Advanced Calculus Class Test 2018

Time Allowed: 55 minutes . .

(1) Find all solutions of the equation $z^7 + 1 = 0$.

(2) Show that the function

$$f(z) = z^2 - 2z$$

is differentiable.

(3) Evaluate $\oint_C xydx + (x - y^2)dy$ where C is the boundary of the following region: $D = \{(x, y) : 0 \leq x \leq 1, 1 \leq y \leq 3$.

(4) Evaluate the contour integral $\int_\gamma 2z^2 dz$ where $\gamma(t) = t - 2it^2$, $t \in [0, 2]$.

(5) Evaluate the integral $\displaystyle\int_\gamma \frac{e^{i\pi z}}{z^2+1} dz$ where γ is the circle of radius 2, centered at $z = i$, taken counterclockwise.

(6) Obtain a Laurent series expansion about $z = 0$ for the function $f(z) = \dfrac{\sin(tz^2)}{z^7}$. What is the order of the pole at $z = 0$? What is the value of the residue at the pole?

(7) Obtain a Taylor series expansion for $f(z) = \dfrac{1}{1-z^2}$ valid for $|z| < 1$ and a Laurent series expansion valid for $|z| > 1$.

Formulas Over Page

Useful information

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots, |z| < 1,$$
$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$
$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$$
$$e^{iz} = \cos z + i \sin z.$$
$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt, \text{ where } \gamma = \gamma(t),\ t \in [a,b],$$
$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_\gamma \frac{f(\xi)}{(\xi - z)^{n+1}} d\xi.$$

ADVANCED CALCULUS – Autum 2018

CLASS TEST

(1) Find all solutions of the equation $z^7+1=0$

$\rightarrow z^7+1=0 \Leftrightarrow z^7=-1=e^{i(\pi+2k\pi)}, \; k\in\mathbb{Z}$

$\Leftrightarrow z=\left(e^{i(\pi+2k\pi)}\right)^{1/7}$

$\Leftrightarrow z=e^{i\left(\frac{\pi}{7}+\frac{2k\pi}{7}\right)}$

$k=-3 \Rightarrow z_1=e^{i\left(\frac{\pi}{7}-\frac{6\pi}{7}\right)}=\boxed{e^{-i\frac{5\pi}{7}}}$

$k=-2 \Rightarrow z_2=e^{i\left(\frac{\pi}{7}-\frac{4}{7}\pi\right)}=\boxed{e^{-i\frac{3\pi}{7}}}$

$k=-1 \Rightarrow z_3=e^{i\left(\frac{\pi}{7}-\frac{2\pi}{7}\right)}=\boxed{e^{-i\frac{\pi}{7}}}$

$k=0 \Rightarrow z_4=\boxed{e^{i\frac{\pi}{7}}}$

$k=1 \Rightarrow z_5=e^{i\left(\frac{\pi}{7}+\frac{2\pi}{7}\right)}=\boxed{e^{i\frac{3\pi}{7}}}$

$k=2 \Rightarrow z_6=e^{i\left(\frac{\pi}{7}+\frac{4}{7}\pi\right)}=\boxed{e^{i\frac{5}{7}\pi}}$

$k=3 \Rightarrow z_7=e^{i\left(\frac{\pi}{7}+\frac{6\pi}{7}\right)}=\boxed{e^{i\pi}}$

(2) show that the function $f(z)=z^2-2z$ is differentiable.

$\rightarrow$ Let $z=x+iy$.

$$f(x+iy)=(x+iy)^2-2(x+iy)=x^2-y^2+2xyi-2x-2iy=$$

$$=\underbrace{x^2-y^2-2x}_{u(x,y)}+i(\underbrace{2xy-2y}_{v(x,y)})$$

- $u_x=2x-2$
- $u_y=-2y$
- $v_x=2y$
- $v_y=2x-2$

$\Rightarrow$ Therefore $u_x=v_y$ and $u_y=-v_x$ for all x and y, so the function F is differentiable for all x and y.

(3) Evaluate $\oint_C xy\,dx+(x-y^2)\,dy$, where C is the boundary of the following region: $D=\{(x,y): 0\le x\le 1,\ 1\le y\le 3\}$

$$\begin{cases} P(x,y) = xy \Rightarrow \dfrac{\partial P}{\partial y} = x \\ Q(x,y) = x - y^2 \Rightarrow \dfrac{\partial Q}{\partial x} = 1 \end{cases}$$

By Green's theorem:

$$\oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \int_1^3 \int_0^1 (1-x)\,dx\,dy = \int_1^3 \left[x - \frac{x^2}{2}\right]_0^1 dy$$

$$= \int_1^3 \left(1 - \frac{1}{2}\right) dy = \int_1^3 \frac{1}{2}\,dy = \frac{1}{2}\left[y\right]_1^3 = \frac{1}{2}\left[3-1\right] = 1$$

(4) Evaluate the contour integral $\int_\gamma 2z^2\,dz$ where $\gamma(t) = t - 2it^2,\ t \in [0,2]$

$\rightarrow \gamma(t) = t - 2it^2 \Rightarrow \gamma'(t) = 1 - 4it$

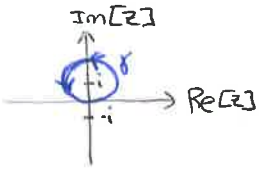
$$\int_\gamma 2z^2\,dz = \int_0^2 2(t-2it^2)^2(1-4it)\,dt = \int_0^2 2(t^2 - 4it^3 - 4t^4)(1-4it)\,dt$$

$$= \int_0^2 2(t^2 - 4it^3 - 4t^4 - 4it^3 - 16t^4 + 16it^5)\,dt = 2\int_0^2 (t^2 - 8it^3 - 20t^4 + 16it^5)\,dt$$

$$= 2\left(\frac{t^3}{3} - 8i\frac{t^4}{4} - 20\frac{t^5}{5} + 16i\frac{t^6}{6}\right]_0^2 = 2\left[\frac{t^3}{3} - 2it^4 - 4t^5 + \frac{8}{3}it^6\right]_0^2 =$$

$$= 2\left[\frac{8}{3} - 2i\,16 - 4(32) + \frac{8}{3}i\,64\right] = 2\left[-\frac{376}{3} + \frac{416}{3}i\right] = \boxed{-\frac{752}{3} + \frac{832}{3}i}$$

(5) Evaluate the integral $\int_\gamma \frac{e^{i\pi z}}{z^2+1}\,dz$, where γ is the circle of radius 1, centered at $z = i$, taken counterclockwise.



$\rightarrow z^2 + 1 = (z+i)(z-i)$.

$$\Rightarrow \int_\gamma \frac{e^{i\pi z}}{(z+i)(z-i)}\,dz = \int_\gamma \frac{e^{i\pi z}/(z+i)}{(z-i)}\,dz$$

Let $f(z) = \dfrac{e^{i\pi z}}{z+i}$, which is analytic inside γ.

The point $z_0 = i$ is inside γ. Therefore, by Cauchy's integral formula, we have that

$$\int_\gamma \frac{e^{i\pi z}/(z+i)}{(z-i)}\,dz = f(i)\,2\pi i = 2\pi i\,\frac{e^{-\pi}}{2i} = \pi e^{-\pi}$$

(6) Obtain a Laurent series expansion about $z=0$ for the function $f(z)=\dfrac{\sin(tz^2)}{z^7}$. What is the order of the pole at $z=0$? What is the value of the residue at the pole?

- $$f(z)=\frac{1}{z^7}\sin(tz^2)=\frac{1}{z^7}\left(tz^2-\frac{(tz^2)^3}{3!}+\frac{(tz^2)^5}{5!}-\frac{(tz^2)^7}{7!}+\dots\right)$$

$$=\frac{t}{z^5}-\frac{t^3}{3!\,z}+\frac{t^5z^3}{5!}-\frac{t^7z^7}{7!}+\dots$$

- f has a pole of order 5 at the origin.
- The residue is $-\dfrac{t^3}{3!}$.

(7) Obtain a Taylor series expansion for $f(z)=\dfrac{1}{1-z^2}$ valid for $|z|<1$ and a Laurent series expansion valid for $|z|>1$.

(i) $$f(z)=\frac{1}{1-z^2}=\sum_{n=0}^{\infty}(z^2)^n=\sum_{n=0}^{\infty}z^{2n}, \quad |z|^2<1 \Rightarrow |z|<1$$

(ii) $$f(z)=\frac{1}{1-z^2}=-\frac{1}{z^2}\,\frac{1}{1-\frac{1}{z^2}}=-\frac{1}{z^2}\sum_{n=0}^{\infty}\left(\frac{1}{z^2}\right)^n=-\frac{1}{z^2}\sum_{n=0}^{\infty}\frac{1}{z^{2n}}=-\sum_{n=0}^{\infty}\frac{1}{z^{2n+2}}$$

This second expansion is valid for $\left|\frac{1}{z^2}\right|<1$ or $|z|^2>1 \Rightarrow |z|>1$