1) a)

$$f(z) = \frac{1-z}{1+z} = \frac{1-(x+iy)}{1+(x+iy)}$$

$$= \frac{1-x-iy}{1+x+iy} \cdot \frac{(1+x-iy)}{(1+x-iy)} \quad 2$$

$$= \frac{1-x^2-y^2-2iy}{(1+x)^2+y^2}$$

$$= \frac{1-(x^2+y^2)}{(1+x)^2+y^2} + i\,\frac{-2y}{(1+x)^2+y^2} \quad 4$$

b) $f(x+iy) = x^4 + 4x^3 - 6x^2y^2 - 12xy^2 + y^4 + i v(x,y)$

$$u = x^4 + 4x^3 - 6x^2y^2 - 12xy^2 + y^4$$

$$u_x = 4x^3 + 12x^2 - 12xy^2 - 12y^2 = v_y \quad 2$$

$$\Rightarrow \int v_y\, dy = 4x^3y + 12x^2y - 4xy^3 - 4y^3 + g(x) = v$$

2

$$v_x = 12x^2y + 24xy - 4y^3 + g'(x)$$

$$= -u_y = -(-12x^2y - 24xy + 4y^3)$$

$$= 12x^2y + 24xy - 4y^3$$

$\therefore \quad g'(x) = 0 \quad \Rightarrow g(x) = A.$

3

$$v = 4x^3y + 12x^2y - 4xy^3 - 4y^3 + A$$

c) $f(x+iy) = u(x,y) + iv(x,y)$ is differentiable

$\Rightarrow$ $u_x = v_y$

$u_y = -v_x$

$\therefore$ $u_{xx} = v_{yx}$ 1

$u_{yy} = -v_{xy}$

$\Rightarrow$ $u_{xx} = -u_{yy}$ 2

or $u_{xx} + u_{yy} = 0.$

Simarlarly,

$u_{xy} = v_{yy}$ ~~4~~ 2

$u_{yx} = -v_{xx}$

$\Rightarrow$ $v_{yy} = -v_{xx}$ 2

or $v_{yy} + v_{xx} = 0$

Both v, u satisfy Laplace's equation.

2) a) $f(z) = z^2 - 2z$, $\quad \gamma_1(t) = t^2 - it, \ t \in [0,1]$

$\gamma_2(t) = (1-i)t, \ t \in [0,1]$

i)
$$\int_{\gamma_1} f(z)\,dz = \int_0^1 f(t^2 - it)\cdot(2t - i)\,dt$$

$$= \int_0^1 \left((t^2 - it)^2 - 2(t^2 - it)\right)(2t - i)\,dt$$

$$= \int_0^1 2t + 7it^2 - 8t^3 - 5it^4 + 2t^5\,dt$$

2

$$= t^2 + \frac{7}{3}it^3 - 2t^4 - it^5 + \frac{1}{3}t^6 \Big|_0^1$$

$$= 1 + \frac{7}{3}i - 2 - i + \frac{1}{3}$$

$$= -\frac{2}{3} + \frac{4}{3}i$$

$$\int_{\gamma_2} f(z)\,dz = \int_0^1 f((1-i)t)\,(1-i)\,dt$$

$$= \int_0^1 \left(((1-i)t)^2 - 2((1-i)t)\right)(1-i)\,dt$$

$$= (2+2i)\int_0^1 (1+i)t - t^2\,dt$$

$$= (2+2i)\left[\frac{(1+i)}{2}t^2 - \frac{1}{3}t^3\right]_0^1$$

2

$$= (2+2i)\left[\frac{1+i}{2} - \frac{1}{3}\right]$$

$$= -\frac{2}{3} + \frac{4}{3}i$$

ii) $$\int_0^{1-i} z^2 - 2z \, dz = \frac{1}{3}z^3 - z^2 \Big|_0^{1-i}$$
$$= \frac{1}{3}(1-i)^3 - (1-i)^2$$
$$= -\frac{2}{3} + \frac{4}{3}i \qquad 2$$

iii) f is differentiable, therefore the integral is path independent. 2

iv) g is not differentiable, therefore the integrals are not path independent. 2

b) $$\oint_C (3 + 6x^2y^3 + y^4)\,dx + (6x^2y^2 + 5x^3)\,dy$$

$$P = 3 + 6x^2y^3 + y^4, \qquad Q = 6x^2y^2 + 5x^3$$

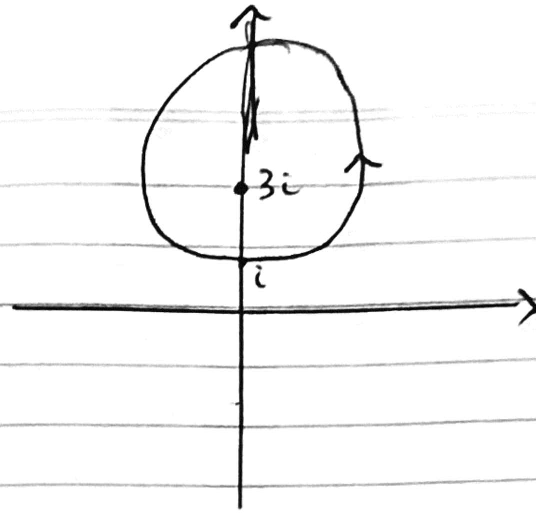
$$\frac{\partial P}{\partial y} = 18x^2y^2 + 4y^3, \qquad \frac{\partial Q}{\partial x} = 12xy^2 + 15x^2$$
2 2

$$\oint_C P\,dx + Q\,dy = \int_0^a \int_0^b \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dx\,dy \qquad 2$$
$$= \int_0^a \int_0^b 12xy^2 + 15x^2 - 18x^2y^2 - 4y^3 \, dx\,dy$$
$$= \int_0^a 6x^2y^2 + 5x^3 - 6x^3y^2 - 4xy^3 \Big|_0^b \, dy$$
$$= \int_0^a 6b^2y^2 + 5b^3 - 6b^3y^2 - 4by^3 \, dy \qquad 2$$
$$= 2b^2y^3 + 5b^3y - 2b^3y^3 - by^4 \Big|_0^a$$
$$= 2b^2a^3 + 5b^3a - 2b^3a^3 - ba^4 \qquad 2$$

4

3) a)

$$\int \frac{e^{\frac{1}{3}\pi z}}{z^2+9}\,dz$$

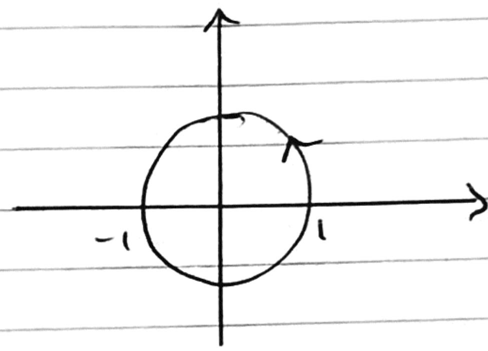


Let $f(z) = \dfrac{e^{\frac{1}{3}\pi z}}{z+3i}$ 2

$$\Rightarrow \quad 2\pi i\, f(3i) = \int \frac{f(z)}{z-3i}\,dz$$ 2

$$= 2\pi i \cdot \frac{e^{\frac{1}{3}\pi 3i}}{6i} = \frac{\pi}{3} e^{i\pi} = -\frac{\pi}{3}$$

b) $\dfrac{1}{2\pi i}\displaystyle\int_c \frac{\sin^2(tz)}{z^3}\,dz$, $t>0$.



Let $f(z) = \sin^2(tz)$

$n=2$. Cauchy's integral formula

$$\Rightarrow \quad \frac{1}{2\pi i}\int \frac{\sin^2(tz)}{z^3}\,dz = f''(0)$$

$$f'(z) = 2\sin(tz)\cdot\cos(tz)\cdot t$$

$$f''(z) = 2t\cos(tz)\cdot t\cos(tz) + 2\sin(tz)\left[-t^2\sin(tz)\right\}$$

$$= 2t^2\cos^2(tz) - 2t^2\sin(tz)$$

$$= 2t^2\cos(2tz). \quad \therefore \frac{1}{2\pi i}\int\frac{\sin^2(tz)}{z^3} = 2t^2$$

c)

$$f'(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^2}\,dz$$

Let $z = z_0 + Re^{i\theta}$, $t \in [0, 2\pi]$,

then,
$$f'(z_0) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0+Re^{i\theta})}{(z_0+Re^{i\theta}-z_0)^2}\cdot iRe^{i\theta}\,d\theta \quad 2$$

$$= \frac{1}{2\pi R}\int_0^{2\pi} e^{-i\theta} f(z_0+Re^{i\theta}) \quad 2$$

d) Let f be differentiable everywhere and suppose

$|f(z)| < M$ for some $M < \infty$, for all z.

From c) we have that

$$f'(z_0) = \frac{1}{2\pi R}\int_0^{2\pi} e^{-i\theta} f(z_0+Re^{i\theta})\,d\theta.$$

Using the boundedness of f, we have

$$|f'(z_0)| = \left|\frac{1}{2\pi R}\int_0^{2\pi} e^{-i\theta} f(z_0+Re^{i\theta})\,d\theta\right|$$

$$\leq \frac{1}{2\pi R}\int_0^{2\pi} |f(z_0+Re^{i\theta})| \leq \frac{M}{R}. \quad 2$$

Letting $R \to \infty$, we have that $f'(z_0) = 0$ for all $z_0 \in \mathbb{C}$

2

Therefore all bounded functions on $\mathbb{C}$ are constants.

4) a)

$$f(z) = \frac{z^2 + \cos z}{z^3}$$

$$= \frac{1}{z^3}\left(z^2 + \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots\right)\right)$$

$$= \frac{1}{z^3} + \frac{1}{2z} + \frac{z}{4!} - \frac{z^3}{6!} + \dots \qquad 3$$

Order of pole at $z=0$ is 3. 1

Residue is $\frac{1}{2}$. 1

b) $f(z) = \dfrac{z^2}{2z^2 - 32}$

ii) $|z| > 4$

$$\frac{z^2}{2z^2-32} = \frac{z^2}{2(z^2-16)} = \frac{z^2}{2(z+4)(z-4)}$$

Now, $$\frac{z^2}{2z^2-32} = \frac{z^2}{2z^2\left(1-\frac{16}{z^2}\right)} = \frac{1}{2\left(1-\frac{16}{z^2}\right)}$$

$$= \frac{1}{2}\left(1 + \frac{16}{z^2} + \left(\frac{16}{z^2}\right)^2 + \left(\frac{16}{z^2}\right)^3 + \dots\right)$$

valid for $\left|\frac{16}{z^2}\right| < 1$

$\Rightarrow 4 < |z|$

3

i) $|z| < 4$

$$f(z) = \frac{z^2}{2z^2 - 32} = \frac{-z^2}{32\left(1 - \frac{2z^2}{32}\right)}$$

$$= \frac{-z^2}{32\left(1 - \frac{z^2}{16}\right)}$$

$$= -\frac{z^2}{32}\left(1 + \frac{z^2}{16} + \left(\frac{z^2}{16}\right)^2 + \left(\frac{z^2}{16}\right)^3 + \dots\right)$$

valid for

$$\left|\frac{z^2}{16}\right| < 1$$

$$\Rightarrow \quad |z| < 4. \qquad 2$$

Poles at $4, -4$.

$$\operatorname{Res}(f, 4) = \lim_{z \to 4} \frac{(z-4)\, z^2}{2(z+4)(z-4)}$$

$$= \lim_{z \to 4} \frac{z^2}{2(z+4)} = \frac{16}{2(8)} = 1 \qquad 1$$

$$\operatorname{Res}(f, -4) = \lim_{z \to -4} \frac{(z+4)\, z^2}{2(z+4)(z-4)}$$

$$= \lim_{z \to -4} \frac{z^2}{2(z-4)} = \frac{16}{2(-8)} = -1. \qquad 1$$

c) $$\int_0^{2\pi} \frac{d\theta}{3+2\sin\theta} = \int_C \frac{1}{3+\frac{2}{2i}\left(z-\frac{1}{z}\right)} \cdot \frac{dz}{iz}$$

$$= \int_C \frac{1}{3iz+z^2-1}\,dz \quad 2$$

Poles at $$\frac{-3i \pm \sqrt{-9+4}}{2} = \frac{-3i \pm \sqrt{-5}}{2}$$

$$= \frac{1}{2}(\sqrt{5}-3)i, \quad 1$$
$$\frac{1}{2}(-\sqrt{5}-3)i.$$

$\frac{1}{2}(\sqrt{5}-3)i$ is inside the unit circle. 1

$$\text{Res}\left(f, \tfrac{1}{2}(\sqrt{5}-3)i\right) = \lim_{z \to \frac{1}{2}(\sqrt{5}-3)i} \frac{\left(z-\frac{1}{2}(\sqrt{5}-3)i\right)}{z^2+3iz-1}$$

$$= \lim_{z \to \frac{1}{2}(\sqrt{5}-3)i} \frac{1}{2z+3i} \quad \text{(l'hopital's rule)}$$

$$= \frac{1}{2\left(\frac{1}{2}(\sqrt{5}-3)i\right)+3i}$$

$$= \frac{1}{(\sqrt{5}-3)i+3i} = \frac{1}{\sqrt{5}i} \quad 2$$

$$\therefore \int_0^{2\pi} \frac{d\theta}{3+2\sin\theta} = \int_C \frac{1}{z^2+3iz-1}\,dz = 2\pi i \cdot \text{Res}\left(f, \tfrac{1}{2}(\sqrt{5}-3)i\right)$$

$$= 2\pi i \cdot \frac{1}{\sqrt{5}i} = \frac{2\pi}{\sqrt{5}} \quad 2$$

5) a) $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{x^4+1}\,dx = \frac{\pi}{\sqrt{2}}$

$$z^4+1=0 \Rightarrow z^4 = -1 = e^{i\pi + 2\pi k i}$$

$$\Rightarrow z = e^{\frac{i\pi}{4} + \frac{\pi k i}{2}}$$

$$z_0 = e^{\frac{i\pi}{4}}, \quad z_1 = e^{\frac{i\pi}{4}+\frac{\pi i}{2}} = e^{\frac{3i\pi}{4}}$$

$$z_2 = e^{-\frac{i\pi}{4}}, \quad z_3 = e^{-\frac{3i\pi}{4}}$$

Only z_0 and z_1 lie above the real axis.

$$\operatorname{Res}\left(\frac{z^2}{z^4+1}, e^{i\frac{\pi}{4}}\right) = \lim_{z\to e^{i\frac{\pi}{4}}} \frac{z^2\left(z-e^{i\frac{\pi}{4}}\right)}{z^4+1}$$

$$= \lim_{z\to e^{i\frac{\pi}{4}}} \frac{2z\left(z-e^{i\frac{\pi}{4}}\right)+z^2}{4z^3}$$ 2

$$= \frac{e^{i\frac{2\pi}{4}}}{4e^{i\frac{3\pi}{4}}} = \frac{1}{4}e^{-\frac{i\pi}{4}}$$

$$\operatorname{Res}\left(\frac{z^2}{z^4+1}, e^{i\frac{3\pi}{4}}\right) = \lim_{z\to e^{i\frac{3\pi}{4}}} \frac{z^2\left(z-e^{i\frac{3\pi}{4}}\right)}{z^4+1}$$

$$= \lim_{z\to e^{i\frac{3\pi}{4}}} \frac{2z\left(z-e^{i\frac{3\pi}{4}}\right)+z^2}{4z^3}$$ 2

$$= \frac{e^{i\frac{6\pi}{4}}}{4e^{\frac{9\pi}{4}}} = \frac{1}{4}e^{-\frac{3\pi}{4}}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{x^2}{x^4+1}\,dx = 2\pi i \sum \text{residues}$$

$$= \frac{2\pi i}{4}\left(e^{-\frac{i\pi}{4}} + e^{-\frac{i3\pi}{4}}\right)$$

$$= \frac{\pi i}{2}\left(\frac{1-i}{\sqrt{2}} - \frac{1+i}{\sqrt{2}}\right)$$

$$= \frac{\pi i}{2}\left(-\frac{2i}{\sqrt{2}}\right)$$

$$= \frac{\pi}{\sqrt{2}}.$$

2

b)

$$\int_\gamma f\,dz = \int_0^R \frac{1}{t^7+1}\,dt + \int_0^{\frac{2\pi}{7}} \frac{iRe^{it}}{R^7e^{i7t}+1}\,dt$$ 2

$$-\int_0^R \frac{e^{2\pi i/7}}{t^7+1}\,dt$$

$$= 2\pi i \,\mathrm{Res}\left(\frac{1}{z^7+1}, e^{\pi i/7}\right)$$ 1

$e^{\pi i/7}$ only pole inside contour. 1

$$\mathrm{Res}\left(\frac{1}{z^7+1}, e^{\frac{\pi i}{7}}\right) = \lim_{z \to e^{i\pi/7}} \frac{(z - e^{\pi i/7})}{z^7+1}$$

$$= \lim_{z \to e^{\pi i/7}} \frac{1}{7z^6}$$

$$= \frac{1}{7e^{6\pi i/7}} = \frac{1}{7}e^{-6\pi i/7}$$ 2

$$\therefore \left(1-e^{2\pi i/7}\right)\int_0^R \frac{1}{t^7+1}\,dt + \int_0^{\frac{2\pi}{7}} \frac{iRe^{it}}{R^7e^{i7t}+1}\,dt$$

$$= \frac{2\pi i}{7}e^{-6\pi i/7}$$ 1

Now, $\left|\frac{Re^{it}}{R^7e^{i7t}+1}\right| \leq \frac{R}{|R^7-1|} \to 0$ as $R \to \infty$.

So, $\int_0^{2\pi/7} \frac{iRe^{it}}{R^7e^{i7t}+1}\,dt \to 0$ as $R \to \infty$ 2

(12), (14)

$$\therefore \quad \left(1 - e^{2\pi i/7}\right) \int_0^{\infty} \frac{1}{t^7+1}\,dt = \frac{2\pi i}{7}\, e^{-6\pi i/7}$$

$$\Rightarrow \int_0^{\infty} \frac{1}{t^7+1} = \frac{2\pi i}{7} \, \frac{e^{-6\pi i/7}}{1 - e^{2\pi i/7}}$$

$$= \frac{2\pi i}{7} \cdot \left(\frac{-i}{2\sin\left(\frac{\pi}{7}\right)} \right)$$

$$= \frac{\pi}{7\sin\left(\frac{\pi}{7}\right)}$$

2