$$\int_\gamma f(z)dz = \int_{-R}^R f(x)dx + \int_0^\pi f(Re^{ix})Rie^{ix}dx$$

we can say that

$$\lim_{R\to\infty} \int_\gamma f(z)dz = \lim_{R\to\infty} \int_{-R}^R f(x)dx = 2\pi i\Sigma.$$

The growth condition $|f(x)| \leq A/x^2$ for large x, is enough to guarantee the convergence of the integral $\int_{-\infty}^\infty f(x)dx$. Thus

$$\int_{-\infty}^\infty f(x)dx = \text{P.V.} \int_{-\infty}^\infty f(x)dx = 2\pi i\Sigma$$

as claimed. □

Example 6.5. Evaluate the integral

$$\int_{-\infty}^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)}$$

where $a, b > 0$ and $a^2 \neq b^2$.

Solution. That $f(x) = \dfrac{1}{(x^2+a^2)(x^2+b^2)}$ satisfies the required bound is clear: For large x, the function behaves roughly like $1/x^4$. The only poles of f in the upper half plane are at $z = ia$, $z = ib$. Both are poles of order one. So

$$\begin{aligned}
\text{Res}(f(z), ia) &= \lim_{z\to ia} \frac{(z-ia)}{(z-ia)(z+ia)(z^2+b^2)} = \frac{1}{2ia(b^2-a^2)} \\
\text{Res}(f(z), ib) &= \lim_{z\to ib} \frac{(z-ib)}{(z^2+a^2)(z+ib)(z-ib)} = \frac{1}{2ib(a^2-b^2)}.
\end{aligned}$$

We therefore conclude that

$$\begin{aligned}
\int_{-\infty}^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)} &= 2\pi i\left(\frac{1}{2ia(b^2-a^2)} + \frac{1}{2ib(a^2-b^2)}\right) \\
&= \frac{\pi}{ab(a+b)}.
\end{aligned}$$

Example 6.6. We show that

$$\int_{-\infty}^\infty \frac{x^2dx}{(x^2+1)^2(x^2+2x+2)} = \frac{7\pi}{50}.$$

The function

$$f(z) = \frac{z^2}{(z^2+1)^2(z^2+2z+2)},$$

has poles of order 2 at $z = \pm i$ and poles of order 1 at the roots of $z^2+2z+2=0$. These are $z = \frac{-2\pm\sqrt{4-8}}{2} = -1 \pm i$. So the poles we want are $z_0 = i$, $z_1 = -1+i$. We calculate the residues.

$$\begin{aligned}
\operatorname{Res}(f(z), i) &= \lim_{z\to i} \frac{d}{dz}\left(\frac{(z-i)^2 z^2}{(z-i)^2(z+i)^2(z^2+2z+2)}\right) \\
&= \lim_{z\to i} \frac{d}{dz}\left(\frac{z^2}{(z+i)^2 2(z^2+2z+2)}\right) \\
&= \lim_{z\to i} -\frac{2z\left(z^3+z^2-iz-2i\right)}{(z+i)^3\left(z^2+2z+2\right)^2} \\
&= -\frac{3}{25} + \frac{9}{100}i
\end{aligned}$$

Next

$$\begin{aligned}
\operatorname{Res}(f(z), -1+i) &= \lim_{z\to -1+i} \frac{(z-(-1+i))z^2}{(z^2+1)^2(z-(-1+i))(z-(-1-i))} \\
&= \lim_{z\to -1+i} \frac{z^2}{(z^2+1)^2(z-(-1-i))} \\
&= \frac{3}{25} - \frac{4}{25}i.
\end{aligned}$$

Thus

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+1)^2(x^2+2x+2)} &= 2\pi i\left(-\frac{3}{25} + \frac{9}{100}i + \frac{3}{25} - \frac{4}{25}i\right) \\
&= \frac{7\pi}{50}.
\end{aligned}$$

Example 6.7. Show that

$$\int_0^{\infty} \frac{dx}{1+x^6} = \frac{\pi}{3}.$$

The function $f(z) = \dfrac{1}{1+z^6}$ has poles at the roots of $z^6+1=0$, or

$$z^6 = e^{\pi i + 2k\pi i}$$

These are $z_1 = e^{i\pi/6}, z_2 = e^{-i\pi/6}$, $z_3 = i$, $z_4 = -i$, $z_5 = e^{5\pi i/6}$ and $z_6 = e^{-5\pi i/6}$.

The poles z_1, z_3 and z_5 are above the axis. So we need the residues at these points.

$$\begin{aligned}
\operatorname{Res}(f(z), z_1) &= \lim_{z\to z_1}(z-z_1)f(z) \\
&= \lim_{z\to e^{i\pi/6}} \frac{1}{6z^5} = \frac{1}{6}e^{-5\pi i/6} \\
\operatorname{Res}(f(z), z_3) &= \lim_{z\to z_1}(z-z_1)f(z) \\
&= \lim_{z\to i} \frac{1}{6z^5} = \frac{1}{6i}
\end{aligned}$$

and

$$\begin{aligned}
\operatorname{Res}(f(z), z_5) &= \lim_{z\to z_5}(z-z_1)f(z) \\
&= \lim_{z\to e^{5\pi i/6}} \frac{1}{6z^5} = \frac{1}{6}e^{-25\pi i/6} = \frac{1}{6}e^{-\pi i/6}
\end{aligned}$$

So we have

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{dx}{1+x^6} &= 2\pi i\left(\frac{1}{6}e^{-5\pi i/6} + \frac{1}{6i} + \frac{1}{6}e^{-\pi i/6}\right) \\
&= 2\pi i\left(-\frac{\sqrt{3}}{12} - \frac{i}{12} - \frac{1}{6}i + \frac{\sqrt{3}}{12} - \frac{i}{12}\right) = \frac{2\pi}{3}.
\end{aligned}$$

Since $f(x)$ is even, we can write

$$\int_{0}^{\infty} \frac{dx}{1+x^6} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{1+x^6} = \frac{\pi}{3}.$$

Exercise 6.3. Verify the values of the following integrals using the method of the previous two examples.

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4} = \frac{\pi}{\sqrt{2}}.$$

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^8} = \frac{\pi}{8\sin\left(\frac{\pi}{8}\right)}.$$

One of the most important applications of infinite integrals arise from the theory of Fourier transforms. If a function $f : \mathbb{R} \to \mathbb{R}$ is integrable, then the Fourier transform of f is defined by the integral

$$\widehat{f}(y) = \int_{-\infty}^{\infty} f(x)e^{-iyx}dx.$$

If $\widehat{f}$ is integrable, then we recover f by *Fourier inversion.*

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(y)e^{iyx}dy.$$

Computing Fourier transforms is an enormously important problem in mathematics, statistics, physics, chemistry, engineering, medical imaging, image processing, astronomy and numerous other areas. In probability theory, the Fourier transform is called the characteristic function.

The methods we have developed here are essential to this task. Let us do an example.

Example 6.8. We calculate the Fourier type integral

$$\int_{-\infty}^{\infty} \frac{e^{iyx}}{(x^2+a^2)(x^2+b^2)}dx,$$

where again a, b and $a \neq b$ and y is real.

Solution. The function $f(z) = \dfrac{e^{iyz}}{(z^2+a^2)(z^2+b^2)}$ decreases at the same rate as the function in Example 6.5 and the poles in the upper half plane are at $z = ia$ and $z = ib$. So

$$\begin{aligned}
\operatorname{Res}(f(z), ia) &= \lim_{z\to ia} \frac{(z-ia)e^{iyz}}{(z-ia)(z+ia)(z^2+b^2)} = \frac{e^{-ay}}{2ia(b^2-a^2)} \\
\operatorname{Res}(f(z), ib) &= \lim_{z\to ib} \frac{(z-ib)e^{iyz}}{(z^2+a^2)(z+ib)(z-ib)} = \frac{e^{-by}}{2ib(a^2-b^2)}.
\end{aligned}$$

Which gives

$$\int_{-\infty}^{\infty} \frac{e^{iyx}}{(x^2+a^2)(x^2+b^2)}dx = \frac{\pi}{b^2-a^2}\left(\frac{e^{-ay}}{a} - \frac{e^{-by}}{b}\right).$$

If we equate the real and imaginary parts we obtain the integrals

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{\cos(yx)}{(x^2+a^2)(x^2+b^2)}dx &= \frac{\pi}{b^2-a^2}\left(\frac{e^{-ay}}{a} - \frac{e^{-by}}{b}\right) \\
\int_{-\infty}^{\infty} \frac{\sin(yx)}{(x^2+a^2)(x^2+b^2)}dx &= 0.
\end{aligned}$$

Integrals of the form $\int_{-\infty}^{\infty} f(x)e^{iyx}dx$ can be handled with the same methods, even if f does not satisfy as strong a condition in terms of its growth. In fact we can prove the following.

Proposition 6.11. *Suppose that f is analytic in the upper half plane $\Im(z) \geq 0$ except at finitely many poles, none of which lie on the real axis. Suppose further that there is a constant $A > 0$ such that for large enough R, $|f(z)| \leq A/|z|$ for z on the semicircle $Re^{it}, t \in [0, \pi]$. Then*

$$\int_{-\infty}^{\infty} f(x)e^{iyx}dx = 2\pi i\Sigma \tag{6.17}$$

in which Σ denotes the sum of the residues of $f(z)e^{iyz}$ in the upper half plane.

Proof. The proof is similar to the previous case, except now the integration is taken around a rectangular contour. The full details are in Stewart and Tall [5]. □

Example 6.9. Calculate the integrals

$$\int_{-\infty}^{\infty} \frac{x^3 \cos x}{(x^2+a^2)(x^2+b^2)} dx \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{x^3 \sin x}{(x^2+a^2)(x^2+b^2)} dx.$$

Solution. We calculate

$$\int_{-\infty}^{\infty} \frac{x^3 e^{ix}}{(x^2+a^2)(x^2+b^2)} dx.$$

The function $f(z) = \dfrac{z^3 e^{iz}}{(z^2+a^2)(z^2+b^2)}$ clearly satisfies the required growth condition, since for large $|z|$ it grows like $1/|z|$. The poles in the upper half plane are again at $z = ia$ and $z = ib$ and they are still simple poles. So

$$\begin{aligned} \operatorname{Res}(f(z), ia) &= \lim_{z \to ia} \frac{(z-ia) z^3 e^{iz}}{(z-ia)(z+ia)(z^2+b^2)} = \frac{a^2 e^{-a}}{2\ (a^2-b^2)} \\ \operatorname{Res}(f(z), ib) &= \lim_{z \to ib} \frac{(z-ib) z^3 e^{iz}}{(z^2+a^2)(z+ib)(z-ib)} = \frac{b^2 e^{-b}}{2\ (b^2-a^2)}. \end{aligned}$$

This gives the value of the integral as

$$\int_{-\infty}^{\infty} \frac{x^3 e^{ix}}{(x^2+a^2)(x^2+b^2)} dx = \pi i \frac{a^2 e^{-a} - b^2 e^{-b}}{a^2-b^2}$$

Equating real and imaginary parts yields the desired integrals.

$$\begin{aligned} &\int_{-\infty}^{\infty} \frac{x^3 \cos x}{(x^2+a^2)(x^2+b^2)} dx = 0 \\ &\int_{-\infty}^{\infty} \frac{x^3 \sin x}{(x^2+a^2)(x^2+b^2)} dx = \pi \frac{a^2 e^{-a} - b^2 e^{-b}}{a^2-b^2}. \end{aligned}$$

Exercise 6.4. Establish the values of the following integrals by residues.

$$\int_0^\infty \frac{\cos 5x}{x^4+a^4}dx = \frac{\pi}{2a^3}e^{-5a/\sqrt{2}}\sin\left(\frac{5a}{\sqrt{2}}+\frac{\pi}{4}\right) \tag{6.18}$$

$$\int_0^\infty \frac{\cos mx}{(1+x^2)^2}dx = \frac{\pi e^{-m}(1+m)}{4} \tag{6.19}$$

$$\int_{-\infty}^\infty \frac{\cos x}{x^2+a^2}dx = \frac{e^{-a}}{a}\pi \tag{6.20}$$

$$\int_{-\infty}^\infty \frac{x\sin x}{x^2+a^2}dx = \pi e^{-a} \tag{6.21}$$

$$\int_0^\infty \frac{\cos(2\pi x)}{1+x^2+x^4}dx = -\frac{\pi}{2\sqrt{3}}e^{-\pi\sqrt{3}} \tag{6.22}$$

$$\int_{-\infty}^\infty \frac{dx}{1+x^2+x^4} = \frac{\pi}{2\sqrt{3}} \tag{6.23}$$

$$\int_0^\infty \frac{x^2\cos 5x}{x^4+a^4}dx = \frac{\pi\left(\cos(\frac{5}{\sqrt{2}}\frac{a}{})-\sin(\frac{5}{\sqrt{2}}\frac{a}{})\right)}{2\sqrt{2}a}e^{-\frac{5}{\sqrt{2}}\frac{a}{}} \tag{6.24}$$

$$\int_{-\infty}^\infty \frac{dx}{(1+x^2)^{n+1}} = \frac{\Gamma(2n)}{2^{2n-1}n\Gamma(n)^2}\pi, \quad n=1,2,3.... \tag{6.25}$$

6.6. **Integration With Different Contours.**

6.6.1. *Contour Integrals with Rectangular Paths.* Not all contour integrals are evaluated around semi circles. A common type of problem arising in contour integration involves taking a rectangular path. We will illustrate with two typical examples. The reader should see that the principle is exactly the same as for a semicircular contour.

The reason why rectangular contours are important is that they change the way that we approach infinity. What this means is that as we take $R\to\infty$ in these examples, the height of the contour does not change. The length does. This is geometrically different to the semicircular case and is useful in situation where there may be infinitely many poles, as in the next example.

Example 6.10. Show that

$$\int_{-\infty}^\infty \frac{e^{ax}}{1+e^x}dx = \frac{\pi}{\sin(\pi a)} \quad 0<a<1$$

Solution. This integral can be evaluated by means of the Gamma function, but it is quite hard to do so. Here we use the residue theorem with a rectangular contour $\gamma=\gamma_1+\gamma_2-\gamma_3-\gamma_4$, where

$$\gamma_1(x)=x, \quad -R\le x\le R, \quad \gamma_2(x)=R+ix, \quad 0\le x\le 2\pi,$$

$$\gamma_3(x)=2\pi i+x, \quad -R\le x\le R, \quad \gamma_4(x)=-R+ix, \quad 0\le x\le 2\pi.$$

This is a rectangle with one edge running along the real axis and the other running above the real axis at a height of 2π. See Figure 4.

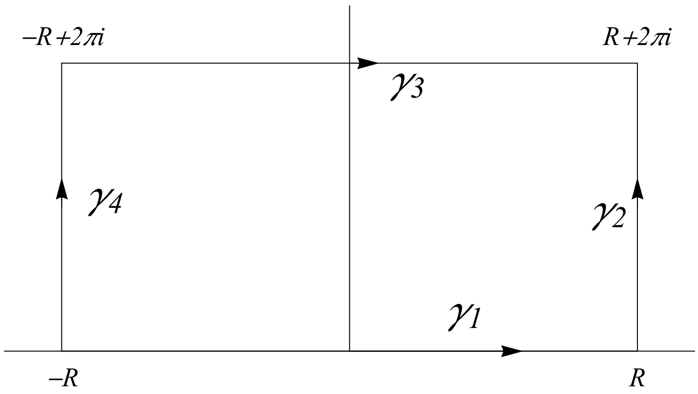

FIGURE 4. A rectangular Contour.

We set $f(z) = \dfrac{e^{az}}{1+e^z}$ and observe that f has a simple pole at $z = \pi i$ which lies inside the contour. There are actually infinitely many poles, but only is in the contour and as $R \to \infty$ the contour still only contains one pole.

The residue at the pole can be computed by means of L'Hôpital's rule. It is

$$\operatorname{Res}(f(z), \pi i) = \lim_{z \to \pi i} \frac{(z - \pi i)e^{az}}{1+e^z} = -e^{\pi a i}.$$

The residue theorem then gives us $\int_\gamma f(z)dz = -2\pi i e^{\pi a i}$.

We now have to extract out the value of the integral we desire in the usual manner. First, we consider the integral along γ_1. Using the definition of γ_1 we have

$$I_R = \int_{\gamma_1} f(z)dz = \int_{-R}^{R} \frac{e^{ax}}{1+e^x} dx.$$

Next we consider the integral along γ_3.

$$\begin{aligned}\int_{\gamma_3} f(z)dz &= \int_{-R}^{R} \frac{e^{(x+2\pi i)a}}{1+e^{x+2\pi i}} dx = e^{2\pi i a} \int_{-R}^{R} \frac{e^{ax}}{1+e^x} dx \\ &= e^{2\pi i a} I_R.\end{aligned}$$

Turning to the integral over γ_4 we see that

$$\int_{\gamma_4} f(z)dz = \int_0^{2\pi} \frac{i e^{a(-R+ix)}}{1+e^{-R+ix}} dx.$$

It is easy to see that

$$\begin{aligned}\left|\int_0^{2\pi} \frac{ie^{a(-R+ix)}}{1+e^{-R+ix}}dx\right| &\leq e^{-Ra}\int_0^{2\pi}\left|\frac{ie^{iax}}{1+e^{-R+ix}}\right|dx\\ &= e^{-Ra}\int_0^{2\pi}\left|\frac{1}{1+e^{-R+ix}}\right|dx\\ &\leq e^{-Ra}\int_0^{2\pi}\frac{1}{1-e^{-R}}dx = \frac{2\pi e^{-Ra}}{1-e^{-R}}.\end{aligned}$$

So $\lim_{R\to\infty}\int_{\gamma_4} f(z)dz = 0$. A similar argument shows that the integral along γ_2 goes to zero as $R \to \infty$.

We thus have

$$\lim_{R\to\infty}\int_\gamma f(z)dz = \text{P.V}\int_{-\infty}^{\infty}\frac{e^{ax}}{1+e^x}dx - e^{2\pi ia}\text{P.V}\int_{-\infty}^{\infty}\frac{e^{ax}}{1+e^x}dx = -2\pi e^{\pi ai}.$$

We know that the integral converges, so we can drop the principle value sign and write

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{e^{ax}}{1+e^x}dx = \frac{-2\pi e^{\pi ai}}{1-e^{2\pi ia}} &= \frac{2\pi i}{e^{\pi ia}-e^{-\pi ia}}\\ &= \frac{\pi}{\sin(\pi a)}.\end{aligned}$$

Example 6.11. By integrating $f(z) = \dfrac{e^{-2\pi i\xi z}}{\cosh(\pi z)}$ around the rectangular contour $\gamma = \gamma_1 + \gamma_2 - \gamma_3 - \gamma_4$, where

$$\gamma_1(x) = x, \quad -R \leq x \leq R, \quad \gamma_2(x) = R + ix, \quad 0 \leq x \leq 2,$$

$$\gamma_3(x) = 2i + x, \quad -R \leq x \leq R, \quad \gamma_4(x) = -R + ix, \quad 0 \leq x \leq 2,$$

show that

$$\int_{-\infty}^{\infty}\frac{e^{-2\pi i\xi x}}{\cosh(\pi x)}dx = \frac{1}{\cosh(\pi\xi)}.$$

First we remark that the integral is convergent since the integrand decays exponentially fast, so that the integral will equal its principal value.

We observe that inside γ, f has poles at $z = i/2$ and $3i/2$, both of order 1. We have

$$\text{Res}(f, i/2) = \lim_{z\to i/2}(z - i/2)f(z) = \frac{1}{i\pi}e^{\pi\xi}.$$

$$\text{Res}(f, 3i/2) = \lim_{z\to 3i/2}(z - 3i/2)f(z) = \frac{i}{\pi}e^{3\pi\xi}.$$

Thus by the residue theorem we have

$$\int_{\gamma_1} + \int_{\gamma_2} - \int_{\gamma_3} - \int_{\gamma_4} = -2e^{\pi\xi}(e^{2\pi\xi} - 1). \tag{6.26}$$

Also $\cosh(\pi(x+2i)) = \cosh(\pi x)$, So that

$$\int_{\gamma_1} f(z)dz = \int_{-R}^{R} \frac{e^{-2\pi i x\xi}}{\cosh(\pi x)}dx$$

and

$$\int_{\gamma_3} f(z)dz = e^{4\pi\xi} \int_{-R}^{R} \frac{e^{-2\pi i x\xi}}{\cosh(\pi x)}dx.$$

Hence

$$\begin{aligned}(1-e^{4\pi\xi})\int_{-R}^{R} \frac{e^{-2\pi i x\xi}}{\cosh(\pi x)}dx + i\int_0^2 \frac{e^{-2\pi i(R+ix)\xi}}{\cosh(\pi(R+ix))}dx \\ - i\int_0^2 \frac{e^{-2\pi i(-R+ix)\xi}}{\cosh(\pi(-R+ix))}dx = -2e^{\pi\xi}(e^{2\pi i}-1).\end{aligned}$$

Now for $0 \le x \le 2$,

$$\left|\frac{e^{-2\pi i(R+ix)\xi}}{\cosh(\pi(R+ix))}\right| \le \frac{e^{4\pi\xi}}{|\cosh(\pi(R+ix))|}.$$

Next notice that

$$\begin{aligned}|\cosh(\pi(R+ix))|^2 &= \cosh^2(\pi R)\cos^2(\pi x) + \sinh^2(\pi R)\sin^2(\pi x) \\ &= \cosh^2(\pi R)\cos^2(\pi x) + \sinh^2(\pi R)(1-\cos^2(\pi x)) \\ &= \cos^2(\pi x) + \sinh^2(\pi R) \\ &\ge \sinh^2(\pi R).\end{aligned}$$

Therefore

$$\begin{aligned}\left|\int_0^2 \frac{e^{-2\pi i(R+ix)\xi}}{\cosh(\pi(R+ix))}dx\right| &\le \int_0^2 \frac{e^{4\pi\xi}}{\sinh(\pi R)}dx \\ &= \frac{2e^{4\pi\xi}}{\sinh(\pi R)} \to 0,\end{aligned}$$

as $R \to \infty$. A similar calculation holds for the integral along γ_4. Taking the limit as $R \to \infty$ we get

$$(1-e^{4\pi\xi})\int_{-\infty}^{\infty} \frac{e^{-2\pi i x\xi}}{\cosh(\pi x)}dx = -2e^{\pi\xi}(e^{2\pi\xi}-1).$$

So

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{e^{-2\pi i x\xi}}{\cosh(\pi x)} dx &= \frac{-2e^{\pi\xi}(e^{2\pi\xi}-1)}{(1-e^{4\pi\xi})} \\ &= \frac{2e^{2\pi\xi}(e^{\pi\xi}-e^{-\pi\xi})}{e^{2\pi\xi}(e^{2\pi\xi}-e^{-2\pi\xi})} \\ &= \frac{2(e^{\pi\xi}-e^{-\pi\xi})}{(e^{\pi\xi}-e^{-\pi\xi})(e^{\pi\xi}+e^{-\pi\xi})} \\ &= \frac{2}{e^{\pi\xi}+e^{-\pi\xi}} \\ &= \frac{1}{\cosh(\pi\xi)}.\end{aligned}$$

Exercise 6.5. Show by a similar argument that for $0 < a < 1$

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi i\xi x}\sin\pi a}{\cosh\pi x + \cos\pi a} dx = \frac{2\sinh(2\pi a\xi)}{\sinh(2\pi\xi)}.$$

There are many results that we can prove that give us formulas for real integrals in terms of residues. We present another one here. We leave it as an exercise to prove it.

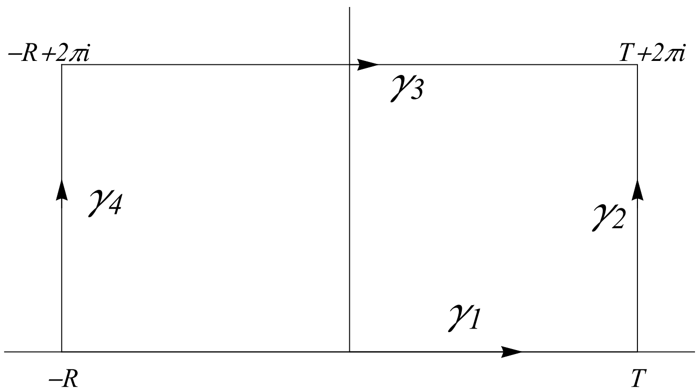

FIGURE 5. Another Rectangular Contour.

Theorem 6.12. *Consider the contour of Figure 5. Suppose that a is not an even integer. If ϕ is such that*

$$\int_{\gamma_2} \frac{e^{az}}{\phi(e^z)} dz \to 0 \tag{6.27}$$

and

$$\int_{\gamma_2} \frac{e^{az}}{\phi(e^z)} dz \to 0 \tag{6.28}$$

as $R, T \to \infty$, then

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{\phi(e^x)} dx = (1 - e^{2\pi i a})^{-1} \sum \tag{6.29}$$

where $\sum$ is the sum of the residues of $\frac{e^{az}}{\phi(e^z)}$ between the lines $\Im(z) = 0$ and $\Im(z) = 2\pi$.

6.6.2. *Poles on the Real Axis.* In the previous two results we assumed that the poles of the function do not lie on the real axis. The obvious question to ask it what do we do when the poles do lie on the real axis? This problem can also be handled by residues and a slight modification of the previous methods.

We illustrate by a famous example.

Example 6.12. Show that

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi.$$

Solution. We consider the function $f(z) = e^{iz}/z$ and integrate around the contour $\gamma = \gamma_1 - \gamma_\epsilon + \gamma_3 + \gamma_R$ where

(1) $\gamma_1(x) = x, \quad -R \le x < \epsilon$

(2) $\gamma_\epsilon(x) = \epsilon e^{ix}, \quad 0 \le x \le \pi$

(3) $\gamma_2(x) = x, \quad \epsilon \le x \le R$

(4) $\gamma_R(x) = Re^{ix}, \quad 0 \le x \le \pi.$

This a contour which is indented at the origin. We have seen it before. It is Figure 2. We choose it because $f(z)$ has a pole at $z = 0$. Notice that because of the indentation, the singularity does not lie inside the contour. So by Cauchy's Theorem $\int_\gamma f(z)dz = 0$. From this we can extract the value of the desired integral. Using the definition of a contour integral we have

$$\int_\gamma f(z)dz = \int_{-R}^{\epsilon} \frac{e^{ix}}{x} dx - \int_{\gamma_\epsilon} \frac{e^{iz}}{z} dz + \int_{\epsilon}^{R} \frac{e^{ix}}{x} dx + \int_0^{\pi} \frac{e^{iRe^{ix}}}{Re^{ix}} Rie^{ix} dx.$$

First we note that

$$\left| \int_0^{\pi} \frac{e^{iRe^{ix}}}{Re^{ix}} Rie^{ix} dx \right| \le \int_0^{\pi} e^{-R\sin x} dx.$$

Since $e^{-R\sin x} \to 0$ as $R \to \infty$ for all $x \ne 0, \pi$ we have by the dominated convergence theorem $\int_0^{\pi} e^{-R\sin x} dx \to 0$ as $R \to \infty$.

Now we calculate the integral around γ_ϵ.

$$\frac{e^{iz}}{z} = \frac{1}{z} + i - \frac{z}{2!} - i\frac{z^2}{3!} + \cdots = \frac{1}{z} + k(z)$$

where k is an entire function. Since $k(z)$ is entire $\int_{\gamma_\epsilon} k(z)dz \to 0$ as $\epsilon \to 0$. Notice that $e^{i\epsilon e^{ix}} \to 1$ as $\epsilon \to 0$. We therefore have

$$\begin{aligned}\int_{\gamma_\epsilon} \frac{e^{iz}}{z}dz &= \int_0^\pi \frac{e^{i\epsilon e^{ix}}}{\epsilon e^{ix}} i\epsilon e^{ix}dx \\ &= i\int_0^\pi e^{i\epsilon e^{ix}}dx \to i\int_0^\pi dx = \pi i,\end{aligned}$$

as $\epsilon \to 0$. We can conclude that

$$\lim_{R\to\infty}\lim_{\epsilon\to 0}\int_\gamma f(z)dz = \text{P.V.}\int_{-\infty}^\infty \frac{e^{ix}}{x}dx = \pi i.$$

Comparing real and imaginary parts we get

$$\begin{aligned}\text{P.V.}\int_{-\infty}^\infty \frac{\sin x}{x}dx &= \pi \\ \text{P.V.}\int_{-\infty}^\infty \frac{\cos x}{x}dx &= 0.\end{aligned}$$

However, the sine integral exists, so we can conclude that

$$\int_{-\infty}^\infty \frac{\sin x}{x}dx = \pi.$$

The cosine integral by contrast, does not converge, so the real part of our integral remains a principle value integral.

The basic principle for evaluating integrals with poles on the real axis is to indent the contour so that it excludes the poles on the real axis and then use the residue theorem or Cauchy's theorem together with a limiting argument. In our next example we have a pole on the real axis and poles on the imaginary axis.

Example 6.13. Show that

$$\int_{-\infty}^\infty \frac{\sin x}{x(x^2+1)}dx = \frac{\pi(e-1)}{e}.$$

Solution. Notice that if we take $f(z) = \dfrac{e^{iz}}{z(z^2+1)}$ then f has poles on the imaginary axis at $z = \pm i$ and a pole at $z = 0$. We use the same contour as in the previous question. The pole at $z = i$ is simple, so that

$$\text{Res}(f(z), i) = \lim_{z\to i}\frac{(z-i)e^{iz}}{z(z+i)(z-i)} = -\frac{1}{2}e^{-1}.$$

By the residue theorem we have

$$\int_\gamma f(z)dz = 2\pi i\text{Res}(f(z), i) = -\frac{\pi}{e}i.$$

As in the previous example $\int_{\gamma_R} f(z)dz \to 0$ as $R \to \infty$. We therefore have to deal with the integral over γ_ϵ. Here we observe that by the dominated convergence theorem

$$\begin{aligned}\int_{\gamma_\epsilon} f(z)dz &= \int_0^\pi \frac{e^{\epsilon e^{ix}}}{\epsilon e^{ix}(\epsilon^2 e^{2ix}+1)} i\epsilon e^{ix} dx \\ &= i\int_0^\pi \frac{e^{\epsilon e^{ix}}}{(\epsilon^2 e^{2ix}+1)} dx \to i\int_0^\pi dx = \pi i,\end{aligned}$$

as $\epsilon \to 0$. Putting it all together we conclude that

$$\lim_{R\to\infty}\lim_{\epsilon\to 0}\int_\gamma f(z)dz = \text{P.V.}\int_{-\infty}^{\infty}\frac{e^{ix}}{x(x^2+1)}dx = \frac{\pi(e-1)}{e}i.$$

Comparing real and imaginary parts, and using the fact that the sine integral is convergent, we get

$$\int_{-\infty}^{\infty}\frac{\sin x}{x(x^2+1)}dx = \frac{\pi(e-1)}{e}.$$

The cosine integral does not converge however, so we can only say that

$$\text{P.V.}\int_{-\infty}^{\infty}\frac{\cos x}{x(x^2+1)}dx = 0.$$

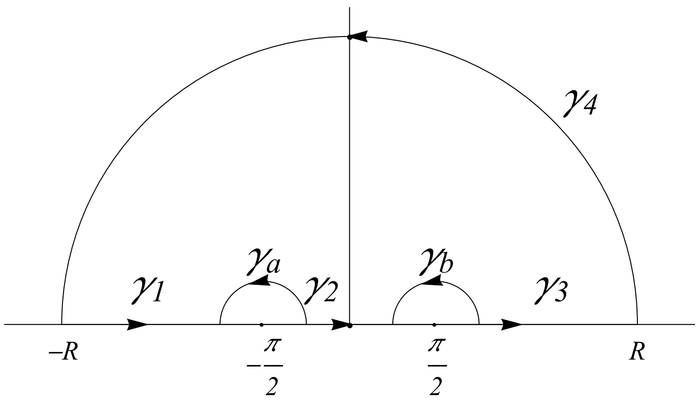

FIGURE 6. A Doubly Indented Contour.

Example 6.14. In this example we will calculate the integral

$$\int_{-\infty}^{\infty}\frac{\cos x}{\pi^2 - 4x^2}dx. \tag{6.30}$$

We will use a contour which is indented at two points. See Figure 6. We notice that $\pi^2 - 4x^2 = 0$ when $x = \pm\frac{\pi}{2}$. However by L'Hôpital's rule

$$\lim_{x\to\frac{\pi}{2}} \frac{\cos x}{\pi^2-4x^2} = \lim_{x\to\frac{\pi}{2}} \frac{-\sin x}{-8x} = \frac{1}{4\pi}.$$

Similarly

$$\lim_{x\to-\frac{\pi}{2}} \frac{\cos x}{\pi^2-4x^2} = \frac{1}{4\pi}.$$

So the singularities are removable and since the function decays as $1/x^2$, the integral is convergent. We then consider the function

$$f(z) = \frac{e^{iz}}{\pi^2-4z^2}, \tag{6.31}$$

which we integrate around the contour $\gamma = \gamma_1 - \gamma_a + \gamma_2 - \gamma_b + \gamma_3 + \gamma_4$. Here $\gamma_a = -\epsilon e^{it} - \frac{\pi}{2}$ and $\gamma_b = \epsilon e^{it} + \frac{\pi}{2}$. $\gamma_1(t) = t,\ -R \le t \le -\epsilon - \frac{\pi}{2}$, etc.

The function f has poles at $\pm\frac{\pi}{2}$, neither of which are inside the contour. So that by Cauchy's Theorem

$$\int_\gamma f(z)dz = 0.$$

Now standard arguments show that $\int_{\gamma_4} f(z)dz \to 0$ as $R \to \infty$. Consider the integral around $\gamma_b = \frac{\pi}{2} - \epsilon e^{it}$, $t \in [0,\pi]$.

$$\begin{aligned}\int_{\gamma_b} f(z)dz &= \int_0^\pi \frac{-ie^{i(\frac{\pi}{2}-\epsilon e^{it})}\epsilon e^{it}}{\pi^2-4(\frac{\pi}{2}-\epsilon e^{it})^2}dt\\ &= \int_0^\pi \frac{-ie^{i(\frac{\pi}{2}-\epsilon e^{it})}\epsilon e^{it}}{\pi^2-\pi^2+4\pi\epsilon e^{it}-4\epsilon^2e^{2it}}dt\\ &\to \int_0^\pi \frac{1}{4\pi}dt = \frac{1}{4}\end{aligned}$$

as $\epsilon \to 0$. Similarly $\int_{\gamma_a} f(z)dz \to \frac{1}{4}$ as $\epsilon \to 0$. The calculation is nearly identical.

Now $\int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3} + \int_{\gamma_4} = \int_{\gamma_a} + \int_{\gamma_b}$ by Cauchy's Theorem. Taking the limits as $R \to \infty$ and $\epsilon \to 0$ we conclude

$$\int_{-\infty}^{\infty} \frac{\cos x}{\pi^2-4x^2}dx = \frac{1}{2}.$$

Exercise 6.6. . Evaluate the following integrals.

(1) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x(x^4+1)}dx$

(2) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x(x^2+4)(x^2+1)}dx.$

6.6.3. *A Logarithmic Integral.*

Example 6.15. Show that

$$\int_0^\infty \frac{\ln(x^2+1)}{x^2+1}dx = \pi \ln 2.$$

Solution. We use our standard semicircular contour $\gamma = \gamma_1 + \gamma_2$ where $\gamma_1(x) = x, -R \le x \le R$, and $\gamma_2(x) = Re^{ix}, 0 \le x \le \pi$. For our function we choose $f(z) = \dfrac{\ln(z+i)}{z^2+1}$.

Why do we not take the actual integrand in the function? The reason we cannot do this is that at $z = i$, $\ln(z^2+1) = \ln 0$. This means that we need to look for a different function. By the (almost) additivity of the complex logarithm, $\ln(z+i)$ is a good choice for the numerator as we will see. Now the pole of f inside γ is at $z = i$ and

$$\text{Res}(f(z), i) = \lim_{z\to i}(z-i)\frac{\ln(z+i)}{(z+i)(z-i)} = \frac{\ln(2i)}{2i}.$$

By the residue theorem

$$\begin{aligned}\int_\gamma f(z)dz = \int_{-R}^{R} \frac{\ln(x+i)}{x^2+1}dx + \int_{\gamma_2} \frac{\ln(z+i)}{z^2+1}dz &= 2\pi i\frac{\ln(2i)}{2i} \\ = \pi\ln(2i) = \pi(\ln 2 + i\text{Arg}(i)) \\ = \pi \ln 2 + \frac{1}{2}\pi^2 i,\end{aligned}$$

since $\text{Arg}(i) = \pi/2$. The integral over the semicircle goes to zero as $R \to \infty$ as is easily seen, since

$$\begin{aligned}\left|\int_{\gamma_2} \frac{\ln(z+i)}{z^2+1}dz\right| &= \left|\int_0^\pi \frac{\ln(Re^{ix}+i)}{R^2e^{2ix}+1}iRe^{ix}dx\right| \\ &\le \int_0^\pi \frac{R|\ln(Re^{ix}+i)|}{R^2-1}dx.\end{aligned}$$

This integral clearly goes to zero as $R \to \infty$. If we wish to prove this rigorously we can observe that $Re^{ix} + i = R\cos x + i(R\sin x + 1)$. Thus

$$|Re^{ix}+i| = \sqrt{R^2+2R\sin x+1} \le \sqrt{R^2+2R+1} = |R+1|.$$

We can therefore write

$$Re^{ix} + i = \sqrt{R^2+2R\sin x+1}e^{i\phi}$$

for some angle ϕ which depends on R and x, but lies in the interval $[-\pi, \pi)$. Hence $\ln(Re^{ix}+i) = \ln(\sqrt{R^2+2R\sin x+1}) + i\phi$.

Thus

$$\int_0^\pi \frac{R\ln(Re^{ix}+i)}{R^2-1}dx = \int_0^\pi \frac{R(\ln(\sqrt{R^2+2R\sin x+1}) + i\phi)}{R^2-1}dx.$$

Now

$$\left|\int_0^\pi \frac{iR\phi}{R^2-1}dx\right| \leq \frac{R}{R^2-1}\int_0^\pi |\phi|dx \leq \frac{\pi^2 R}{R^2-1} \to 0,$$

as $R \to \infty$, since $|\phi| \leq \pi$.

Also,

$$\begin{aligned}\left|\int_0^\pi \frac{R\ln(\sqrt{R^2+2R\sin x+1})}{R^2-1}dx\right| &\leq \int_0^\pi \frac{R\ln|R+1|}{R^2-1}dx \\ &= \frac{R\ln|R+1|}{R^2-1}\pi \to 0,\end{aligned}$$

as $R \to \infty$.

Now we look a bit more carefully at the integral over the interval $[-R, R]$.

$$\begin{aligned}\int_{-R}^R \frac{\ln(x+i)}{x^2+1}dx &= \int_{-R}^0 \frac{\ln(x+i)}{x^2+1}dx + \int_0^R \frac{\ln(x+i)}{x^2+1}dx \\ &= \int_0^R \frac{\ln(i-x)}{x^2+1}dx + \int_0^R \frac{\ln(x+i)}{x^2+1}dx \\ &= \int_0^R \frac{\ln(i-x)+\ln(x+i)}{x^2+1}dx.\end{aligned}$$

The natural logarithm is not quite additive when we deal with complex numbers. That is, for two complex numbers z_1, z_2 in general $\ln(z_1 z_2) \neq \ln z_1 + \ln z_2$. However $\ln(z_1 z_2) = \ln z_1 + \ln z_2 + iC$. where C is some real constant that depends on the argument of z_1 and z_2. Thus

$$\ln(i-x)+\ln(x+i) = \ln(-1-x^2)+iC = \ln(-1)+\ln(x^2+1)+iC.$$

Since $\ln(-1) = \pi i$ we have

$$\int_0^R \frac{\ln(i-x)+\ln(x+i)}{x^2+1}dx = \int_0^R \frac{\ln(x^2+1)+iD}{x^2+1}dx,$$

where $D = C + \pi$ is real. Hence

$$\int_0^R \frac{\ln(x^2+1)+iD}{x^2+1}dx + \int_{\gamma_2} f(z)dz = \pi\ln 2 + \frac{1}{2}\pi^2 i$$

Taking $R \to \infty$ and comparing real parts gives

$$\int_0^\infty \frac{\ln(x^2+1)}{x^2+1}dx = \pi\ln 2.$$

Comparing imaginary parts tells us that in fact $C = 0$ in this calculation.

6.6.4. *Miscellaneous Integrals.* Evaluating real integrals by residues is something of an art. The key is to find the right contour. While the standard semicircular and rectangular contours suffice to evaluate a very large variety of integrals, they are not the only kinds of contours which can be used.

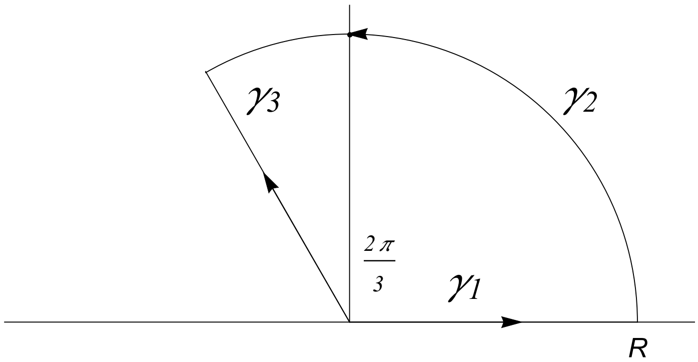

FIGURE 7. An nonstandard Contour.

Example 6.16. Show that for $a > 0$

$$\int_0^\infty \frac{dx}{x^3 + a^3} = \frac{2\pi}{3\sqrt{3}a^2}.$$

The integrand here has a pole at $x = -a$, which lies on the real axis, so we would like to avoid this pole. The key is to use the following contour. $\gamma = \gamma_1 + \gamma_2 - \gamma_3$ in which $\gamma_1 = x, 0 \le x \le R$; the contour γ_2 is a circular arc given by $\gamma_2(x) = Re^{ix}, 0 \le x \le 2\pi/3$ and $\gamma_3(x) = xe^{2\pi i/3}, 0 \le x \le R$. See Figure 7. The contour is chosen so that only the pole at $z = ae^{i\pi/3}$ lies inside it. Also, observe that $(xe^{2\pi i/3})^3 = x^3$.

The choice of contours is not so cryptic when we understand where it comes from. The third contour is chosen so that the integral over γ_3 is a multiple of the integral over γ_1.

First our integrand is $f(z) = \dfrac{1}{z^3 + a^3}$ which has poles at $z = -a, z = ae^{\pi i/3}$ and $z = ae^{-\pi i/3}$.

The residue at $z = ae^{i\pi/3}$ is given by

$$\operatorname{Res}(z, ae^{i\pi/3}) = \lim_{z \to ae^{i\pi/3}} \frac{(z - ae^{i\pi/3})}{(z-a)(z - ae^{i\pi/3})(z - ae^{-i\pi/3})} = \frac{e^{-2\pi i/3}}{3a^2}.$$

By the residue theorem

$$\begin{aligned}\int_\gamma f(z)dz &= \int_0^R \frac{dx}{x^3 + a^3} + \int_{\gamma_2} \frac{dz}{z^3 + a^3} - \int_0^R \frac{e^{2\pi i/3}dx}{x^3 + a^3} \\ &= (1 - e^{2\pi i/3}) \int_0^R \frac{dx}{x^3 + a^3} + \int_{\gamma_2} \frac{dz}{z^3 + a^3} = 2\pi i \frac{e^{-2\pi i/3}}{3a^2}.\end{aligned}$$

As usual it is not hard to show that the integral over the circular arc γ_2 goes to zero as $R \to \infty$. Taking the limit as $R \to 0$ we have

$$\int_0^\infty \frac{dx}{x^3+a^3} = 2\pi i \frac{e^{-2\pi i/3}}{3a^2(1-e^{2\pi i/3})}$$
$$= \frac{-\pi}{3a^2}\left(\frac{2i}{e^{-i\pi/3}-e^{i\pi/3}}\right) = \frac{2\pi}{3\sqrt{3}a^2},$$

since $e^{-i\pi/3} - e^{i\pi/3} = -2i\sin(\pi/3)$.

Exercise 6.7. Prove that for $a > 0$

$$\int_0^\infty \frac{dx}{x^5+a^5} = \frac{\pi}{5a^4\sin(\pi/4)}.$$

Use a contour similar to the previous example, but with the angle $2\pi/3$ replaced by $2\pi/5$ in the definition of γ_2 and γ_3.

6.6.5. *Integration with a Keyhole Contour.* There are numerous contours that have been employed for the evaluation of contour integrals. One of the more frequently occurring non standard contours is the keyhole contour seen in Figure 8. We will illustrate how such a contour might arise. The principles are exactly the same as those that we have been using.

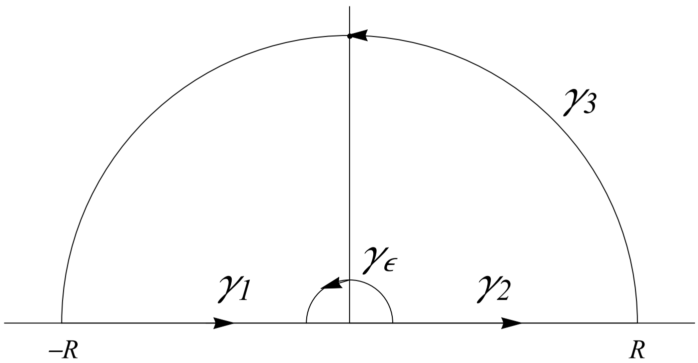

FIGURE 8. A Keyhole Contour.

Example 6.17. We are going to evaluate the integral

$$\int_0^\infty \frac{x^{m-1}}{x^2+1}dx, 0 < m < 2.$$

To do this we consider the contour integral $I = \displaystyle\int_C \frac{z^{m-1}}{z^2+1}dz$, in which C is the keyhole contour of Figure 8.

We need to choose a branch of the power function z^α and we will let $z = re^{i\theta}$, for $\theta \in [0, 2\pi)$. This places the branch cut along the positive real axis which our contour misses. If we choose the usual branch for the argument, we would have a branch cut along the negative axis and we would have to flip the contour. This can be done, but we will not do so here.

Notice that there are two poles inside C at $z = \pm i$. Neither are on the real axis. The branch cut along the positive real axis means that the value of the argument jumps by 2π as we cross the axis. So we can think of the contour on the way in having an argument of 2π and the argument on the way out being zero.

We leave it as an exercise to show that $\int_{C_R} \frac{z^{m-1}}{z^2+1} dz \to 0$ as $R \to \infty$. We also can show that $\int_{C_\epsilon} \frac{z^{m-1}}{z^2+1} dz \to 0$ as $\epsilon \to 0$.

Let the contour above the axis on the way out be C_a, then

$$\int_{C_a} \frac{z^{m-1}}{z^2+1} dz = \int_\epsilon^R \frac{x^{m-1}}{x^2+1} dx. \tag{6.32}$$

If C_b is the contour running below the axis we have

$$\int_{C_b} \frac{z^{m-1}}{z^2+1} dz = \int_R^\epsilon \frac{(xe^{2\pi i})^{m-1}}{(xe^{2\pi i})^2+1} e^{2\pi i} dx. \tag{6.33}$$

What we have done here is right $x = xe^{2\pi i}$ to indicate that we are below the axis. Then we also have $dx \to e^{2\pi i} dx$. The choice of branch only makes its presence felt when we are dealing with non integer powers. Remember integer powers are unique.

Obviously we need to compute the residues at the two poles. For the pole at $z = i = e^{\pi i/2}$ we have

$$\begin{aligned} \text{Res}(f(z), i) &= \lim z \to e^{i\frac{\pi}{2}} \frac{(z-i)z^{m-1}}{(z-i)(z+i)} \\ &= \frac{(e^{i\frac{\pi}{2}})^{m-1}}{2i}. \end{aligned}$$

Notice that it is only at the non integer power of z that out choice of branch plays a role.

For the pole at $z = -i = e^{3\pi i/2}$ we have

$$\begin{aligned} \text{Res}(f(z), -i) &= \lim z \to e^{i\frac{3\pi}{2}} \frac{(z+i)z^{m-1}}{(z-i)(z+i)} \\ &= \frac{(e^{i\frac{3\pi}{2}})^{m-1}}{-2i}. \end{aligned}$$

Now by the residue theorem

$$\int_C = \int_{C_R} + \int_{C_\epsilon} + \int_\epsilon^R + \int_R^\epsilon = 2\pi i \left(\frac{(e^{i\frac{\pi}{2}})^{m-1}}{2i} - \frac{(e^{i\frac{3\pi}{2}})^{m-1}}{2i} \right).$$

Letting $R \to \infty$ and $\epsilon \to 0$ we have

$$\int_C f(z)dz \to (1 - e^{2\pi i m}) \int_0^\infty \frac{x^{m-1}}{x^2+1}dx$$
$$= 2\pi i \left(\frac{(e^{i\frac{\pi}{2}})^{m-1}}{2i} - \frac{(e^{i\frac{3\pi}{2}})^{m-1}}{2i} \right)$$

So that

$$\int_0^\infty \frac{x^{m-1}}{x^2+1}dx = \frac{-i\pi e^{\frac{im\pi}{2}}(1+e^{im\pi})}{1-e^{2\pi i m}}$$
$$= \frac{\pi}{2\sin\left(\frac{\pi m}{2}\right)}.$$

The final reduction is left to the reader.

It is of course possible to handle special cases without the need for a keyhole contour. For example, if $m = 3/2$ then the integral is

$$I = \int_0^\infty \frac{x^{1/2}}{x^2+1}dx. \tag{6.34}$$

We can put $x = u^2$, then $dx = 2udu$ and the integral becomes

$$I = \int_0^\infty \frac{2u^2}{u^4+1}du = \int_{-\infty}^\infty \frac{u^2}{u^4+1}du, \tag{6.35}$$

which can be evaluated using our earlier results.

6.7. **Summation of Series.** There are many applications of the residue theorem. Evaluating real integrals is perhaps the most important, but there are many others. One interesting one is to the evaluation of infinite sums. The key results are as follows.

Theorem 6.13. *Let C_N be the path traversing the square with vertices at the points $(N+1/2)(1+i), (N+1/2)(-1+i), (N+1/2)(-1-i), (N+1/2)(1-i)$ counterclockwise. Let f be a function which is analytic except at its poles, $z_1, z_2, ...z_m$, which are contained in C_N. Suppose that on C_N, f satisfies $|f(z)| \leq \frac{A}{|z|^k}, k > 1$ and has no poles at the points $n = 0, \pm 1, \pm 2, ...$ Then*

$$\sum_{n=-\infty}^{\infty} f(n) = -\sum_{k=1}^{m} \mathrm{Res}(\pi \cot(\pi z) f(z), z_k). \tag{6.36}$$

Proof. The function $\cot(\pi z)$ has simple poles at $z = 0, \pm 1, \pm 2, ...$ Hence $\pi \cot(\pi z) f(z)$ has poles at $z = n, n = 0, \pm 1, \pm 2, ...$ and $z_1, ..., z_m$. Now

$$\mathrm{Res}(\pi \cot(\pi z) f(z), n) = \pi \lim_{n \to \pi} \left(\frac{z-n}{\sin(\pi z)} \right) \cos(\pi z) f(z)$$
$$= f(n),$$

after an application of L'Hôpital's rule. Let $S_N = \sum_{n=-N}^{N} f(n)$. By the residue theorem

$$\int_{C_N} \pi \cot(\pi z) f(z) dz = 2\pi i \left(S_N + \sum_{k=1}^{m} \operatorname{Res}(\pi \cot(\pi z) f(z), z_k) \right).$$

Now C_N has four sides of length $2N+1$, so the length of the path C_N is $8N+4$. We can show that there is a constant A such that $|\cot(\pi z)| \leq M$ on C_N. By the ML inequality

$$\left| \int_{C_N} \pi \cot(\pi z) f(z) dz \right| \leq \frac{\pi A M}{(N+1/2)^k}(8N+4) \to 0,$$

as $N \to \infty$. As $N \to \infty, S_N \to \sum_{n=-\infty}^{\infty} f(n)$. Thus

$$\begin{aligned} \lim_{N\to\infty} \int_{C_N} \pi \cot(\pi z) f(z) dz &= \lim_{N\to\infty} 2\pi i \left(S_N + \sum_{k=1}^{m} \operatorname{Res}(\pi \cot(\pi z) f(z), z_k) \right) \\ &= 0. \end{aligned}$$

Which gives

$$\sum_{k=1}^{m} \operatorname{Res}(\pi \cot(\pi z) f(z), z_k) + \sum_{n=-\infty}^{\infty} f(n) = 0.$$

□

It is interesting to note that this result is still true if f has *infinitely* many poles $z_1, z_2, ...$ In this case

$$\sum_{n=-\infty}^{\infty} f(n) = -\sum_{k=1}^{\infty} \operatorname{Res}(\pi \cot(\pi z) f(z), z_k). \tag{6.37}$$

Example 6.18. Show that

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2+a^2} = \frac{\pi}{a} \coth(\pi a).$$

We take the function $f(z) = \frac{1}{z^2+a^2}$ which has simple poles at $\pm ia$. Now

$$\begin{aligned} \operatorname{Res}\left(\frac{\pi \cot(\pi z)}{z^2+a^2}, ia \right) &= \lim_{z\to ia} \frac{(z-ia)\pi \cot(\pi z)}{(z-ia)(z+ia)} \\ &= \frac{\pi \cot(i\pi a)}{2ia} \\ &= \frac{\pi \cos(i\pi a)}{2ia \sin(i\pi a)} \\ &= \frac{\pi \cosh(\pi a)}{-2a \sinh(\pi a)}. \end{aligned}$$

Similarly

$$\begin{aligned}\operatorname{Res}\left(\frac{\pi\cot(\pi z)}{z^2+a^2},-ia\right) &= \lim_{z\to -ia}\frac{(z+ia)\pi\cot(\pi z)}{(z-ia)(z+ia)}\\ &= \frac{-\pi\cot(i\pi a)}{-2ia}\\ &= \frac{\pi\cosh(\pi a)}{-2a\sinh(\pi a)}.\end{aligned}$$

Thus

$$\begin{aligned}\sum_{n=-\infty}^{\infty}\frac{1}{n^2+a^2} &= -\left(\frac{\pi\cosh(\pi a)}{-2a\sinh(\pi a)}+\frac{\pi\cosh(\pi a)}{-2a\sinh(\pi a)}\right)\\ &= \frac{\pi}{a}\coth(\pi a).\end{aligned}$$

We can also sum other kinds of series.

Theorem 6.14. *Let C_N be the path traversing the square with vertices at the points $(N+1/2)(1+i),(N+1/2)(-1+i),(N+1/2)(-1-i),(N+1/2)(1-i)$ counterclockwise. Let f be a function which is analytic except at its poles, $z_1,z_2,....z_m$, which are contained in C_N. Suppose that on C_N, f satisfies $|f(z)|\le\frac{A}{|z|^k}, k>1$ and has no poles at the points $n=0,\pm1,\pm2,....$ Then*

$$\sum_{n=-\infty}^{\infty}(-1)^n f(n) = -\sum_{k=1}^{m}\operatorname{Res}(\pi\operatorname{cosec}(\pi z)f(z),z_k). \tag{6.38}$$

The proof is similar to that of Theorem 6.13. This result is also valid if there are infinitely many poles.

Example 6.19. We sum the series

$$\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{(n+a)^2}.$$

The function $f(z)=\frac{1}{(z+a)^2}$ has a pole of order 2 at $z=-a$. Then

$$\begin{aligned}\operatorname{Res}\left(\frac{\pi\mathrm{cosec}(\pi\mathrm{z})}{(z+a)^2},-a\right) &= \lim_{z\to -a}\frac{d}{dz}\left(\frac{(z+a)^2\pi\mathrm{cosec}(\pi\mathrm{z})}{(z+a)^2}\right)\\ &= -\frac{\pi^2\cos(\pi a)}{\sin^2(\pi a)}.\end{aligned}$$

And so

$$\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{(n+a)^2}=\frac{\pi^2\cos(\pi a)}{\sin^2(\pi a)}.$$

6.8. **Counting Zeroes.** One of the many applications of Residues is to a simple problem. Given a function f we often need to know the zeroes of the function. That is the values of z such that $f(z) = 0$. Think of the problem of factorising a polynomial or determining the maxima and minima of a function.

A rather startling result allows us to determine how many zeroes a function h as inside a given closed contour. The proof is not particularly hard.

Theorem 6.15. *Suppose that $f : S \to \mathbb{C}$ is differentiable, except at a finite number of poles. Suppose that none of the poles or zeroes of f lie on the simple closed contour γ, which is contained in S. Let the number of zeroes of f inside γ be N, counted according to multiplicity. Let the number of poles be P, counted according to order. Then*

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = N - P. \tag{6.39}$$

We suppose that γ is traversed counterclockwise.

Proof. We establish this in two parts. The key result is

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = \sum_{k=1} \operatorname{Res}\left(\frac{f'(z)}{f(z)}, z_k\right)$$

where the z_k are the poles of the integrand.

First we prove that if f has a zero of order k at z_1, then f'/f has a pole of order 1 at z_1 with residue equal to k.

To see this, notice that since f has a zero of order k_1 at z_1, then we can write $f(z) = (z - z_1)^{k_1}\phi(z)$, where ϕ is analytic in the neighbourhood of z_1 and $\phi(z_1) \neq 0$. Now a simple calculation gives

$$\frac{f'(z)}{f(z)} = \frac{k_1}{z - z_1} + \frac{\phi'(z)}{\phi(z)}.$$

Hence z_1 is a pole of order 1 and the residue is k_1.

Now we suppose that f has a pole of order n_p at z_p. Then f'/f has a pole at z_p and the residue is $-n_p$.

Adapting our previous argument, we see that we can write

$$f(z) = \frac{\psi(z)}{(z - z_p)^{n_p}},$$

in which ψ is differentiable around z_p and $\psi(z_p) \neq 0$. We then have

$$f'(z) = \frac{-n_p\psi(z)}{(z - z_p)^{n_p+1}} + \frac{\psi'(z)}{(z - z_p)^{n_p}}.$$

Hence

$$\frac{f'(z)}{f(z)} = \frac{-n_p}{z - z_p} + \frac{\psi'(z)}{\psi(z)}.$$

Hence z_p is a pole of f'/f and the residue is $-n_p$.

Now let the zeroes of f be $z_1, ..., z_{p-1}$ and the multiplicities of the zeroes be $k_1, ..., k_{p-1}$ and let $N = k_1 + \cdots + k_{p-1}$. Let the poles of f be $z_p, ..., z_m$ and the orders of the poles be $n_p, ..., n_m$. Let $P = n_p + \cdots + n_m$. Then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = \sum_{k=1} \text{Res}\left(\frac{f'(z)}{f(z)}, z_k\right) = N - P.$$

□

There is an immediate corollary of this. If f has no poles, then we easily have the number of zeroes inside γ.

Corollary 6.16. *Suppose that $f : S \to \mathbb{C}$ is differentiable except at a finite number of poles. Suppose that none of the poles or zeroes of f lies on the simple closed contour γ, which is contained in S. Suppose further that there are no poles inside γ. Let the number of zeroes of f inside γ be N, counted according to multiplicity. Then*

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = N. \tag{6.40}$$

We suppose that γ is traversed counterclockwise.

Proof. This is the previous result with $P = 0$. □

This has lots of applications. For example, suppose that f has no poles inside the closure of $\Omega \subseteq \mathbb{C}$. Suppose further that f'/f is analytic in the closure of Ω. Then for any simple closed contour γ in $\mathbb{C}$, we have

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = 0,$$

by Cauchy's Theorem. Hence f has no zeroes inside Ω.

For example, take $f(z) = e^z$. Then $f'(z) = e^z$. So that for any simple closed curve γ in the complex plane,

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{f'(z)}dz = \frac{1}{2\pi i}\int_\gamma \frac{e^z}{e^z}dz \tag{6.41}$$

$$= \frac{1}{2\pi i}\int_\gamma dz = 0. \tag{6.42}$$

Since this holds for any simple closed curve, it follows that e^z has no zeroes anywhere.

Example 6.20. We take $f(z) = z^n - 1$ where n is a positive integer. Let $\gamma = 2e^{i\theta}, \theta \in [0, 2\pi)$. Then

$$\begin{aligned}
\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{f'(z)}dz &= \frac{1}{2\pi i}\int_\gamma \frac{nz^{n-1}}{z^n-1}dz \\
&= \frac{1}{2\pi i}\int_0^{2\pi} \frac{2^n nie^{ni\theta}}{2^n e^{ni\theta}-1}d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \frac{2^n nie^{ni\theta}-1+1}{2^n e^{ni\theta}-1}d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} nd\theta + \frac{1}{2\pi}\int_0^{2\pi}\frac{1}{2^n e^{ni\theta}-1}d\theta \\
&= n.
\end{aligned}$$

We leave it as an exercise to prove that $\int_0^{2\pi}\frac{1}{2^n e^{ni\theta}-1}d\theta = 0$. So we have recovered the fact that $z^n = 1$ has n solutions inside the circle γ. In fact these are the only zeroes.

This result is useful in many areas. Often it is used in conjunction with a numerical method. Suppose we want to solve $f(z) = 0$. We can use Newton's method or some other iterative scheme to obtain zeroes. If we know that there are exactly N zeroes, then when we have obtained N values for the zeroes, we know that we can stop because we have found all of them.

6.9. **Inversion of Laplace Transforms.** The Laplace transform is one of the most useful tools in analysis. It is defined by the integral

$$F(s) = \int_0^\infty f(t)e^{-st}dt.$$

A sufficient condition for the Laplace transform of f to exist is that f is continuous and there exists constants $K > 0$ and a such that $|f(t)| \le Ke^{at}$. Laplace transforms play an important role in differential equations, probability theory, the theory of linear systems and many other areas.

Example 6.21. Let us calculate the Laplace transform of $f(t) = t^a, a > -1$. The restriction on a is to ensure that the integral converges. We have via the change of variables $st = u$

$$\begin{aligned}
\int_0^\infty t^a e^{-st}dt &= \frac{1}{s^{a+1}}\int_0^\infty u^a e^{-u}du \\
&= \frac{\Gamma(a+1)}{s^{a+1}}.
\end{aligned}$$

So $\mathcal{L}(f)(s) = \dfrac{\Gamma(a+1)}{s^{a+1}}$. Here $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$ is the Gamma function.

Exercise 6.8. Compute the Laplace transform of $f(t) = e^{at}$.

Example 6.22. Find the Laplace transform of $f(t) = \sin(at)$.
Solution. To do this we observe that by Euler's formula

$$\sin(at) = \frac{1}{2i}(e^{iat} - e^{-iat}).$$

Hence

$$\begin{aligned}\mathcal{L}(f) = \frac{1}{2i}\mathcal{L}(e^{iat}) - \frac{1}{2i}\mathcal{L}(e^{-iat}) &= \frac{1}{2i}\left(\frac{1}{s-ia} - \frac{1}{s+ia}\right)\\ &= \frac{1}{2i}\left(\frac{s+ia-(s-ia)}{s^2+a^2}\right)\\ &= \frac{a}{s^2+a^2}.\end{aligned}$$

Given a Laplace transform $F(s)$, we would like to know the original function f. That is, invert the Laplace transform. We can do this in several ways, but one of the most powerful methods is to use contour integration. We need a preliminary definition first.

Definition 6.17. A function f is said to have bounded variation on $[a, b]$ if for every partition $P = \{x_0, x_1, ..., x_n\}$ of $[a, b]$ the quantity

$$\text{variation}(f) = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|,$$

is bounded.

Theorem 6.18 (Laplace Transform Inversion)**.** *If $f(t)$ is a locally integrable function on $[0, \infty)$ such that,*

(1) *f is of bounded variation in a neighborhood of a point $t_0 \geq 0$ (a right hand neighborhood if $t_0 = 0$).*
(2) *The integral $\int_0^\infty f(t)e^{-st}dt$ converges absolutely on $Re(s) = \gamma$ then,*

$$\lim_{T\longrightarrow\infty} \frac{1}{2\pi i}\int_{\gamma-iT}^{\gamma+iT} F(s)e^{st}ds = \begin{cases} 0 & t_0 < 0\\ f(0^+)/2 & t_0 = 0\\ \frac{1}{2}(f(t_0^+) + f(t_0^-)) & t_0 > 0.\end{cases}$$

In particular if f is differentiable on $(0, \infty)$ and satisfies (1) and (2) then,

$$\lim_{T\longrightarrow\infty} \frac{1}{2\pi i}\int_{\gamma-iT}^{\gamma+iT} F(s)e^{st}ds = f(t) \quad 0 < t < \infty.$$

Proof. This result can be established from the Fourier inversion theorem. We will prove the special case when f is continuous. We suppose that $F(s) = \int_0^\infty f(t)e^{-st}dt$. So applying the inversion formula we have

$$\begin{aligned}
&\lim_{T\to\infty} \frac{1}{2\pi i}\int_{\gamma-iT}^{\gamma+iT} e^{st}F(s)ds \\
&= \lim_{T\to\infty} \frac{1}{2\pi i}\int_{\gamma-iT}^{\gamma+iT} e^{st}\left(\int_0^\infty e^{-su}[e^{\gamma u}f(u)]du\right)ds.
\end{aligned}$$

We now make the substitution $s = \gamma + iy$. So $ds = idy$ and

$$\begin{aligned}
&\lim_{T\to\infty} \frac{1}{2\pi i}\int_{\gamma-iT}^{\gamma+iT} e^{st}\left(\int_0^\infty e^{-su}f(u)du\right)ds \\
&= \lim_{T\to\infty} \frac{1}{2\pi}e^{\gamma t}\int_{-T}^{T} e^{iyt}\left(\int_0^\infty e^{-iyu}f(u)du\right)dy \\
&= e^{\gamma t}\begin{cases} e^{-\gamma t}f(t), & t \geq 0 \\ 0, & t < 0, \end{cases}
\end{aligned}$$

by the Fourier Inversion Theorem applied to the function

$$g(t) = \begin{cases} e^{-\gamma t}f(t), & t \geq 0 \\ 0, & t < 0, \end{cases}.$$

□

Remark 6.19. The integral is taken in the principal value sense since in general $\int_{\gamma-i\infty}^{\gamma+i\infty} F(s)e^{st}ds$ does not converge.

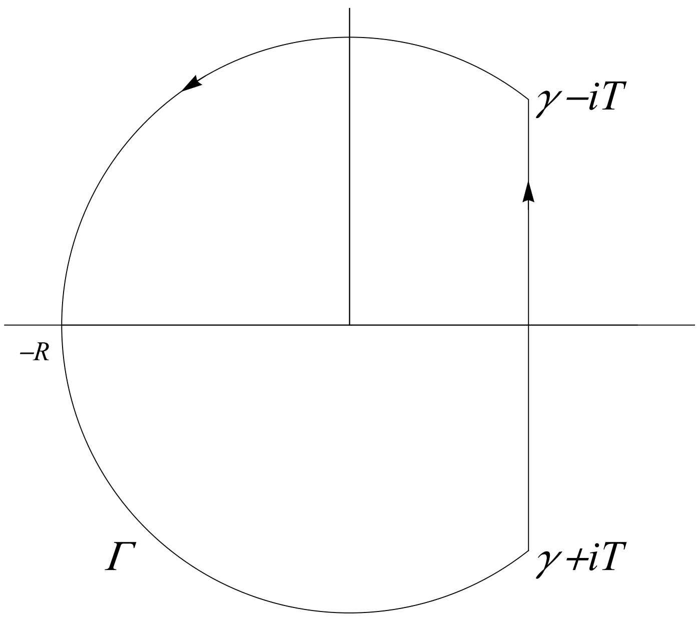

FIGURE 9. The Bromwich Contour.

The integral is usually evaluated using the so called *Bromwich contour*. We want the integral along the line running from $\gamma - iT$ to $\gamma + iT$. To obtain this, we integrate around the Bromwich contour and use the residue Theorem.

We have the following result, which we will not prove.

Theorem 6.20. *If we can find constants $M > 0$ and $k > 0$ such that on Γ with $s = Re^{i\theta}$ we have*

$$|F(s)| < \frac{M}{R^k}, \tag{6.43}$$

then the integral around Γ of $e^{st}F(s)$ approaches 0 as $R \to \infty$. ie.

$$\lim_{R\to\infty} \int_\Gamma e^{st} F(s) ds = 0. \tag{6.44}$$

Thus taking $R \to \infty$ and $T \to \infty$ we should obtain the value of the inverse Laplace transform. In practice this means that we have to sum the residues of $e^{st}F(s)$ at every pole of $F(s)$.

Example 6.12. Let

$$F(s) = \frac{1}{(s^2+1)}.$$

We observe that there are poles at $s = \pm i$. So that

$$\mathrm{Res}\left(\frac{e^{st}}{s^2+1}, s = i\right) = \lim_{s\longrightarrow i}\left(\frac{(s-i)e^{st}}{(s-i)(s+i)}\right) = \frac{1}{2i}e^{it}$$

$$\mathrm{Res}\left(\frac{e^{st}}{s^2+1}, s = -i\right) = \lim_{s\longrightarrow -i}\left(\frac{(s+i)e^{st}}{(s-i)(s+i)}\right) = -\frac{1}{2i}e^{-it}$$

By the Cauchy residue theorem we then have

$$\begin{aligned}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}F(s)ds &= 2\pi i \sum Residues \\ &= 2\pi i\left(\frac{1}{2i}(e^{it} - e^{-it})\right) = 2\pi i \sin t\end{aligned}$$

So

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) = \frac{1}{2\pi i} 2\pi i \sin t = \sin t$$

Notice that the $2\pi i$ from the Theorem and the $1/2\pi i$ from the Theorem cancel. So we just need to sum the residues.

Example 6.23. Compute the inverse Laplace transform of

$$F(s) = \frac{s}{(s^2+4)(s^2+1)}$$

using residues.

Solution The poles are at $\pm 2i$ and $\pm i$. The residues are then

$$\begin{aligned}\mathrm{Res}\left(\frac{se^{st}}{(s^2+1)(s^2+4)}, s=i\right) &= \lim_{s\to i}\left(\frac{(s-i)se^{st}}{(s-i)(s+i)(s^2+4)}\right)\\ &= \frac{1}{6}e^{it},\end{aligned}$$

$$\begin{aligned}\mathrm{Res}\left(\frac{se^{st}}{(s^2+1)(s^2+4)}, s=-i\right) &= \lim_{s\to -i}\left(\frac{(s+i)se^{st}}{(s-i)(s+i)(s^2+4)}\right)\\ &= \frac{1}{6}e^{-it},\end{aligned}$$

and

$$\begin{aligned}\mathrm{Res}\left(\frac{se^{st}}{(s^2+1)(s^2+4)}, s=2i\right) &= \lim_{s\to 2i}\left(\frac{(s-2i)se^{st}}{(s-2i)(s+2i)(s^2+4)}\right)\\ &= -\frac{1}{6}e^{2it},\\ \mathrm{Res}\left(\frac{se^{st}}{(s^2+1)(s^2+4)}, s=-2i\right) &= \lim_{s\to -2i}\left(\frac{(s+2i)se^{st}}{(s-2i)(s+2i)(s^2+4)}\right)\\ &= -\frac{1}{6}e^{-2it}.\end{aligned}$$

As the $2\pi i$ from the Residue theorem and the $1/(2\pi i)$ from the inversion theorem cancel, we have

$$\begin{aligned}\mathcal{L}^{-1}(F(s)) &= \left(\frac{1}{6}e^{it}+\frac{1}{6}e^{-it}-\frac{1}{6}e^{2it}-\frac{1}{6}e^{-2it}\right)\\ &= \frac{1}{3}(\cos t-\cos(2t)).\end{aligned}$$

It is possible for a Laplace transform to have infinitely many poles. In this case we have to sum all the residues.

Example 6.24. Find the inverse Laplace transform of

$$F(s)=\frac{\sinh(sx)}{s^2\cosh(sa)},\ 0<x<a.$$

First note that

$$\frac{\sinh(sx)}{s^2}=\frac{x}{s}+\frac{sx^3}{3!}+\frac{s^3x^5}{5!}+\cdots$$

so that there is a pole of order 1 at $s=0$ and

$$\mathrm{Res}(e^{st}F(s),0)=x.$$

Now $\cosh(sa) = 0$ when $s = (k+\frac{1}{2})\frac{\pi i}{a}$. So there are poles at $s_k = (k+\frac{1}{2})\frac{\pi i}{a}$, $k = 0, \pm 1, \pm 2, ...$ These are poles of order 1. Now

$$\begin{aligned}\mathrm{Res}(e^{st}F(s), s_k) &= \lim_{s\to s_k}(s-s_k)e^{st}\frac{\sinh(sx)}{s^2\cosh(sa)} \\ & \lim_{s\to s_k} e^{st}\frac{\sinh(sx)}{s^2}\lim_{s\to s_k}\frac{s-s_k}{\cosh(sa)} \\ &= e^{(k+\frac{1}{2})\frac{\pi i}{a}t}\frac{\sinh((k+\frac{1}{2})\frac{\pi i}{a}x)}{((k+\frac{1}{2})\frac{\pi i}{a})^2}\frac{1}{a\sinh((k+\frac{1}{2})\pi i},\end{aligned}$$

where we used L'Hôpital's rule. This simplifies to

$$\mathrm{Res}(e^{st}F(s), s_k) = (-1)^{k+1}a\frac{e^{(k+\frac{1}{2})\frac{\pi i}{a}t}\sin((k+\frac{1}{2})\frac{\pi}{a}x)}{\pi^2(k+\frac{1}{2})^2}.$$

So if $F(s) = \mathcal{L}[f_x(t)]$ we have

$$\begin{aligned}f_x(t) &= x + \sum_{k=-\infty}^{\infty}(-1)^{k+1}a\frac{e^{(k+\frac{1}{2})\frac{\pi i}{a}t}\sin((k+\frac{1}{2})\frac{\pi}{a}x)}{\pi^2(k+\frac{1}{2})^2} \\ &= x + \frac{8a}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^2}\sin\left(\frac{(2n-1)\pi x}{2a}\right)\cos\left(\frac{(2n-1)\pi t}{2a}\right).\end{aligned}$$

Using the Weierstrass M test we can prove that this series converges uniformly to a continuous function in both x and t. It is quite common for an inverse Laplace transform to be computed as an infinite series.

It must be cautioned that the inversion of Laplace transforms can be a difficult problem. There are numerous examples where the method used here is not sufficient to invert a transform, often because calculating the poles and residues of certain functions can be quite difficult. Laplace transform inversion is typically carried out with the aid of tables of transforms. Numerical integration is also used. Residues can play a role here as well, but that is not something that we can discuss here.

7. Appendix: Cardano and the Cubic

A formula for the roots of a cubic in terms of its coefficients was first published in 1545 in a book by the Italian physician and mathematician Girolamo Cardano(1501-1576), entitled *Artis magnae sive de regulis algebraicis liber unus (The Great Art or the First Book about Regular Algebra)*. This is often called the *Ars Magna* for short. Cardano was the illegitimate son of Fazio Cardano and Chiara Micheria. Fazio Cardano was a Milanese lawyer and an accomplished mathematician, as well as a friend of Leonardo da Vinci. Leonardo often consulted Fazio about problems in geometry. Indeed the elder Cardano actually taught geometry at the University of Pavia, where the junior Cardano later studied. The son seems to have learned a good deal of mathematics from his father. Girolamo Cardano did interesting(if not entirely correct) work in mechanics and hydrodynamics. He also appears to be the first person to have studied what we would call probability theory.

The formula which Tartaglia gave to Cardano was for the solution of a so called *depressed cubic*, which we studied at the beginning of these notes. One of the many strange features of this story is that Tartaglia delivered the formula to Cardano in the form of a poem, rather than as a straight formula.

Cardano had tried to solve the cubic himself and failed. Tartaglia, however, had been using his solution of the cubic to win competitions with fellow mathematicians. It may seem bizarre, but in the Renaissance, it was common for one mathematician to challenge another to a problem solving contest, with prize money on the outcome. And people followed these contests.

Becoming aware that Tartaglia had the formula Cardano tried to persuade him to reveal the secret. However every attempt was met with refusal. It was not until Cardano told Tartaglia that his name had been mentioned to Alfonso d'Avalos, the Marchese del Vasto, who was Cardano's patron and a military governor for the emperor in Milan, that Tartaglia changed his tune. Cardano offered to provide Tartaglia with an introduction to the Marchese. Since Tartaglia was a poorly paid mathematics teacher looking for a way to move up in the world, this was too good an opportunity to pass up.

So Tartaglia travelled to Milan in March 1539 and stayed in Cardano's house. Cardano was the perfect host, tending to Tartaglia's every need, while all the time trying to extract the secret of the cubic. When Tartaglia finally relented and gave Cardano the method, he delivered it in the form of a poem. This was really a kind of code, designed to protect the secret from prying eyes.

Unfortunately for Tartaglia, the Marchese was away and Tartaglia never got to meet him. Cardano did provide a letter of introduction for Tartaglia, but by the end of the trip, Tartaglia was regretting the whole business and was becoming very suspicious of his host.

The formula was delivered on the condition that Cardano would keep it a secret until Tartaglia published it himself. With the method in hand, Cardano and his student Lodivico Ferrari (1522-1565) set to work studying and extending it and in 1540 Ferrari discovered a method for solving the general *quartic*! As we have seen, the method of solving the quartic actually depended on the solution of the cubic, however, and so Ferrari and Cardano could not publish this great discovery.

However Cardano, after duly keeping the secret for several years, grew tired of waiting and included a formula for the cubic in his master work, the *Ars Magna*. Why did Cardano break his oath? Cardano argued that he didn't break the oath at all. Because Cardano and Ferrari had discovered a rather interesting fact from a very obscure mathematician named Hannibal della Nave.

Tartaglia was furious with Cardano, even though the later fully acknowledges in his writings that Tartaglia had solved the cubic and told him the secret. However Cardano also tells us in the book that Tartaglia didn't discover the cubic formula first. Tartaglia had actually learned of the existence of a method for the depressed cubic from Antonio Maria Fior around the year 1535. As it happens Fior, who is usually described as a fairly mediocre mathematician by historians of mathematics, didn't discover it either. He was a student of Scipione del Ferro (ca. 1465-1526), who appears to be the true discoverer of the formula. Scipione del Ferro was the holder of the Chair of Arithmetic and Geometry at the University of Bologna from 1496 onwards. From our imperfect knowledge he appears to have discovered the method for finding roots of the equation $x^3 + bx = c$. This is an example of a so called depressed cubic. In his book, Cardano gives equal credit to del Ferro for the discovery. In the Ars Magna he writes:

Scipione Ferro of Bologna, almost thirty years ago, discovered the solution of the cube and things equal to a number [that is, the solution of the depressed cubic $x^3 + bx = c$], a really beautiful and admirable accomplishment. In distinction this discovery surpasses all mortal ingenuity, and all human subtlety. It is truly a gift from heaven, although at the same time a proof of the power of reason, and so illustrious that whoever attains it may believe himself capable of solving any problem.

Historians argue that because del Ferro didn't accept negative numbers, he would not have understood that he could use the formula to solve an equation like $x^3 = -bx + c$, because the negative coefficient would have confused him. However, at least one historian, Bortolotti, argues that del Ferro could solve both $x^3 + bx = c$ and $x^3 = bx + c$.

It appears strange to us that mathematicians at the time had such a problem with negative numbers. The reason is that they were still thinking of numbers geometrically. A negative length had no meaning. The idea that a number could also indicate a direction did not occur till much later. Thus an equation we might write as $x^2 - 2x + 4 = 0$ they would write as $x^2 + 4 = 2x$. In antiquity things were worse, because mathematicians would think of this equation in words, rather than symbols. Thus they would seek a number which, when four is added to its square yields twice the original number. This cumbersome way of thinking made what to us is elementary mathematics, extremely hard. And the rejection of negative quantities meant that instead of considering one single case for a polynomial, mathematicians had to solve all sorts of different special cases. Thus $x^3 + bx = c$ was regarded as quite different from $x^3 = bx + c$, whereas to us they are exactly the same. Moreover, zero was not in widespread use. Mathematics was held back for centuries by this basic lack of understanding of what are, to us at least, relatively simple concepts.

Whether or not del Ferro understood that his method really solved all cubics or not, he passed the secret onto Fior and several other people, including Hannibal della Nave. Fior was quite happy for people to know that he had it. How much he divulged about the method while making his boasts is unclear. It is unlikely that del Fior really understood the method well. Some years earlier del Fior had challenged Tartaglia to a public contest solving cubics and lost badly. There were thirty equations to solve and the contestants had forty days to do them. Tartaglia did the lot in two hours. If del Ferro really understood how the method worked, he should also have been able to solve the problems easily.

Yes, there really was a public contest solving cubic equations. It is hard to imagine a public contest these days with two mathematicians competing against one another to see who could solve the most equations in a given time, though it would probably be better than most reality tv. At the time though, these 'debates' were very common and a way for a scholar to establish a public reputation. There was also often money involved. Tartaglia was very experienced in this format.

The reason for the debate was that Tartaglia had actually discovered a method for solving the cubic $x^3 + bx^2 = c$ and made this fact known, though not the method. Fior thought that Tartaglia was bluffing and so issued the challenge, which Tartaglia accepted. Working furiously, Tartaglia managed to discover the method of solving $x^3 = bx+c$, (which was what del Fiore was supposed to know how to do) just before the contest. How much of Ferro's method did Tartaglia glean from del Fiore before the contest? We don't actually know. Tartaglia may well have made the discovery entirely independently. Or, he may have picked up vital clues from del Fiore.

Whatever the truth is, mathematics cannot be kept secret. Over the history of mathematics when an idea is in the air, sooner or later more than one mathematician will discover it. The reason is that important discoveries don't come from nowhere. They develop out of what is already known. So that if one person can take pre-existing knowledge and push it in some new direction, then so can someone else. In the history of mathematics it is not unusual for the same results to be discovered more or less independently by different people around the same time. This can however lead to some childishly silly disputes.

For example, Sir Isaac Newton (1642-1727)insisted that Gottfried Wilhelm von Leibniz (1646-1716) had stolen calculus from him, and Leibniz in turn accused Newton of the same. It seems however that they discovered it largely independently, though Newton did it first. Leibniz, however, published it first. Both men benefited from the fact that the ground work for the invention of calculus had already been done by others. Years before Newton, Pierre de Fermat (1607-1665) could essentially differentiate a polynomial and both René Descartes (1596-1650) and Fermat had methods for calculating tangents to curves, which itself lead to a bitter dispute between Fermat and Descartes. Fermat's method based on something similar to differentiation was quite superior to Descartes' clumsy method of trying to make a circle touch the curve at the desired point and use the fact that the tangent to a circle can be computed. Indeed in 1659 the Dutch mathematician Johann Hudde used Fermat's ideas to show that if $y = \sum_{k=0}^{n} a_k x^k$, then the extreme points occur where $\sum_{k=1}^{n} k a_k x^{k-1} = 0$. That is, where $y' = 0$, though they did not think in terms of the modern derivative concept. Despite the superiority of Fermat's ideas, the mathematician Girard Desargues (1591-1661) was called to artbitrate this dispute and his judgment is a famous masterpiece of diplomacy: "Monsieur Descartes is right and Monsieur Fermat is not wrong."

As for integration, John Wallis (1616-1703) could effectively integrate powers of x and so find areas under curves bounded by polynomials. Therefore, someone was bound to discover calculus proper sooner or later. That is, see the link between the problem of finding areas and the problem of finding tangents to curves. That person turned out to be Newton, closely followed by Leibniz. For the record, we use Leibniz's name for the subject (calculus) as well as his notation for integrals and derivatives. Newton actually called Calculus, 'the method of fluents and fluxions.' Newton's genius was for mathematics and physics, not the creation of catchy names.

Returning to the cubic, it was from Tartaglia's victory in his contest with del Fiore that Cardano learned that Tartaglia had solved the problem of the cubic, or at least several special cases of it. This motivated Cardano to try and solve the cubic himself, but he failed and so asked Tartaglia for the secret, which is where we came in.

In 1543, Cardano and Ferrari met Hannibal della Nave, who just happened to be del Ferro's son in law and he showed them del Ferro's notebook with the secret of the depressed cubic in it. Knowing that Tartaglia was not the original discoverer of the method, Cardano decided that he was no longer obliged to keep Tartaglia happy. What he did however, was publish del Ferro's methods, not the method as Tartaglia gave it to him. Technically he argued, he had kept his oath.

Cardano' argument, that he was presenting del-Ferro's method, did not impress Tartaglia. So incensed was Tartaglia that he even tried to persuade Cardano that the formula he had been given was wrong. Since it plainly is not wrong, (one can easily check that it works by trying it out), this shows a considerable desperation. For the rest of his life, Tartaglia nursed a deep hatred for Cardano and wrote a book giving his version of events in which he heaped abuse on Cardano. Since Tartaglia was not well known and Cardano was Europe's leading mathematician, Cardano was not much bothered by this. However his student Ferarri defended Cardano in the dispute with Tartaglia and challenged him to a contest similar to the one with Fior. Tartaglia wanted to debate Cardano, but was ignored. Eventually Tartaglia did debate Ferrari, though only when he was forced to as part of his ambitions to take up a university lectureship. This time Tartaglia was soundly beaten and the defeat seemed to break him. He died nine years later. Never before or since has mathematics been such a brutal business.

Cardano it must be said, was a very good mathematician in his own right and he did make a major contribution to the problem of the cubic. What Cardano did was show how to extend the methods of Tartaglia-del Ferro to all cubics. That is, he showed how to solve the equation that we would write as $x^3 + bx^2 + cx + d = 0$ for any choice of b, c, d. Since he was not including negative coefficients, he had to consider a whole range of special cases to pull this off. At the time it was an important achievement in its own right.

Some historians still claim that Cardano was not above pinching other people's results. Because his student Ferrari had derived a formula for the roots of a quartic equation, which Cardano published in the *Ars Magna*. Some have claimed that this was a form of theft. However it is likely that Cardano made a real contribution to the solution of the quartic too. It is also true that Cardano credits Ferrari with the solution of the quartic.

The solution of the quartic made a name for Ferrari, but he did not live long enough to enjoy it. Leaving mathematics, he became a tax inspector, which was considerably more lucrative. When he retired at a young age as a very wealthy man, he appeared to be moving back into mathematics, but died in 1565 at the age of 43. It is said that he was killed by white arsenic, delivered by his sister. Such was life in the Italian Renaissance. This story brings to mind a famous line from the movie, 'The Third Man', where Harry Lime (played by Orson Welles) tells Holly Martin (Joseph Cotten) that

"You know what the fellow said–in Italy, for thirty years under the Borgias, they had warfare, terror, murder and bloodshed, but they produced Michelangelo, Leonardo da Vinci and the Renaissance. In Switzerland, they had brotherly love, they had five hundred years of democracy and peace - and what did that produce? The cuckoo clock."

Today del Ferro's formula is usually, somewhat bizarrely, referred to as Cardano's formula. Many famous results in mathematics end up being named after people who did not actually discover them. (L'Hôpital's rule is due to Johann Bernoulli, Stokes' Theorem is due to Lord Kelvin, Cotes discovered Euler's formula decades before Euler, Taylor series were known to Gregory 45 years earlier,.....)

The history of mathematics is a fascinating subject. For those wanting to know more, Stillwell's book is a good starting point, [6].

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