Advanced Calculus 35232: Basic Skills Make up Test

Time Allowed: 1 hour.

(1) Write $z = -6 + 8i$ in the form $z = re^{i\theta}$ and plot the point in the complex plane. In determining θ pay careful attention to the quadrant in which z lies.

(2) Show that $f(z) = z^3 + 3z$ is an entire function.

(3) Evaluate the contour integral $\int_\gamma f(z)dz$, where $f(z) = 2z^2 + 3z$ and $\gamma(t) = -t + it^2, t \in [0, 1]$.

(4) Evaluate

$$\oint_C (\cos^3 x + 4y)dx + (x^2 + \frac{y}{\ln(y^2 + 3)})dy ,$$

where C is the path given by $x^2 + y^2 = 4$, traversed in the anti-clockwise direction.

(5) Evaluate the contour integral

$$\int_C \frac{e^{3z}}{(z-2)^2} dz$$

where C is a circle of radius 1 centered at $z = 2$.

(6) Find a Laurent series expansion for $f(z) = \dfrac{\sin(2z)}{z^6}$ about the point $z = 0$ and find the order of the pole and the value of the residue at that point.

(7) By letting $z = e^{i\theta}$ and converting to an integral around the unit circle, show that

$$\int_0^{2\pi} \frac{d\theta}{2 + \sin\theta} = \frac{2\pi}{\sqrt{3}}.$$

(8) Show that

$$\int_0^\infty \frac{dx}{(x^2 + 1)(x^2 + 4)(x^2 + 9)} = \frac{\pi}{120}.$$

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Time Allowed: 25 minutes .

(1) (a) Identify the poles of

$$f(z) = \frac{e^{iz}}{(z^2+4)(z^2+9)}.$$

(b) Calculate the residues at the poles in the upper half plane.

(c) Evaluate the integral

$$\int_0^\infty \frac{\cos x dx}{(x^2+4)(x^2+9)}.$$

(2) Find the inverse Laplace transform of the function

$$F(s) = \frac{s+2}{(s+9)(s^2+1)}.$$

Note: For a pole of order 1

$$\mathrm{Res}(f(z), z_0) = \lim_{z\to z_0}(z-z_0)f(z).$$

$$f(t) = \lim_{T\to\infty}\frac{1}{2\pi i}\int_{c-iT}^{c+iT} F(s)e^{st}ds.$$

Advanced Calculus 35232: Basic Skills Test 8

Time Allowed: 25 minutes .

(1) (a) Let $z = e^{i\theta}$. Show that $\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$. Also show that $d\theta = \frac{dz}{iz}$

(b) Use the results for (a) to show that

$$\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \frac{2}{i}\int_C \frac{dz}{1+4z+z^2}$$

where C is the circle of radius 1 centered at 0. Identify the pole of $f(z) = \frac{1}{1+4z+z^2}$ inside the unit cirle. Obtain the value of the residue at this pole.

(c) Use the residue theorem to show that

$$\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \frac{2\pi}{\sqrt{3}}.$$

(2) (a) Identify the poles of

$$f(z) = \frac{z}{(z^2+1)(z^2+4)}.$$

(b) Calculate the residues at the poles above the real axis.

(c) Use residues to evaluate the integral

$$\int_{-\infty}^{\infty} \frac{x\,dx}{(x^2+1)(x^2+4)}.$$

Note: For a pole of order 1

$$\text{Res}(f(z), z_0) = \lim_{z\to z_0}(z - z_0)f(z).$$

Advanced Calculus 35232: Basic Skills Test 7

Time Allowed: 30 minutes .

(1) Obtain a Laurent series expansion for the function $f(z) = \dfrac{2}{2-z}$ about the point $z = 0$ valid for $|z| > 2$.

(2) Obtain a Laurent series expansion for

$$f(z) = \frac{\cos z + \sin z}{z^3}.$$

What is the order of the pole at $z = 0$? What is the value of the residue at the pole?

(3) Given that $\tan^{-1} z = \displaystyle\int_0^z \frac{dt}{1+t^2}$ and $\dfrac{1}{1+z} = 1 - z + z^2 - z^3 + \cdots$, determine a Laurent series for the function

$$f(z) = \frac{\tan^{-1} z}{z^4}.$$

(4) Evaluate the contour integral

$$\int_\gamma \frac{z}{z-i} dz$$

where γ is a simple closed counterclockwise contour centered at $z = i$.

Advanced Calculus 35232: Basic Skills Test 6

Time Allowed: 25 minutes

(1) Let f be a differentiable function and C_R the boundary of the circle of radius R centered at z. Write down the integral

$$\frac{1}{2\pi i}\int_{C_R}\frac{f(\xi)}{\xi - z}d\xi$$

in parametric form. For the curve C_R take $\gamma(t) = z + Re^{it}, t \in [0, 2\pi)$.

(2) Explicitly evaluate the integral in (1) in the case when $f(z) = z$.

(3) What would the integral in (1) give if we take $f(z) = e^{z^2}$? Explain your answer without attempting to evaluate the integral.

(4) Obtain a Laurent series expansion for

$$f(z) = \frac{e^{2z}}{z^4}$$

about $z = 0$. What is the order of the pole at $z = 0$? What is Residue?

Advanced Calculus 35232: Basic Skills Test 5

Time Allowed: 30 minutes.

(1) Evaluate

$$\int\int_R (x+y)dxdy$$

where R is the region bounded by the rectangle joining the points $(1,0),(3,0),(3,6)$ and $(1,6)$. Show that reversing the order of integration gives the same answer.

(2) Evaluate the double integral

$$\int_1^2 \int_{1-x}^{\sqrt{x}} x^2 y dy dx.$$

(3) Use polar coordinates to evaluate the integral

$$\int\int_A (1+x^2+y^2)dxdy,$$

where A is the region in the *first* quadrant satisfying $x^2+y^2 \leq 4$.

Advanced Calculus 35232: Basic Skills Test 4

Time Allowed: 25 minutes.

(1) Show that the function

$$f(z) = z - z^2$$

is entire. That is, differentiable everywhere.

(2) In what domain, if any, is the function

$$f(z) = z^2 - \bar{z}^2$$

differentiable? Here $\bar{z}$ is the complex conjugate of z.

(3) Identify the points where the following functions are *not* differentiable?

(a)

$$f(z) = \frac{1}{z + 2i}$$

(b)

$$f(z) = \frac{1}{z^3 - 27}$$

(4) Evaluate the contour integral

$$\int_\gamma f(z)dz$$

for $\gamma(t) = 1 + it^2, t \in [0, 1]$, $f(z) = z^2$.

(5) Given two entire complex functions $f(z)$ and $g(z)$, show that the linear combination

$$h(z) = af(z) - bg(z),$$

where a, b are constant, is also entire.

Advanced Calculus 35232: Basic Skills Test 3

Time Allowed: 25 minutes .

(1) Plot the following curves in the complex plane:
a)

$$z(t) = i + e^{-2it}, \ t \in [0, \pi]$$

Hint: What is the curve $z(t) = e^{-2it}$, $t \in [0, \pi]$? What effect does adding i have?
b)

$$z(t) = x(t) + iy(t)$$

where $x(t) = t$, $y(t) = -2t^2$, for $t \in [0, 2]$ Hint: Express y as a function of x

(2) Find a parametrization of the straight line segment joining $1-i$ to $1+i$.

(3) Find a parametrization of the circle with radius 2 centred at the point $z_0 = 2 - i$. You should have something similar to 1 a)

(4) Explain why $f(z) = |z|$ is not a differentiable function for any nonzero values of z.

(5) Write the function

$$f(z) = z^3$$

in the form $f(x + iy) = u(x, y) + iv(x, y)$, with u and v real functions. Show that f is an *entire function.* That is, show that it is differentiable for every z.

Advanced Calculus 35232: Basic Skills Test 2

Time Allowed: 25 minutes.

(1) Find the polar form of the the complex number $z = 4 + 3i$.

(2) Find all 4th roots of i.

(3) Find the real and imaginary parts of $z = \dfrac{i}{1+i}$.

(4) Find all points $z = x + iy$ such that $|z - 2 - i| = 3$. (Hint: Use the definition of the modulus $|a + ib| = \sqrt{a^2 + b^2}$, square both sides and collect real and imaginary parts to determine a curve on which x, y lie).

(5) Plot the region $|z - i + 1| > \frac{1}{2}$ in the complex plane.

(6) Find all solutions to the equation

$$z^6 + 2z^3 + 1 = 0.$$

Hint: This is really a quadratic in disguise. Also, the roots occur in complex conjugate pairs, so you only need to find three different roots to determine all six.

(7) Let $\omega = e^{2\pi i/5}$. Explain why

$$1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0.$$

Hint:

$$z^5 - 1 = (z - 1)(1 + z + z^2 + z^3 + z^4).$$

Advanced Calculus 35232: Basic Skills Test 1

Time Allowed: 25 minutes.

(1) Evaluate the derivatives of the following functions:

a) $f(x) = 2x^3 + 2\sqrt{x+1} + 1$

b) $f(x) = \cos(2x)$

c) $f(x) = \sin(2x^3)$

d) $f(x) = \ln(\sin 3x))$

(2) Evaluate the following definite integrals:

a)

$$\int_1^2 \left(x^3 + \frac{1}{x^3}\right) \mathrm{d}x$$

b)

$$\int_0^{\pi/2} \sin x \cos^3 x \mathrm{d}x$$

(3) Evaluate

$$\lim_{x \to 0} \frac{\tan x}{x} .$$

(4) Given the scalar function $f(x, y, z) = 3x^4 + 2z^2y^2 + xyz^2$ calculate the partial derivatives

a) $\dfrac{\partial f}{\partial x}$

b) $\dfrac{\partial f}{\partial y}$

(5) Write the complex number $\dfrac{2+4i}{-3+5i}$ in the form $x + iy$.

(6) Write $z = -2 - i$ in the form $z = re^{i\theta}$ and plot the point in the complex plane. In determining θ pay careful attention to the quadrant in which z lies.