Complex Analysis 37234: Tutorial 1. Basic revision

1. Show that the function $f(x) = |x-1|$ has a discontinuous first derivative at $x = 1$.

2. Use L'Hopital's rule to evaluate
   (a) $\lim_{x \to 0} \dfrac{\tan x}{2x}$

   (b) $\lim_{x \to -2} \dfrac{x+2}{x^3+8}$.

3. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for the functions
   (a) $f(x, y) = x^2 + y^2$

   (b) $f(x, y) = x^3 y$

   (c) $f(x, y) = x \ln(xy^2)$

   (d) $f(x, y) = e^{-x^2-y^2}$.

   (e) $f(x, y) = \sin(xy^2)$.

4. Write $1/i$ in the form $re^{i\theta}$. Indicate where the point lies in the complex plane.

5. Plot the following sets of points in the complex plane.
   (a) $|z| = 2$.

   (b) $|z+1| = 1$.

   (c) $|z-2i| \le 1$.

   (d) $\mathrm{Arg} z = 2\pi/3$.

   (e) The set of points for which $|z-2| > 1$ and $|z-2| < 3$.

6. Use Euler's formula to show that
   $$\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$$
   $$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz}).$$
   Verify that $\cos^2 z + \sin^2 z = 1$.

7. Find all solutions of the equation $z^3 = 1 + i$.

Complex Analysis 37234: Tutorial 2

(1) Show that the zeroes of $\sin z$ and $\cos z$ are all real.

(2) Show that the series expansions for e^z, $\sin z$ and $\cos z$ converge for all $z \in \mathbb{C}$.

(3) Let
$$\tanh z = \frac{\sinh z}{\cosh z}.$$
Find a formula for $\tanh^{-1} z$ in terms of the natural logarithm.

(4) Find all possible values of $\cosh^{-1}(i)$.

(5) Find all solutions of $z^6 = 1$.

(6) Use the polar form of a complex number to find $(2 - 2i)^{1/4}$.

(7) Show that
$$\cosh^2 z - \sinh^2 z = 1.$$

(8) Show, for all complex numbers z with imaginary part in the range $(-\pi, \pi]$, that
$$\operatorname{Log}(e^z) = z\ .$$

(9) Evaluate $i^{i/2}$

(10) Use Euler' formula to sum the series
$$\sum_{k=1}^{n} \sin(k\theta) \quad \text{and} \quad \sum_{k=1}^{n} \cos(k\theta).$$
(Hint: Evaluate the geometric sum $\sum_{k=1}^{n} e^{ik\theta}$).

(11) Calculate the Real part of
$$f(\theta) = \frac{1 + i\tan(\theta/2)}{1 - i\tan(\theta/2)},$$
where $\theta \in (-\pi/2, \pi/2)$.

(12) If $z = x + iy$ find an expression for the real part of $\sin^{-1}(z)$.

Complex Analysis 37234: Tutorial 3

(1) Plot the following curves in the complex plane:

(a) $z(t) = 2t + 1 + i(t-1)$ with $t \in [0, 2]$,

(b) $z(t) = t + 2i$ with $t \in [0, \infty]$,

(c) $z(t) = \cos t + i \sin t$ with $t \in [0, 2\pi/3]$,

(d) $z(t) = te^{i\pi/4}$ with $t \in [0, 5]$.

(2) Find a parametrization of the straight line joining the points :
a) 0 and $4 + i$,
b) i and $3 + i$,
c) $1 - 4i$ and $1 + 4i$.

(3) Find a parameterisation of the following curves:

(a) A circle of radius 2, centred at 3, traversed in the *anticlockwise* direction.

(b) A quarter circle, centred at $z_0 = 1 + i$, starting at $z_1 = 1$ and ending at $z_2 = i$.

(4) Prove that if a complex valued function is differentiable at z_0, then it is also continuous as z_0. (Hint: f is continuous at z_0 if for every sequence $z_n \to z_0$, $f(z_n) \to f(z_0)$. Now look at the definition of the derivative).

(5) Is $f(z) = \bar{z}^2$ differentiable at any point?

(6) Find the real constants a, b, c and d such that the function

$$f(x + iy) = x^2 + axy + by^2 + i(cx^2 + dxy + y^2)$$

is differentiable for all x and y.

(7) A function which is differentiable in the entire complex plane is said to be an *entire* function. Prove that $f(z) = e^{-z^2}$ is entire.

(8) Let $u(x, y) = x^3 - 3xy^2$. Does there exist $v(x, y)$ such that $f(x + iy) = u(x, y) + iv(x, y)$ is differentiable? If there does, find v.

Complex Analysis 37234: Tutorial 4

(1) Given the function $f(z) = u(x, y) + iv(x, y)$, with $z = x + iy$, with

$$u(x, y) = 2x + y$$

find a function $v(x, y)$ such that f is entire. (That, is differentiable everywhere).

(2) Show directly that $\cos z$ is not a bounded function when $z \in \mathbb{C}$.

(3) Show that real part of the function

$$f(z) = e^{iz^2}$$

is *harmonic*. That is, it satisfies Laplace's equation

$$u_{xx} + u_{yy} = 0.$$

(4) Show that $f(z) = ze^{iz}$ and $f(z) = z^3$ are entire functions. Show also that their real and imaginary parts are harmonic.

(5) Let $f(z) = 2z + z^2$ and $\gamma(t) = t + it^2$, $t \in [0, 1]$. Evaluate $\displaystyle\int_\gamma f(z)dz$.

(6) Let $f(z) = z^3$ and $\gamma(t) = e^{it}$, $t \in [0, 2\pi)$. Evaluate $\displaystyle\int_\gamma f(z)dz$.

(7) Evaluate the double integral

$$\int_D (x^4 + y^2)dA,$$

where D is the region bounded by $y = x^3$ and $y = x^2$. Show that reversing the order of integration does not change the result.

Complex Analysis 37234: Tutorial 5

(1) Evaluate
$$\oint_C (\sin^3 x + 3y)dx + (x^2 - \frac{1}{\ln(y+3)})dy \ ,$$
where C is the path given by $(x+2)^2 + (y-2)^2 = 4$, traversed in the anti-clockwise direction.

(2) Evaluate
$$\oint_C 2y^4 dx + 4xy^3 dy$$
where C is the path along the triangle with vertices at $(0,0)$, $(1,3)$, and $(0,3)$, traversed in the anti-clockwise direction.

(3) Evaluate
$$\int_C |z|^2 dz$$
where the contour C joins the points $z_0 = 1$ and $z_1 = 2+i$, and is a straight line.

(4) Evaluate
$$\int_C \frac{1}{z-3-i} dz$$
where C is the circle of radius 3 in the complex plane, centred at $z_0 = 3+i$ and traversed anti-clockwise.

(5) Show that, when C is a circle of any radius in the complex plane,
$$\int_C \frac{1}{z^2} dz = 0 \ .$$

(6) Evaluate the integral
$$\int_0^1 \int_{2y}^2 e^{x^2} dx dy$$
by reversing the order of integration.

(7) Use polar coordinates to evaluate the integral
$$\int\int_R \frac{1}{x^2+y^2} dx dy$$
where R is the region within the ring $a^2 \leq x^2 + y^2 \leq b^2$.

(8) Use polar coordinates to evaluate the integral
$$\int_{-\infty}^{\infty} e^{-x^2} dx$$

(9) Let $f : D \to \mathbb{C}$ be a differentiable function. Fix $z \in D$ and define
$$F(\xi) = \frac{f(\xi) - f(z)}{\xi - z}.$$

Let C_R be a circle or radius R centered at z and S_ϵ be a circle of radius $\epsilon > 0$ centered at z. Show that

$$\int_{C_R} F(\xi) d\xi = \int_{S_\epsilon} F(\xi) d\xi.$$

Complex Analysis 37234: Tutorial 6

(1) Use Green's theorem in the plane to evaluate

$$\oint_C (\log \frac{1}{\sqrt{x}} + 3y^2)dx + (x - \frac{\cos y}{y^4+1})dy$$

where C is the path given by equation $(x-1)^2 + (y-2)^2 = 25$, traversed in the anti-clockwise direction.

(2) By integrating explicitly in the complex plane, evaluate

$$\int_C (z^3 + iz)dz$$

where C is the quartercircle centered at the origin, starting at 2 and ending at $2i$. Show that the Fundamental Theorem of Calculus holds for this example.

(3) Evaluate

$$\int_C \frac{1}{z-i}dz$$

where C is the path joining the points -1,2,$2+i$,$-3+2i$ in the positive sense. i.e. Counterclockwise.

(4) Evaluate

$$\int_C \frac{\ln(z-i)}{z+i}dz$$

where C is the circle described by $|z-2| = 2$ and traversed in the positive sense. The principle value of the logarithm is used here.

(5) Let $u(x,y)$ be a continuous function with continuous first and second partial derivatives on a positively-oriented smooth closed path C and throughout the interior of D. Use Green's theorem in the plane to show that

$$\int_D \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] dxdy = \oint_C \left(-\frac{\partial u}{\partial y}\right) dx + \frac{\partial u}{\partial x}dy.$$

(6) Let F, G be continuous and have continuous first partial derivatives in a simply connected region R bounded by C. With C traversed counterclockwise, show that

$$\oint_C F\left(\frac{\partial G}{\partial y}dx - \frac{\partial G}{\partial x}dy\right) = -\int\int_R \left[F\Delta G + \nabla F \cdot \nabla G\right] dxdy.$$

Here $\Delta G = G_{xx} + G_{yy}$ and $\nabla G = G_x\mathbf{i} + G_y\mathbf{j}$ and similarly for ∇F.

(7) Evaluate

$$\oint_C \frac{z^3-6}{2z-i}dz$$

where C is any anticlockwise-oriented contour enclosing the point $z_0 = i/2$.

(8) Evaluate

$$\int_C \frac{2z^3}{(z-2)^2}dz$$

where C is the positively oriented rectangle with vertices $4 \pm i$ and $-4 \pm i$.

(9) Using Cauchy's integral formula, evaluate the integral

$$\int_C \frac{e^z}{z}dz$$

where C is the unit circle around the origin. Then, by integrating the above integral directly in the complex plane, evaluate the real integral

$$\int_0^{2\pi} e^{\cos\theta}\cos(\sin\theta)d\theta \ .$$

(10) Prove that

$$\left|\int_C \frac{e^{iz}}{z^3+z^2}dz\right| \leq \frac{2\pi}{R}$$

where C is the circle in the upper half-plane with radius $R > 2$, starting at R and ending at $-R$.

Complex Analysis 37234: Tutorial 7

(1) Use the Cauchy integral formula to evaluate the integrals

(a)

$$\int_{C_R} \frac{e^{3z}}{(z-1)^2} dz$$

where C_R is the circle of radius R centered at $z = 1$.

(b)

$$\int_{C_R} \frac{e^{3z}}{(z^2-1)} dz$$

where C_R is the circle of radius R centered at $z = 1$.

(2) Find the Taylor series expansion about $z = 0$ for the following functions, expressing the answer in series notation (i.e. in terms of a sum $\sum_{n=0}^{\infty} c_n z^n$):

(a) e^{2z}

(b) $\sin 3z$

(c) $(z+1)\cos z$

(3) Find the Laurent expansion about $z = 0$ of $f(z) = \dfrac{\cosh z}{z^3}$ and determine the residue at $z = 0$.

(4) Find the Taylor series expansion about $z = 0$ for

$$f(z) = \frac{1}{2-z}$$

What is the domain of convergence for this series?

(5) Find the Taylor series expansion, centred at $z = 0$, for

$$f(z) = \frac{z}{1-z}$$

Using the root test for series, show that the domain of convergence of this series is $|z| < 1$.

(6) Find a Laurent series expansion for $f(z) = z^{-2}e^z$, valid for the annular domain $|z| > 0$.

(7) Expand

$$f(z) = \frac{z}{z^2 - z - 2}$$

in a Laurent series valid for the domain

a) $0 < |z+1| < 3$

b) $1 < |z| < 2$.

(8) Expand the function

$$f(z) = \frac{2z}{1+z^2}$$

in a Laurent series about the point $z_0 = i$, and find the residue at this point.

(9) Find the Laurent series for

$$f(z) = \frac{\cos(3z)\sin(2z)}{z^3}$$

around $z = 0$ and determine the residue at this point.

Complex Analysis 37234: Tutorial 8

(1) For the function

$$f(z) = \frac{1}{(z-5)(z-3)}$$

a) Find a Laurent series expansion for $f(z)$ valid in the domain $0 < |z-3| < 2$.
b) What is the residue of f at $z = 3$?

(2) For the function

$$f(z) = \frac{1}{(z-4)(z+1)}$$

a) Find a Laurent series expansion for $f(z)$ valid in the domain $0 < |z+1| < 5$.
b) What is the residue of f at $z = -1$?
c) Hence or otherwise evaluate $\oint_C f(z)dz$, where C is the positively-oriented circular path $|z+1| = 2$.

(3) Use the substitution $z = e^{i\theta}$ to show that

(a)

$$\int_0^{2\pi} \frac{d\theta}{(5 - 3\sin\theta)^2} = \frac{5\pi}{32}.$$

(b)

$$\int_0^{2\pi} \frac{d\theta}{3 + \cos\theta - 2\sin\theta} = \pi.$$

(c)

$$\int_0^{2\pi} \frac{\cos(3\theta)}{5 - 4\cos\theta} = \frac{\pi}{12}.$$

(4) Use residues to evaluate the following integrals

(a)

$$\int_0^{\infty} \frac{\cos(2x)}{4 + x^2} dx.$$

(b)

$$\int_{-\infty}^{\infty} \frac{\cos(2x)dx}{(x^2+1)(x^2+4)(x^2+9)}.$$

(c)

$$\int_0^{\infty} \frac{\cos x}{1 + x^4} dx.$$

(d)

$$\int_0^{\infty} \frac{dx}{x^5 + 1}.$$

Complex Analysis 37234: Tutorial 9

(1) Evaluate the integral
$$\int_0^\infty \frac{x^{1/4}}{x^2+16}dx.$$

(2) Calculate
$$\int_0^\infty \frac{x}{x^5+1}dx.$$

(3) Suppose that f is analytic. In terms of f, obtain

(a) $\int_0^{2\pi} f(e^{i\theta})d\theta$

(b) $\int_0^{2\pi} f(e^{i\theta})\cos\theta d\theta$

(4) Evaluate the infinite series

(a) $\displaystyle\sum_{n=-\infty}^{\infty} \frac{1}{(n^2+1)^2}.$

(b) $\displaystyle\sum_{n=-\infty}^{\infty} \frac{1}{n^4+4}.$

(5) Use residues to invert the Laplace Transforms

(a) $F(s) = \dfrac{s^2-1}{(s^2+4)(s^2+9)}.$

(b) $F(s) = \dfrac{1}{s(e^s+1)}.$

Note that in (b) there are infinitely many poles or order 1.