4 + 4 + 5 + 5 + ~~[illegible]~~ = ~~20~~ 25

7

① $f(z) = z^3 + 3i z^2$

$z = x + iy$

$\Rightarrow f(x+iy) = (x+iy)^3 + 3i(x+iy)^2$

$= x^3 - 6xy - 3xy^2 + i(3x^2y + 3x^2 - 3y^2 - y^3)$

$u(x,y) = x^3 - 6xy - 3xy^2$ $\cancel{4}$ 1

$v(x,y) = 3x^2y + 3x^2 - 3y^2 - y^3$ $\cancel{4}$ 1

CR equations: $u_x = v_y$
$u_y = -v_x$

$u_x = 3x^2 - 6y - 3y^2 = v_y$ 1

$u_y = -6x - 6xy = -v_x$ 1

CR equations satisfied $\Rightarrow$ f is differentiable.

2

$$\oint_C (x^3+y^3)\,dx + (2y^3-x^3)\,dy$$

C: unit circle boundary traversed counterclockwise

$$P = x^3+y^3 \Rightarrow \frac{\partial P}{\partial y} = 3y^2 \qquad 1$$

$$Q = 2y^3 - x^3 \Rightarrow \frac{\partial Q}{\partial x} = -3x^2$$

$$\Rightarrow \oint_C (x^3+y^3)\,dx + (2y^3-x^3)\,dy = \iint_D -3x^2 - 3y^2\,dx\,dy$$

$$= -3\iint_D x^2+y^2\,dx\,dy \qquad 1$$

Let $x = r\cos\theta$, $y = r\sin\theta$

$$\Rightarrow dx\,dy = r\,dr\,d\theta \qquad 1$$

where $0 \le r \le 1$, $0 \le \theta < 2\pi$.

Then we have

$$-3\int_0^{2\pi}\int_0^1 \left((r\cos\theta)^2 + (r\sin\theta)^2\right) r\,dr\,d\theta$$

$$= -3\int_0^{2\pi}\int_0^1 r^2(\cos^2\theta+\sin^2\theta)\,r\,dr\,d\theta$$

$$= -3\int_0^{2\pi}\int_0^1 r^3\,dr\,d\theta = -3\int_0^{2\pi} \frac{r^4}{4}\Big|_0^1\,d\theta$$

$$= -\frac{3}{4}\int_0^{2\pi} d\theta = -\frac{3}{4}\theta\Big|_0^{2\pi} = -\frac{3}{4}(2\pi) = -\frac{3\pi}{2} \qquad 1$$

(3) $$\int_\gamma \frac{e^{i\pi z}}{z^2-9}\,dz$$

γ: circle of radius 1, centred at $z=3$, taken counterclockwise.

$$\int_\gamma \frac{e^{i\pi z}}{z^2-9}\,dz = \int_\gamma \frac{e^{i\pi z}}{(z-3)(z+3)}\,dz$$

Pole of order 1 contained within γ at $z=3$. 1

So rewrite integral as

$$\int_\gamma \frac{\frac{e^{i\pi z}}{z+3}}{z-3}\,dz = 2\pi i\, f(3)$$ 1 1

where $f(z) = \dfrac{e^{i\pi z}}{z+3}$

$$= 2\pi i \cdot \frac{e^{i\pi \cdot 3}}{3+3} = -\frac{\pi i}{3}$$ 2

By Cauchy's integral formula.

④

$$f(z) = \frac{z^4 + \sin(tz)}{z^6}$$

Laurent series of $f(z)$ around $z=0$:

$$\sin(tz) = tz - \frac{(tz)^3}{3!} + \frac{(tz)^5}{5!} - \frac{(tz)^7}{7!} + \dots$$

$$\frac{\sin(tz)}{z^6} = \frac{t}{z^5} - \frac{t^3}{3!\,z^3} + \frac{t^5}{5!\,z} - \frac{t^7 z}{7!} + \dots$$

$$\frac{z^4 + \sin(tz)}{z^6} = \frac{t}{z^5} - \frac{t^3}{6z^3} + \frac{1}{z^2} + \frac{t^5}{120z} - \frac{t^7 z}{7!} + \dots$$ 3

1

Order of pole at $z=0$ is 5

Value of residue at the pole is $\frac{t^5}{120}$ 1

④

(5)

$$I = \int_0^{2\pi} (\cos^2 t + \sin^3 t)\, dt$$

Take $z = e^{it}$ $\Rightarrow$ $\cos t = \frac{1}{2}\left(z + \frac{1}{z}\right)$, $\sin t = \frac{1}{2i}\left(z - \frac{1}{z}\right)$,

$$dt = \frac{dz}{iz}$$

$$\Rightarrow I = \int_C \left[\frac{1}{2}\left(z + \frac{1}{z}\right)\right]^2 + \left[\frac{1}{2i}\left(z - \frac{1}{z}\right)\right]^3 \cdot \frac{dz}{iz}$$

$$= \int_C \left[\left(\frac{1}{2} + \frac{1}{4z^2} + \frac{z^2}{4}\right) + \left(-\frac{i}{8z^3} + \frac{3i}{8z} - \frac{3iz}{8} + \frac{iz^3}{8}\right)\right] \cdot \frac{dz}{iz}$$

2

$$= \int_C -\frac{1}{8z^4} - \frac{i}{4z^3} + \frac{z^2}{8} + \frac{3}{8z^2} - \frac{iz}{4} - \frac{i}{2z} - \frac{3}{8}\, dz$$ Pole at $z=0$.

Where C is the unit circle traversed counterclockwise.
Now by the Residue theorem:

Pole at $z = 0$ $\Rightarrow$
$\mathrm{Res}(f, 0) = -\frac{i}{2}$

$$\therefore \int_C f(z)\, dz = 2\pi i \cdot \mathrm{Res}(f, 0)$$

$$= 2\pi i \cdot -\frac{i}{2} = \pi$$ ans

□

(5)