Advanced Calculus Class Test

Time Allowed: 55 minutes .

(1) Obtain the real and imaginary parts of the function

$$f(z) = 2z^2 + 3z,$$

where $z = x+iy$. Show that the real and imaginary parts satisfy the Cauchy-Riemann equations.

(2) Evaluate $\oint_C (3xy + 4y^2)dx + xydy$ where C is the boundary of the following region: $D = \{(x, y) : -1 \leq x \leq 2, 0 \leq y \leq 4\}$.

(3) Evaluate the contour integral $\int_\gamma (\overline{z})^2 dz$. Here γ is the contour $\gamma(t) = t - it^2, t \in [0, 1]$ and $\overline{z}$ is the complex conjugate of z.

(4) State Cauchy's Theorem.

(5) Evaluate the integral $\displaystyle\int_\gamma \frac{e^{i\pi z}}{z^2+4} dz$ where γ is the circle of radius 1, centered at $z = 2i$, taken counterclockwise.

(6) Obtain a Laurent series expansion about $z = 0$ for the function $f(z) = \dfrac{\cos(tz)}{z^5}$. What is the order of the pole at $z = 0$? What is the value of the residue at the pole?

(7) Obtain a Taylor series expansion of

$$f(z) = \frac{1}{1 - 3z^2}$$

valid for $|z| < 1/3$. Find a Laurent series expansion of f valid for $|z| > 1/3$.

Formulas Over Page

Useful information

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots, \quad |z| < 1.$$

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$$

$$e^{iz} = \cos z + i \sin z.$$

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt, \text{ where } \gamma = \gamma(t), \ t \in [a, b],$$

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_\gamma \frac{f(\xi)}{(\xi - z)^{n+1}} d\xi.$$

ADVANCED CALCULUS - Autumn 2018

CLASS TEST

(1) Obtain the real and imaginary parts of the function $f(z)=2z^2+3z$, where $z=x+iy$. Show that the real and imaginary parts satisfy the Cauchy-Riemann equations.

$\Rightarrow f(x+iy)=2(x+iy)^2+3(x+iy)=2(x^2-y^2+2ixy)+3(x+iy)$

$$=\underbrace{2x^2-2y^2+3x}_{u(x,y)}+i(\underbrace{4xy+3y}_{v(x,y)})$$

- $u_x=4x+3$ • $v_x=4y$
- $u_y=-4y$ • $v_y=4x+3$

so the Cauchy-Riemann equations $\begin{cases} u_x=v_y \\ u_y=-v_x \end{cases}$ are satisfied for all x and y.

(2) Evaluate $\oint_C (3xy+4y^2)dx+xy\,dy$, where C is the boundary of the following region: $D=\{(x,y):-1\le x\le 2,\ 0\le y\le 4\}$

$\begin{cases} P(x,y)=3xy+4y^2 \rightarrow \frac{\partial P}{\partial y}=3x+8y \\ Q(x,y)=xy \rightarrow \frac{\partial Q}{\partial x}=y \end{cases}$

By Green's theorem

$$\oint_C Pdx+Qdy=\iint_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy=\int_0^4\int_{-1}^2(y-3x-8y)dxdy=\int_0^4\int_{-1}^2(-3x-7y)dxdy$$

$$=\int_0^4\left[-\frac{3x^2}{2}-7yx\right]_{-1}^2dy=\int_0^4\left(-6-14y+\frac{3}{2}-7y\right)dy=\int_0^4\left(-\frac{9}{2}-21y\right)dy=$$

$$=\left[-\frac{9}{2}y-\frac{21y^2}{2}\right]_0^4=-18-168=\boxed{-186}$$

(3) Evaluate the contour integral $\int_\gamma (\bar{z})^2dz$. Here γ is the contour $\gamma(t)=t-it^2$, $t\in[0,1]$ and $\bar{z}$ is the complex conjugate of z.

$\rightarrow \gamma(t) = t - it^2$, $t \in [0,1]$

$\bar{\gamma}(t) = t + it^2$

$\gamma'(t) = 1 - 2it$

Then $\displaystyle\int_\gamma (\bar{z})^2 dz = \int_0^1 (t+it^2)^2(1-2it)\,dt = \int_0^1 (t^2+2it^3-t^4)(1-2it)\,dt =$

$\displaystyle = \int_0^1 (t^2 + \cancel{2it^3} - t^4 - \cancel{2it^3} + 4t^4 + 2it^5)\,dt = \int_0^1 (t^2+3t^4+2it^5)\,dt = \left[\frac{t^3}{3} + \frac{3}{5}t^5 + 2i\frac{t^6}{6}\right]_0^1$

$\displaystyle = \frac{1}{3} + \frac{3}{5} + \frac{i}{3} = \boxed{\frac{14}{15} + \frac{i}{3}}$

(4) State Cauchy's theorem

Cauchy's Theorem: Let f be analytic in a simply connected region D. Then for any simple closed contour γ in D,

$$\int_\gamma f(z)\,dz = 0.$$

(5) Evaluate the integral $\displaystyle\int_\gamma \frac{e^{i\pi z}}{z^2+4}\,dz$ where γ is the circle of radius 1, centered at $2i$, taken $\circlearrowleft$.

$\rightarrow z^2+4 = (z+2i)(z-2i)$

$$\int_\gamma \frac{e^{i\pi z}}{z^2+4}\,dz = \int_\gamma \frac{e^{i\pi z}}{(z+2i)(z-2i)}\,dz = \int_\gamma \frac{e^{i\pi z}/(z+2i)}{(z-2i)}\,dz$$

Let $f(z) = e^{i\pi z}/(z+2i)$, which is analytic inside γ.

The point $z_0 = 2i$ is inside γ. Then, by Cauchy's integral formula with $n=0$, we have that

$$\int_\gamma \frac{e^{i\pi z}/(z+2i)}{(z-2i)}\,dz = 2\pi i\, f(2i) = 2\pi \cancel{i}\,\frac{e^{-2\pi}}{4\cancel{i}} = \boxed{\frac{\pi}{2}e^{-2\pi}}$$

(6) Obtain a Laurent series expansion about $z=0$ for the function $f(z)=\frac{\cos(tz)}{z^5}$. What is the order of the pole at $z=0$? What is the value of the residue at the pole?

$$\Rightarrow f(z)=\frac{1}{z^5}\cos(tz)=\frac{1}{z^5}\left(1-\frac{(tz)^2}{2!}+\frac{(tz)^4}{4!}-\frac{(tz)^6}{6!}+\frac{(tz)^8}{8!}-\cdots\cdots\right)$$

$$=\frac{1}{z^5}-\frac{t^2}{2!z^3}+\frac{t^4}{4!z}-\frac{t^6 z}{6!}+\frac{t^8 z^3}{8!}-\cdots\cdots$$

$\Rightarrow$ f has a pole of order 5 at the origin.

The residue is $\frac{t^4}{4!}$

(7) Obtain a Taylor series expansion of $f(z)=\frac{1}{1-3z^2}$ valid for $|z|<\frac{1}{3}$. Find a Laurent series expansion of f valid for $|z|>\frac{1}{3}$.

(i) $f(z)=\frac{1}{1-3z^2}=\sum_{n=0}^{\infty}(3z^2)^n=\sum_{n=0}^{\infty}3^n z^{2n}$, $|3z^2|<1 \Rightarrow |z|^2<\frac{1}{3}$

(ii) $f(z)=\frac{1}{1-3z^2}=\frac{1}{-3z^2\left(1-\frac{1}{3z^2}\right)}=-\frac{1}{3z^2}\frac{1}{1-\frac{1}{3z^2}}=-\frac{1}{3z^2}\sum_{n=0}\left(\frac{1}{3z^2}\right)^n=$

$=-\frac{1}{3z^2}\sum_{n=0}^{\infty}\frac{1}{3^n z^{2n}}=-\sum_{n=0}^{\infty}3^{-(n+1)}z^{-2(n+1)}$, $\left|\frac{1}{3z^2}\right|<1 \Rightarrow |z|^2>\frac{1}{3}$