2015 Exam Solutions Advanced Calculus

(1)(a) Solve $z^3+1=0$ Let $z_k^3 = e^{2\pi i k+\pi i}$
as $e^{2k\pi i}=1$ all integers k, and $e^{\pi i}=-1$
~~z~~ $k=0$ gives $z_0^3 = e^{\pi i}$, so $z_0 = e^{\pi i/3}$
Since roots are in complex conjugate pairs $e^{-\pi i/3}$
is another root. $k=1$ gives $z_1^3 = e^{3\pi i} \Rightarrow z_1 = e^{\pi i} = -1$
So roots are

$$-1,\ e^{\pi i/3},\ e^{-\pi i/3}$$

b) $$f(x+iy) = \frac{1}{x+iy+1} = \frac{1}{1+x+iy}\,\frac{1+x-iy}{1+x-iy} = \frac{1+x}{(1+x)^2+y^2} - \frac{iy}{(1+x)^2+y^2}$$
$$= u+iv$$

(c) ~~$f(z) = x^2-y^2$~~ $f(x+iy) = x^2-y^2+iv(x,y)$ f is analytic
$= u+iv$

by CR equations
$$u_x = 2x = v_y$$
$$u_y = -2y = -v_x$$
$$\therefore\ v = 2xy + C\ ,\ C \text{ a constant}$$
Thus $f(x+iy) = x^2-y^2+i2xy+C$
$$= (x+iy)^2 + C = z^2+C$$

d) $f(x+iy) = u(x,y)+ic$. F is analytic.
$= u+iv$ So $u_x = v_y = 0$
with $v=c$ $u_y = -v_x = 0$
$\therefore\ u_x = u_y = 0$.
Since $u_x = 0$ we have $u = f(y)$ some function f
and $u_y = f'(y) = 0$. Thus f is constant
$\therefore\ u = a$ constant

2) $f(z) = z^2$, $\gamma(t) = 1+it^3$, $t\in[0,1]$

$$\int_\gamma f = \int_1^{1+i} (1+it^3)^2\, 3it^2\,dt$$
(ii) $$\int_1^{1+i} z^2\,dz = \left[\frac{z^3}{3}\right]_1^{1+i} = \frac{(1+i)^3}{3} - \frac{1}{3} = -1 + \frac{2i}{3}$$

(iii) By the Fundamental Theorem of contour integration these integrals are equal, as f is differentiable.

(b) $$\oint_C (5+10xy+y^2)\,dx + (6xy+5y^2)\,dy = \oint_C P\,dx + Q\,dy$$

$$= \int_0^a\int_0^b \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$= \int_0^a\int_0^b (6y-2y-10x)\,dx\,dy$$

$$= \int_0^a\int_0^b (4y-10x)\,dx\,dy$$

$$= \int_0^a \left[4xy-5x^2\right]_0^b dy$$

$$= \int_0^a (4by-5b^2)\,dy = 2by^2-5b^2y\Big|_0^a$$

$$= 2ba^2-5b^2a$$

(c) $$\left|\int_{\gamma_R} \frac{dz}{z^2+4}\right| = \left|\int_0^\pi \frac{iRe^{it}}{R^2e^{2it}+4}\,dt\right|$$

$$\leq \int_0^\pi \frac{R}{R^2-4}\,dt = \frac{\pi R}{|R^2-4|} \to 0 \text{ as } R\to\infty$$

(3) (a) $\int_\gamma f = 0$ by Cauchy's Theorem as f is analytic everywhere

(b) $f(z) = e^{2z}$

$$f(5) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-5}\,dz$$

So $$\int_C \frac{e^{2z}}{z-5} = 2\pi i e^{10}.$$ by Cauchy integral formula

c) $$\frac{1}{2\pi i}\int_C \frac{e^{2z}}{(z+3)^2}\,dz = f'(-3) \quad f(z)=e^{2z}$$

$$= 2e^{-6}.$$ By Cauchy int form.

since curve contains point $z=-3$, even though curve is not centered at -3.

(d) $f(z) = \frac{1}{2\pi i}\int_{C_R} \frac{f(\zeta)}{\zeta - z}\,d\zeta$. Put $\zeta(t) = z + Re^{it}$

$t \in [0, 2\pi]$

Then $f(z) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z+Re^{it})}{z+Re^{it}-z}\, iRe^{it}\,dt$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(z+Re^{it})\,dt$$

Q4

(a) $\frac{\cos(2z)}{4z^4} = \frac{1}{4z^4}\left(1 - \frac{(2z)^2}{2!} + \frac{(2z)^4}{4!} - \frac{(2z)^6}{6!} + \dots\right)$

$$= \frac{1}{4z^4} - \frac{1}{2z^2} + \frac{1}{6} - \frac{z^2}{45} + \frac{z^4}{630} - + \dots$$

Pole is of order 4. Residue is 0 as there is no term ~~in~~ of form $\frac{a_{-1}}{z}$.

(b) $\frac{z}{4z-3} = \frac{z}{-3}\frac{1}{\left(1-\frac{4z}{3}\right)} = -\frac{z}{3}\left(1 + \left(\frac{4z}{3}\right) + \left(\frac{4z}{3}\right)^2 + \dots\right)$

For $\left|\frac{4z}{3}\right| < 1$ or $|z| < \frac{3}{4}$.

Now $\frac{z}{4z-3} = \frac{z}{4z\left(1-\frac{3}{4z}\right)} = \frac{1}{4}\cdot\frac{1}{1-3/4z}$

$$= \frac{1}{4}\left(1 + \frac{3}{4z} + \left(\frac{3}{4z}\right)^2 + \left(\frac{3}{4z}\right)^3 + - \right)$$

for $|3/4z| < 1$ or $|z| > 3/4$

(c) $\int_0^{2\pi} \frac{d\theta}{4+2\cos\theta} = \int_C \frac{1\,dz}{4+2\times\frac{1}{2}(z+\frac{1}{z})}\cdot\frac{1}{iz}$ C unit circle

$z = e^{i\theta}$

$\cos\theta = \frac{1}{2}(z+1/z)$

$d\theta = \frac{dz}{iz}$

$= \frac{1}{i}\int \frac{dz}{z^2+4z+1}$. Poles at points where

$z^2+4z+1=0$

$z = \frac{-4 \pm \sqrt{16-4}}{2} = -2 \pm \sqrt{3}$

Only pole at $z = -2+\sqrt{3}$ is in unit circle.

Thus we want $\text{Res}(f, -2+\sqrt{3}) = \lim_{z\to -2+\sqrt{3}} \dfrac{\cancel{z-(-2+\sqrt{3})}}{i\cancel{(z-(-2+\sqrt{3}))}(z+2+\sqrt{3})}$

$$= \frac{1}{2\sqrt{3}i}$$

$$\therefore \int_0^{2\pi} \frac{d\theta}{4+2\cos\theta} = 2\pi i \times \frac{1}{2\sqrt{3}i} = \frac{\pi}{\sqrt{3}}.$$

Q5 (a) $f(x) = \dfrac{1}{(x^2+1)(x^2+9)}$ decays as $\dfrac{1}{x^4}$ for x large.

No poles on axis,

So by a theorem in lectures

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x^2+9)} = 2\pi i \sum \text{Res. above axis}$$

Poles are $x = \pm i, \pm 3i$.

$$\text{Res}(f, i) = \lim_{x\to i} \frac{x-i}{(x+i)(x-i)(x^2+9)} = \frac{-i}{16}$$

$$\text{Res}(f, 3i) = \lim_{x\to 3i} \frac{x-3i}{(x+3i)(x-3i)(x^2+1)} = \frac{i}{48}$$

$$\text{So } \int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x^2+9)} = 2\pi i\left(\frac{i}{48} - \frac{i}{16}\right) = \frac{\pi}{12}.$$

(b) $f(z) = \dfrac{z e^{i\pi z}}{z^2+2z+5} = \dfrac{z e^{i\pi z}}{(z+1)^2+4}$

poles at $(z+1)^2 = -4$ or $z = -1 \pm 2i$

Pole at $z = -1+2i$ is above axis

f is oscillatory and decays as $1/x$

So

integral is $2\pi i \times$ sum of residues in poles above axis

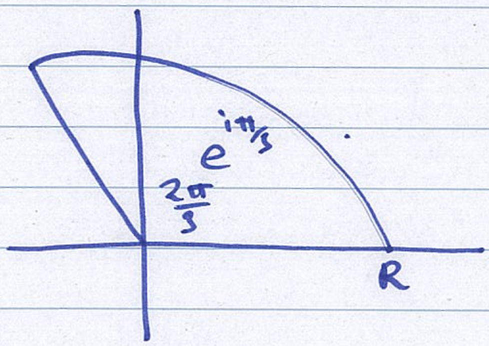
$$\mathrm{Res}(f, -1+2i) = \lim_{z\to -1+2i} \frac{(z+1-2i)\, z e^{\pi i z}}{z^2+2z+5}$$

$$= \lim_{z\to -1+2i} \frac{z e^{\pi i z}}{z-(-1-2i)} = -\left(\frac{1}{2}+\frac{i}{4}\right)e^{-2\pi}$$

$$\therefore \int_{-\infty}^{\infty} \frac{z e^{i\pi z}}{z^2+2z+5}\, dz = 2\pi i\left(-\left(\frac{1}{2}+\frac{i}{4}\right)\right)e^{-2\pi} = \frac{\pi}{2}e^{-2\pi} - \pi i e^{-2\pi}$$

Hence $\displaystyle\int_{-\infty}^{\infty} \frac{x\cos(\pi x)}{x^2+2x+5}\, dx = \frac{\pi}{2} e^{-2\pi}$

(c)



$$\int_\gamma f = \int_0^R \frac{dx}{x^3+1} + \int_0^{\frac{2\pi}{3}} \frac{iRe^{ix}}{R^3e^{3ix}+1}\, dx - \int_0^R \frac{e^{2\pi i/3}}{x^3+1}\, dx$$

$$= 2\pi i\, \mathrm{Res}(f, e^{i\pi/3})$$

only $e^{i\pi/3}$ is inside circle

$$\mathrm{Res}(f, e^{i\pi/3}) = \lim_{z\to e^{i\pi/3}} \frac{z-e^{i\pi/3}}{z^3+1}$$

From Q 1 poles are $-1, e^{i\pi/3}, e^{-i\pi/3}$

$$= \frac{1}{3e^{2\pi i/3}}$$

$$\therefore \left(1-e^{2\pi i/3}\right)\int_0^R \frac{dx}{x^3+1} + \int_0^{\frac{2\pi}{3}} \frac{iRe^{ix}}{R^3e^{3ix}+1}\, dx = \frac{2\pi i}{3e^{2\pi i/3}}$$

$$\left|\int_0^{\frac{2\pi}{3}} \frac{iRe^{ix}}{R^3e^{3ix}+1}\, dx\right| \leq \int_0^{\frac{2\pi}{3}} \frac{R}{|R^3-1|}\, dx = \frac{R\,2\pi/3}{|R^3-1|}$$

$\to 0$ as $R \to \infty$

Thus as $R \to \infty$

$$\left(1 - e^{2\pi i/3}\right)\int_0^\infty \frac{dx}{x^3+1} = \frac{2\pi}{3} i e^{-2\pi i/3}$$

or $$\int_0^\infty \frac{dx}{x^3+1} = \frac{2\pi}{3} i \frac{e^{-2\pi i/3}}{1 - e^{2\pi i/3}}$$

$$= \frac{2\pi}{3} i \frac{e^{-\pi i}}{e^{-\pi i/3} - e^{\pi i/3}}$$

$$= \frac{2\pi}{3} \frac{i}{e^{\pi i/3} - e^{-\pi i/3}} = \frac{\pi}{3} \frac{1}{\sin(\pi/3)}$$

$$= \frac{2\pi}{3\sqrt{3}}$$

(d) $$F(s) = \frac{s^2+2}{(s+4)(s^2+9)} = \mathcal{L}(f(t))$$

$f(t) =$ sum of residues of $F(s)e^{st}$

$$\mathrm{Res}\left(F(s)e^{st}, s=-4\right) = \lim_{s \to -4} \frac{(s^2+2)\cancel{(s+4)}e^{st}}{\cancel{(s+4)}(s^2+9)}$$

$$= e^{-4t}\frac{18}{25}$$

$$\mathrm{Res}(F(s)e^{st}, s = 3i) = \left(\frac{7}{50} + \frac{14i}{75}\right)e^{3it}$$

$$\mathrm{Res}(F(s)e^{st}, s = -3i) = \left(\frac{7}{50} - \frac{14i}{75}\right)e^{-3it}$$

I got lazy and did these in Mathematica

$$\therefore \quad f(t) = \frac{18}{25}e^{-4t} + \frac{7}{50}\left(e^{3it} + e^{-3it}\right) + \frac{14i}{75}\left(e^{3it} - e^{-3it}\right)$$

$$= \frac{18}{25}e^{-4t} + \frac{7}{25}\cos(3t) - \frac{28}{75}\sin(3t)$$