Question One.

Total marks: 5+5+5+5=20.

(a) Obtain all solutions of the equation $z^3 + 1 = 0$.

(b) Let $z = x + iy$. Obtain the real and imaginary parts of the function

$$f(z) = \frac{1}{1+z}.$$

(c) Let $f(x+iy) = x^2 - y^2 + iv(x, y)$. Determine a function $v$ such that $f$ is differentiable in the whole complex plane. Express $f$ as a function of a single complex variable $z$.

(d) Let $f(x+iy) = u(x, y) + ic$ be analytic, where $c$ is a constant. Show that $u$ must also be a constant. (Hint: Use the Cauchy-Riemann equations).

Question Two.

Total marks: 6+8+6=20.

(a) Let $f(z) = z^2$ and $\gamma(t) = 1 + it^3, t \in [0, 1]$.

(i) Write out the contour integral $\int_\gamma f(z)dz$ as an integral with respect to $t$. You do not need to evaluate this integral.

(ii) Evaluate the integral

$$\int_1^{1+i} z^2 dz.$$

(iii) What is the relationship between the integrals in (i) and (ii)? Give reasons for your answer.

(b) Use Green's Theorem to evaluate the integral

$$\oint_C (5 + 10xy + y^2)dx + (6xy + 5y^2)dy$$

where $C$ is the rectangle with vertices $(0, 0)$, $(0, a)$, $(b, a)$ and $(b, 0)$ traversed counterclockwise.

(c) Consider the contour $\gamma_R(t) = Re^{it}$, $t \in [0, \pi]$. Show that

$$\int_{\gamma_R} \frac{dz}{z^2 + 4} \to 0,$$

as $R \to \infty$.

Question Three.

Total marks: 4+5+6+5=20.

(a) Let $\gamma$ be a closed, simple contour in the complex plane. What is the value of the contour integral

$$\int_{\gamma} (z^3 + 3z^2 + 4z + 5) dz \ ?$$

Justify your answer.

(b) Evaluate the contour integral

$$\int_C \frac{e^{2z}}{z-5} dz,$$

where $C$ is the circle of radius 2 centered at $z = 5$, traversed counterclockwise.

(c) Evaluate the contour integral

$$\frac{1}{2\pi i} \int_C \frac{e^{2z}}{(z+3)^2} dz,$$

where $C$ is the circle of radius 5 centered at the origin, traversed counterclockwise.

(d) Let $f(z)$ be an entire function. Show that

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(z + Re^{it}) dt,$$

for every $R > 0$. (Hint: Cauchy integral formula).

Question Four.

Total marks: 6+7+7=20.

(a) Obtain a Laurent series expansion for the function

$$f(z) = \frac{\cos(2z)}{4z^4}$$

around $z = 0$. What is the order of the pole at $z = 0$? What is the value of the residue at the pole?

(b) Obtain a Laurent series expansion for

$$f(z) = \frac{z}{4z - 3}$$

valid for: (i) $|z| < 3/4$, (ii) $|z| > 3/4$.

(c) Use residues to show that

$$\int_0^{2\pi} \frac{d\theta}{4 + 2\cos\theta} = \frac{\pi}{\sqrt{3}}.$$

Hint: $\cos\theta = \frac{1}{2}(z + 1/z)$ if $z = e^{i\theta}$.

Question Five. 4+5+6+5=20 Marks

(a) Using residues, establish the following result.

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+9)} dx = \frac{\pi}{12}.$$

(b) (i) Identify the pole of $f(z) = \dfrac{ze^{i\pi z}}{z^2+2z+5}$ lying above the real axis.

(ii) Show that

$$\int_{-\infty}^{\infty} \frac{x\cos(\pi x)}{x^2+2x+5} dx = \frac{\pi}{2}e^{-2\pi}.$$

(c) Let $\gamma = \gamma_1 + \gamma_R - \gamma_3$ Where

$$\gamma_1(x) = x, \qquad 0 \le x \le R,$$
$$\gamma_R(x) = Re^{ix}, \quad 0 \le x \le \frac{2\pi}{3},$$
$$\gamma_3(x) = xe^{2\pi i/3}, \ 0 \le x \le R.$$

Taking careful note of which poles lie inside $\gamma$, use the residue theorem to evaluate the contour integral

$$\int_\gamma \frac{dz}{z^3+1}.$$

Hence show that

$$\int_0^\infty \frac{dx}{x^3+1} = \frac{2\pi}{3\sqrt{3}}.$$

(Your answer to Question One (a) will be useful). If you cannot reduce your answer to the exact answer given, use your calculator to show that the numerical value of your answer is 1.2092.

(d) Use residues to invert the Laplace Transform

$$F(s) = \frac{s^2+2}{(s+4)(s^2+9)}.$$

Useful information

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots \qquad |z| < 1,$$

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$$

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \frac{1}{2!} f''(z_0)(z - z_0)^2 + \frac{1}{3!} f'''(z_0)(z - z_0)^3 + \cdots$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt, \text{ where } \gamma = \gamma(t),\ t \in [a, b],$$

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_\gamma \frac{f(\xi)}{(\xi - z)^{n+1}} d\xi.$$

$$f(t) = \lim_{R\to\infty} \frac{1}{2\pi i} \int_{R-ic}^{R+ic} F(s)e^{st}ds,\ F \text{ is the Laplace transform of } f$$

$$\text{Residue}(f(z), z_0) = \frac{1}{(n-1)!} \lim_{z\to z_0} \frac{d^{n-1}}{dz^{n-1}} \left((z - z_0)^n f(z)\right),$$

for a pole or order $n$.

$$\text{Residue}(f(z), z) = \lim_{z\to z_0} (z - z_0)f(z),$$

for a pole of order 1.

$$\int_\gamma f(z)dz = 2\pi i \sum_{k=1}^{N} \text{Residue}(f(z), z_k)$$

$z_1, ...z_N$ are the poles of $f$ inside the closed simple curve $\gamma$.

The Cauchy-Riemann equations are $u_x = v_y$, $u_y = -v_x$.

$$\oint_C Pdx + Qdy = \int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy,$$

where $C$ is the boundary of $D$.