Advanced Calculus Class Test

Time Allowed: 50 minutes.

(1) Obtain the real and imaginary parts of the function

$$f(z) = z^3 + 3iz^2,$$

where $z = x + iy$. Prove that f is a differentiable function.

(2) Evaluate $\oint_C (x^3 + y^3)dx + (2y^3 - x^3)dy$ where C is the boundary of the unit circle traversed counterclockwise. You will need to use polar coordinates.

(3) Evaluate the integral $\displaystyle\int_\gamma \frac{e^{i\pi z}}{z^2 - 9} dz$ where γ is the circle of radius 1, centered at $z = 3$, taken counterclockwise.

(4) Obtain a Laurent series expansion about $z = 0$ for the function $f(z) = \dfrac{z^4 + \sin(tz)}{z^6}$. What is the order of the pole at $z = 0$? What is the value of the residue at the pole?

(5) Let

$$I = \int_0^{2\pi} (\cos^2 t + \sin^3 t)dt.$$

Show that setting $z = e^{it}$, $t \in [0, 2\pi)$, converts the integral to $I = \int_C f(z)dz$ where C is the unit circle taken counterclockwise and

$$f(z) = -\frac{1}{8z^4} - \frac{i}{4z^3} + \frac{z^2}{8} + \frac{3}{8z^2} - \frac{iz}{4} - \frac{i}{2z} - \frac{3}{8}.$$

Where is the pole of f located and what is the value of the residue at the pole? Use this to show that the integral $I = \pi$.

Formulas Over Page

Useful information

$$\begin{aligned}
\frac{1}{1-z} &= 1 + z + z^2 + z^3 + \cdots, \ |z| < 1. \\
e^z &= 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \\
\sin z &= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \\
e^{iz} &= \cos z + i \sin z. \\
\int_\gamma f(z)dz &= \int_a^b f(\gamma(t))\gamma'(t)dt, \text{ where } \gamma = \gamma(t), \ t \in [a, b], \\
f^{(n)}(z) &= \frac{n!}{2\pi i} \int_\gamma \frac{f(\xi)}{(\xi - z)^{n+1}} d\xi.
\end{aligned}$$