ADVANCED CALCULUS

CLASS TEST

(1). Find all solutions of $z^4=16$.

$16=16e^{i(2\pi k)}$ $\quad k=0,\pm1,\pm2,\dots$

we need to find all z such that $z^4=16=16e^{i(2\pi k)} \Leftrightarrow z=\left(16e^{i(2\pi k)}\right)^{1/4}$

$\Leftrightarrow z=16^{1/4}e^{i\frac{2\pi k}{4}}=2e^{i\frac{\pi k}{2}}$

$\rightarrow k=0 \rightarrow z_1=2e^0=2$

$\rightarrow k=-1 \rightarrow z_2=2e^{-i\frac{\pi}{2}}$

$\rightarrow k=1 \rightarrow z_3=2e^{i\frac{\pi}{2}}$

$\rightarrow k=2 \rightarrow z_4=2e^{i\pi}=-2$

(2) Let $f(z)=z^2+e^z$. Show that f is differentiable everywhere.

Let $z=x+iy$. Then

$$f(z)=f(x+iy)=(x+iy)^2+e^{(x+iy)}=x^2-y^2+2ixy+e^xe^{iy}\overset{\boxed{\text{Euler's formula}}}{=}$$

$$=x^2-y^2+2ixy+e^x(\cos y+i\sin y)=\underbrace{(x^2-y^2+e^x\cos y)}_{u(x,y)}+i\underbrace{(2xy+e^x\sin y)}_{v(x,y)}$$

Now let us calculate the partial derivatives of u,v:

$$\frac{\partial u}{\partial x}=2x+e^x\cos y \qquad \frac{\partial v}{\partial x}=2y+e^x\sin y$$

$$\frac{\partial u}{\partial y}=-2y-e^x\sin y \qquad \frac{\partial v}{\partial y}=2x+e^x\cos y$$

$\left\{\begin{array}{l} u_x=v_y \\ u_y=-v_x \end{array}\right.$ for all x,y $\Rightarrow$ The Cauchy-Riemann equations are satisfied for all x and y $\Rightarrow$ the function is differentiable everywhere.

(3) For what values of a and b is there a function v, such that $f(x+iy)=\underbrace{(ax^3+bxy^2)}_{u(x,y)}+iv(x,y)$ is analytic?

If f is analytic then the Cauchy-Riemann equations must hold for all x and y, i.e. $u_x = v_y$, $u_y = -v_x$ for all x and y.

$$u(x,y) = ax^3 + bxy^2 \Rightarrow \begin{cases} u_x = 3ax^2 + by^2 \overset{①}{=} v_y \\ u_y = 2bxy \overset{②}{=} -v_x \end{cases}$$

$$v(x,y) \overset{\boxed{\text{using ①}}}{=} \int (3ax^2 + by^2)\,dy + g(x) = 3ax^2y + \frac{by^3}{3} + g(x)$$

$$v_x = 6axy + g'(x) \overset{\boxed{\text{using ②}}}{=} -2bxy \Leftrightarrow \begin{cases} g'(x) = 0 \Rightarrow g(x) = C \\ 6a = -2b \Rightarrow a = \frac{-b}{3} \end{cases}$$

So, for $a = -\frac{b}{3}$, there exists $v(x,y) = 3ax^2y + \frac{by^3}{3} + C =$

$= -bx^2y + \frac{b}{3}y^3 + C$ such that $f = u + iv$ is analytic.

(4) (a) Evaluate the contour integral

$$\int_\gamma z^2\,dz, \text{ where } \gamma(t) = t^2 - it, \quad t \in [0,1]$$

$$\hookrightarrow \gamma'(t) = 2t - i$$

$$\int_\gamma z^2\,dz = \int_0^1 (\gamma(t))^2 \gamma'(t)\,dt = \int_0^1 (t^2 - it)^2 (2t - i)\,dt = \int_0^1 (t^4 - t^2 - 2it^3)(2t - i)\,dt$$

$$= \int_0^1 (2t^5 - 2t^3 - 4it^4 - it^4 + it^2 - 2t^3)\,dt = \int_0^1 \left[(2t^5 - 4t^3) + i(t^2 - 5t^4)\right] dt =$$

$$= \left[\left(\frac{1}{3}t^6 - t^4\right) + i\left(\frac{t^3}{3} - t^5\right)\right]_0^1 = \left(\frac{1}{3} - 1\right) + i\left(\frac{1}{3} - 1\right) = -\frac{2}{3} - \frac{2}{3}i$$

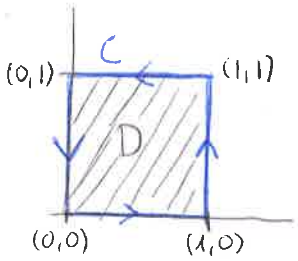
OR

$$\int_\gamma z^2\,dz = \int_0^{1-i} z^2\,dz = \left[\frac{z^3}{3}\right]_0^{1-i} = \frac{(1-i)^3}{3} = \frac{1 - 3i + 3i^2 - i^3}{3} = \frac{1 - 3i - 3 + i}{3} =$$

$$= -\frac{2}{3} - \frac{2}{3}i.$$

(5). (a) Evaluate the line integral

$\oint_C \underbrace{x^2y}_{P}\,dx + \underbrace{2y^2}_{Q}\,dy$, where C is the square with vertices $(0,0), (1,0), (1,1)$ and $(0,1)$ traversed counterclockwise.



$P(x,y) = x^2y \Rightarrow \frac{\partial P}{\partial y} = x^2$

$Q(x,y) = 2y^2 \Rightarrow \frac{\partial Q}{\partial x} = 0$

So, using Green's theorem,

$$\oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \int_0^1\int_0^1 -x^2\,dx\,dy = \int_0^1 -\left[\frac{x^3}{3}\right]_0^1 dy =$$

$$= \int_0^1 \left(-\frac{1}{3}\right) dy = -\frac{1}{3}\left[y\right]_0^1 = \boxed{-\frac{1}{3}}$$

(b) Use the Cauchy integral formula to evaluate

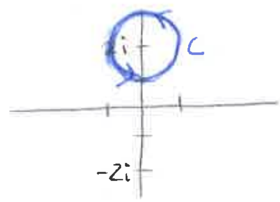
$\int_C \frac{ze^{\frac{\pi}{2}z}}{z^2+4}\,dz$, where C is a circle of radius 1 centered at $z = 2i$, traversed counterclockwise.

First, we write

$$\frac{1}{z^2+4} = \frac{A}{z+2i} + \frac{B}{z-2i} \Leftrightarrow A(z-2i) + B(z+2i) = 1$$

At $z = 2i \Rightarrow B(4i) = 1 \Rightarrow B = \frac{1}{4i} = -\frac{i}{4}$

At $z = -2i \Rightarrow -4iA = 1 \Rightarrow A = -\frac{1}{4i} = \frac{i}{4}$



So we can write

$$\int_C \frac{ze^{\frac{\pi}{2}z}}{z^2+4}\,dz = \frac{i}{4}\int_C \frac{ze^{\frac{\pi}{2}z}}{z+2i} - \frac{i}{4}\int_C \frac{ze^{\frac{\pi}{2}z}}{z-2i} = -\frac{i}{4}\int_C \frac{ze^{\frac{\pi}{2}z}}{z-2i} = -\frac{i}{4}\left[2\pi i\, f(2i)\right] =$$

"$f(z)$", which is entire.

this function is analytic everywhere except at $z=-2i$, but $-2i$ is not enclosed by C, so by Cauchy's theorem this integral is 0.

by Cauchy's integral formula with $n=0$

$$= \frac{\pi}{2} 2i e^{\frac{\pi}{2}2i} = \pi i e^{\pi i} = \boxed{-\pi i}$$

(6)(a) Using the definition of $\cosh u$ and the series for e^u, obtain a Taylor series expansion around $u=0$ for $f(u)=\cosh u$.

$$\cosh z = \frac{e^z+e^{-z}}{2} = \frac{\left(1+\cancel{z}+\frac{z^2}{2!}+\cancel{\frac{z^3}{3!}}+\cdots\right)+\left(1-\cancel{z}+\frac{z^2}{2!}-\cancel{\frac{z^3}{3!}}+\cdots\right)}{2}$$

$$= \frac{\left(1+\frac{z^2}{2!}+\frac{z^4}{4!}+\cdots\right)+\left(1+\frac{z^2}{2!}+\frac{z^4}{4!}+\cdots\right)}{2} = \frac{\cancel{2}\left(1+\frac{z^2}{2!}+\frac{z^4}{4!}+\frac{z^6}{6!}+\cdots\right)}{\cancel{2}}$$

$$= 1+\frac{z^2}{2!}+\frac{z^4}{4!}+\frac{z^6}{6!}+\cdots = \sum_{n=0}^{\infty}\frac{z^{2n}}{(2n)!}$$

(b) Let $f(z)=\frac{\cosh(z)}{3z^5}$. Obtain a Laurent series expansion for $f(z)$ about $z=0$. What is the order of the pole at $z=0$? What is the value of the residue at $z=0$?

$$f(z)=\frac{\cosh(z)}{3z^5} = \frac{1}{3z^5}\sum_{n=0}^{\infty}\frac{z^{2n}}{(2n)!} = \frac{1}{3}\sum_{n=0}^{\infty}\frac{z^{2n-5}}{(2n)!} = \frac{1}{3}\left(\frac{1}{z^5}+\frac{1}{2z^3}+\frac{1}{24z}+\frac{z}{6!}+\cdots\right)$$

$$= \frac{1}{3z^5}+\frac{1}{6z^3}+\frac{1}{72z}+\cdots$$

the pole at $z=0$ is of order 5

the residue of f at $z=0$ is $\frac{1}{72}$