Advanced Calculus Class Test

Time Allowed: 55 minutes .

(1) Obtain the real and imaginary parts of the function

$$f(z) = z^3 + 4z^2,$$

where $z = x+iy$. Show that the real and imaginary parts satisfy the Cauchy-Riemann equations.

(2) Evaluate $\oint_C y^2dx + 3xydy$ where C is the boundary of the following region: $D = \{(r, \theta) : 1 \le r \le 2; 0 \le \theta \le \pi\}$.

(3) State Cauchy's Theorem.

(4) Evaluate the integral $\displaystyle\int_\gamma \frac{e^{4i\pi z}}{z^2+16}dz$ where γ is the circle of radius 2, centered at $z = 4i$, taken counterclockwise.

(5) Obtain a Laurent series expansion about $z = 0$ for the function $f(z) = \dfrac{e^z \sin z}{z^4}$. What is the order of the pole at $z = 0$? What is the value of the residue at the pole?

(6) (i) Let C be the unit circle, traversed counter-clockwise. By setting $z = e^{i\theta}$, show that

$$\int_0^{2\pi} \frac{d\theta}{3 - 2\cos\theta + \sin\theta} = \int_C \frac{2dz}{(1-2i)z^2 + 6iz - 1 - 2i}.$$

(ii) Show that $(1-2i)z^2 + 6iz - 1 - 2i = 0$ when $z = 2-i$ and $z = (2-i)/5$.

(iii) Calculate the residue of $f(z) = \frac{2}{(1-2i)z^2+6iz-1-2i}$ at the only pole inside the unit circle. Using L'Hôpital's rule will make it easier.

(iv) Show that $\displaystyle\int_0^{2\pi} \frac{d\theta}{3 - 2\cos\theta + \sin\theta} = \pi.$

Formulas Over Page

Useful information

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$$

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt, \text{ where } \gamma = \gamma(t),\ t \in [a, b],$$

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_\gamma \frac{f(\xi)}{(\xi - z)^{n+1}} d\xi.$$

$$\text{Residue}(f(z), z) = \lim_{z \to z_0} (z - z_0) f(z),$$

for a pole of order 1.

$$\int_\gamma f(z)dz = 2\pi i \sum_{k=1}^{N} \text{Residue}(f(z), z_k)$$

$z_1, ...z_N$ are the poles of f inside the closed simple curve γ.

$$\oint_C P dx + Q dy = \int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx dy,$$

where C is the boundary of D.