Advanced Calculus 35232: Basic Skills Test 1

Time Allowed: 20 minutes - NO calculators may be used.

(1) Evaluate the derivatives of the following functions:

a) $f(x) = x^2 + 2\sqrt{x} + 1$

b) $f(x) = \sin(2x)$

c) $f(x) = \cos(2x^3)$

d) $f(x) = \ln(\cos 3x))$

(2) Evaluate the following definite integrals:

a)

$$\int_1^2 \left(x^2 + \frac{1}{x^2}\right) \mathrm{d}x$$

b)

$$\int_0^{\pi/2} \cos x \sin^3 x \mathrm{d}x$$

(3) Evaluate

$$\lim_{x\to\pi} \frac{\sin x}{\pi - x} \; .$$

Is the function $f(x) = \frac{\sin x}{\pi - x}$ continuous at $x = \pi$? Why or why not?

(4) Given the scalar function $f(x, y, z) = 3x^3 + 2zy^2 + xyz^2$ calculate the partial derivatives

a) $\frac{\partial f}{\partial x}$

b) $\frac{\partial f}{\partial y}$

(5) Write $\frac{2-4i}{3+5i}$ in the form $x + iy$.

(6) Write $-2 + i$ in the form $|z|e^{i\mathrm{Arg}(z)}$ and plot the point in the complex plane.

Advanced Calculus 35232: Basic Skills Test 2

Time Allowed: 20 minutes - NO calculators may be used.

(1) Find all cube roots of $-i$.

(2) Show that $1/i = -i$.

(3) Plot the curve $|z - 2 - i| = 3$ in the complex plane.

(4) Plot the region $|z - i + 1| > \frac{1}{2}$ in the complex plane.

(5) Find all solutions to the equation

$$z^6 + 2z^3 + 1 = 0.$$

Hint: This is really a quadratic in disguise.

(6) Plot the path $z(t) = 2e^{-2i\pi t}$, with $t \in [0, \frac{3}{8})$, in the complex plane.

(7) Let $\omega = e^{2\pi i/5}$. Explain why

$$1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0.$$

Hint:

$$z^5 - 1 = (z - 1)(1 + z + z^2 + z^3 + z^4).$$

Advanced Calculus 35232: Basic Skills Test 3

Time Allowed: 30 minutes - NO calculators may be used.

(1) Plot the following curves in the complex plane:

a)

$$z(t) = i + e^{-2it} \text{ for } t \in [0, \pi/2]$$

b)

$$z(t) = x(t) + iy(t) \text{ where } x(t) = t,\ y(t) = -t^2, \text{ for } t \in [0, 2]$$

(2) Find a parametrization of the curve joining $1 - i$ to $1 + i$.

(3) Find a parametrization of the circle with radius 2 centred at the point $z_0 = 2 - i$.

(4) From the definition of the complex logarithm, evaluate

$$\ln\left[(1 - i)^3\right]$$

Here we mean the principle value.

(5) Write the function

$$f(z) = \cos 2z$$

in the form $f = u + iv$, with u and v real functions. Show that f is an entire function.

Advanced Calculus 35232: Basic Skills Test 4

Time Allowed: 30 minutes - NO calculators may be used.

(1) Show that the function

$$f(z) = 2z + z^2$$

is entire.

(2) Show that the function $f(z) = |z|^2$ is *not* analytic.

(3) In what domain, if any, is the function

$$f(z) = z - \bar{z}$$

analytic?

(4) Identify the points where the following functions are *not* analytic:

(a)

$$f(z) = \frac{1}{z + 2i}$$

(b)

$$f(z) = \frac{1}{z^3 - 27}$$

(5) Given two entire complex functions $f(z)$ and $g(z)$, show that the linear combination

$$h(z) = af(z) - bg(z),$$

where a, b are constant, is also entire.

Advanced Calculus 35232: Basic Skills Test 5

Time Allowed: 30 minutes - NO calculators may be used.

(1) Evaluate

$$\int\int (x-y)dxdy$$

where R is the region bounded by the rectangle joining the points (1,0), (3,0), (3,6) and (1,6).

(2) Use polar coordinates to evaluate the integral

$$\int\int_A (9-x^2-y^2)dxdy,$$

where A is the region in the *first* quadrant satisfying $x^2+y^2 \leq 4$.

(3) Show that the real part of the function

$$f(z) = \ln(1+z)$$

is *harmonic* except when $z=-1$. Here the log is the principle value.

(4) Use Green's Theorem to calculate

$$\oint_C e^x \cos y dx + e^x \sin y dy$$

where C is the boundary of the triangle with vertices at $(0,0)$, $(1,0)$ and $(0,1)$.

Advanced Calculus 35232: Basic Skills Test 6

Time Allowed: 25 minutes - NO calculators may be used.

(1) Let f be a differentiable function and C_R the boundary of the circle of radius R centered at z. Write down the integral

$$\frac{1}{2\pi i}\int_{C_R}\frac{f(\xi)}{\xi - z}d\xi$$

in parametric form. For the curve C_R take $\gamma(t) = z + Re^{it}, t \in [0, 2\pi)$.

(2) Explicitly evaluate the integral in (1) in the case when $f(z) = z$ and $f(z) = z^2$.

(3) What would the integral in (1) give if we take $f(z) = e^{-z^2}$? Explain your answer without attempting to evaluate the integral.

(4) With $f(z) = z^2$ what is the value of

$$\frac{1}{2\pi i}\int_{C_R}\frac{f(\xi)}{(\xi - z)^2}d\xi?$$

Once more, for the curve C_R take $\gamma(t) = z + Re^{it}, t \in [0, 2\pi)$.

Advanced Calculus 35232: Basic Skills Test 7

Time Allowed: 25 minutes - NO calculators may be used.

(1) Use the Cauchy Integral formula to evaluate the integral

$$\int_{\gamma_R} \frac{e^z dz}{z - i}$$

where γ_R is the circle of radius $R > 0$ centered at $z = i$.

(2) Use the Cauchy Integral formula to evaluate the integral

$$\int_{\gamma_R} \frac{\sin(z) dz}{(z - i)^2}$$

where γ_R is the circle of radius $R > 0$ centered at $z = i$.

(3) Find the first three nonzero terms of the Taylor series expansion of $f(z) = z^2 \sin(z)$ about the point $z = 0$.

(4) Identify the singularities of the function

$$f(z) = \frac{\cos(z)}{\sin(z)}.$$

(5) Identify the order of the pole at $z = 0$ of the function

$$f(z) = \frac{\cos(z)}{z^3}.$$

What is the residue of f at $z = 0$?

Advanced Calculus 35232: Basic Skills Test 8

Time Allowed: 25 minutes .

(1) Given that $\ln(1+z) = \int_0^z \frac{dt}{1+t}$ and $\frac{1}{1+t} = 1 - t + t^2 - t^3 + \cdots$, determine a Laurent series for the function

$$f(z) = \frac{\ln(1+z)}{z^3}.$$

(2) Determine the poles and residues of the function

$$f(z) = \frac{e^{iz}}{(z^2+a^2)(z^2+b^2)}.$$

(3) Suppose $|a| > |b|$. Determine the value of the integral

$$\int_{\gamma_R} \frac{e^{iz}}{(z^2+a^2)(z^2+b^2)} dz$$

where $\gamma_R(t) = Re^{it}$, $t \in [0, 2\pi)$ and $0 < R < |a|$.

(4) Determine the value of the integral

$$\int_{\gamma_R} \frac{e^{iz}}{(z^2+a^2)(z^2+b^2)} dz$$

where $\gamma_R(t) = Re^{it}$, $t \in [0, 2\pi)$ and $R > |a|$ and also $R > |b|$.

Advanced Calculus 35232: Basic Skills Test 9

Time Allowed: 25 minutes .

(1) (a) Let $z = e^{i\theta}$. Express $\cos\theta$ in terms of z and $1/z$. Also express $d\theta$ in terms of dz.

(b) Use the results for (a) to show that

$$\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \frac{2}{i}\int_C \frac{dz}{1+4z+z^2}$$

where C is the circle of radius 1 centered at 0.

(c) Use the residue theorem to show that

$$\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \frac{2\pi}{\sqrt{3}}.$$

(2) (a) Identify the poles of

$$f(z) = \frac{e^{iz}}{(z^2+4)(z^2+9)}.$$

(b) Calculate the residues at the poles in the upper half plane.

(c) Evaluate the integral

$$\int_0^{\infty} \frac{\cos x dx}{(x^2+4)(x^2+9)}.$$