UTS

SEAT NUMBER:

STUDENT NUMBER:

SURNAME:
(FAMILY NAME)

OTHER NAMES:

This paper and all materials issued must be returned at the end of the examination.
They are not to be removed from the exam centre.

Examination Conditions:

It is your responsibility to fill out and complete your details in the space provided on all the examination material provided to you. Use the time before your examination to do so as you will not be allowed any extra time once the exam has ended.

You are **not** permitted to have on your desk or on your person any unauthorised material. This includes but not limited to:

- Mobile phones
- Smart watches and bands
- Electronic devices
- Draft paper (unless provided)
- Textbooks (unless specified)
- Notes (unless specified)

You are **not** permitted to obtain assistance by improper means or ask for help from or give help to any other person.

If you wish to **leave and be re-admitted** (including to use the toilet), you have to wait until **90 mins** has elapsed.

If you wish to **leave the exam room permanently**, you have to wait until **60 mins** has elapsed.

You are not permitted to leave your seat (including to use the toilet) during the final 15 mins.

During the examination **you must first seek permission** (by raising your hand) from a supervisor before:

- Leaving early
- Using the toilet
- Accessing your bag

Misconduct action will be taken against you if you breach university rules.

Declaration: I declare that I have read the advice above on examination conduct and listened to the examination supervisor's instructions for this exam. In addition, I am aware of the university's rules regarding misconduct during examinations. I am not in possession of, nor do I have access to, any unauthorised material during this examination. I agree to be bound by the university's rules, codes of conduct, and other policies relating to examinations.

Signature:

Date:

37234 Advanced Calculus

Time Allowed: 120 minutes.

Reading time: 10 minutes.

Reading time is for reading only. You are not permitted to write, calculate or mark your paper in any way during reading time.

Closed Book
Non-programmable Calculators Only

Permitted materials for this exam:

None

Materials provided for this exam:

1 x 5 Page Booklet
1 x 20 Page Booklet
1 formula sheet.

Students please note:

All answers are to be entered in the 20 page booklet. The 5 page booklet is for rough work only. It must be handed in with the rest of the exam.

Do not open your exam paper until instructed.

Question 1 Total marks: 6+7+7=20.

(a) Let $z = x + iy$. Obtain the real and imaginary parts of the function

$$f(z) = \frac{z}{1+z^3}.$$

(b) Let $f(x+iy) = u(x,y) + iy\,(5x^4 - 10x^2y^2 + 2x + y^4)$. Determine a function u such that f is differentiable in the whole complex plane. Express f as a function of $z = x + iy$.

(c) Let $f(x+iy) = u(x,y) + iv(x,y)$, be differentiable everywhere. Prove that the function $g(x+iy) = u(x,y) - iv(x,y)$ *cannot* be differentiable everywhere, unless u and v are constants. (Hint: Use the Cauchy-Riemann equations).

Question 2 Total marks: 10+10=20.

(a) Let $f(z) = z^4 + 2z$ and consider two contours: $\gamma_1(t) = t^2 - it$, $t \in [0,1]$ and $\gamma_2(t) = (1-i)t,\ t \in [0,1]$.

(i) Show by directly evaluating both integrals that

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz.$$

(ii) Evaluate the integral

$$\int_0^{1-i} f(z)dz.$$

(iii) Explain why the integrals in (i) and (ii) are the same.

(iv) Let $g(z) = e^{-|z|^2}$. Let $I_1 = \int_{\gamma_1} g(z)dz$, $I_2 = \int_{\gamma_2} g(z)dz$. It can be shown that $I_1 \neq I_2$. (You do NOT have to do this). Explain why I_1 and I_2 can have different values.

(b) Use Green's Theorem to evaluate the integral

$$\oint_C (5xy^2 + 3x^3 + 7y)dx + (6x^2y^2 + 5x^3y)dy$$

where C is the rectangle with vertices $(0,0)$, $(b,0)$, (b,a) and $(0,a)$ which is traversed counterclockwise.

Question 3 Total marks: 5+5+6+4=20.

(a) Evaluate the contour integral

$$\int_C \frac{ze^{\frac{1}{3}\pi z}}{z^2+1}dz,$$

where C is the circle of radius 3 centered at $z = 0$, traversed counterclockwise.

(b) Evaluate the contour integral

$$\frac{1}{2\pi i}\int_C \frac{\cos^2(tz)}{z^2+1}dz, \ t > 0,$$

where C is the circle of radius 1 centered at $z = i$, traversed counterclockwise.

(c) Use the Cauchy integral formula to show that if f is differentiable inside a circle of radius R centered at z_0, then

$$f'(z_0) = \frac{1}{2\pi R}\int_0^{2\pi} e^{-i\theta} f(z_0 + Re^{i\theta})d\theta.$$

(d) Show that if f is differentiable everywhere and there is a positive constant $M < \infty$ such that $|f(z)| \leq M$, for all z, then $f'(z) = 0$ everywhere. (Hint: Use (c) and the fact that $|\int_a^b f(x)dx| \leq \int_a^b |f(x)|dx$. Notice that (c) holds for all $R > 0$.)

Question 4 Total marks: 7+5+8=20.

(a) Obtain a Laurent series expansion for the function

$$f(z) = \frac{\sin(tz) + z^2}{z^4},$$

around $z = 0$. What is the order of the pole at $z = 0$? What is the value of the residue at the pole? Let γ be a circular contour of radius 1 centered at zero, traversed counterclockwise. Find the value of $\int_\gamma f(z)dz$.

(b) Obtain a Laurent series expansion for

$$f(z) = \frac{z}{z^2 + 16},$$

valid for: (i) $|z| < 4$, (ii) $|z| > 4$. Identify the poles of f and the values of the residues at the poles.

(c) Show that

$$\int_0^{2\pi} \frac{d\theta}{4 + 2\sin\theta} = \int_C \frac{1}{z^2 + 4iz - 1} dz,$$

where C is the unit circle taken counterclockwise. Hence show that

$$\int_0^{2\pi} \frac{d\theta}{4 + 2\sin\theta} = \frac{\pi}{\sqrt{3}}.$$

Hint: $\sin\theta = \frac{1}{2i}(z - 1/z)$ if $z = e^{i\theta}$.

Question 5 Total marks 8+12=20 Marks

(a) Using residues, establish the following result.

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^4+1} dx = \frac{e^{-\frac{1}{\sqrt{2}}}\pi \left(\sin\left(\frac{1}{\sqrt{2}}\right) + \cos\left(\frac{1}{\sqrt{2}}\right)\right)}{\sqrt{2}}.$$

Hint: Find the poles of the function $f(z) = \dfrac{e^{iz}}{z^4+1}$.

(b) Consider the contour $\gamma = \gamma_1 + \gamma_2 - \gamma_3$.

$$\begin{aligned} \gamma_1(t) &= t, & 0 \leq t \leq R, \\ \gamma_2(t) &= Re^{it}, & 0 \leq t \leq 2\pi/5, \\ \gamma_3(t) &= te^{2\pi i/5}, & 0 \leq t \leq R. \end{aligned}$$

The function $f(z) = \dfrac{z^2}{z^5+1}$ has simple poles at

$$z_k = e^{(2k+1)\pi i/5}, \ k = 0, \pm 1, \pm 2.$$

Taking careful note of which pole lies inside γ, use the residue theorem to evaluate the contour integral

$$\int_\gamma f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz - \int_{\gamma_3} f(z)dz,$$

and hence show that

$$\int_0^{\infty} \frac{x^2}{x^5+1} dx = \frac{\pi}{5\sin\left(\frac{3\pi}{5}\right)}.$$

Useful information

$$
\begin{aligned}
\frac{1}{1-z} &= 1 + z + z^2 + z^3 + \cdots \qquad\qquad |z| < 1, \\
e^z &= 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \\
\cos z &= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots \\
\sin z &= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \\
f(z) &= f(z_0) + f'(z_0)(z - z_0) + \frac{1}{2!} f''(z_0)(z - z_0)^2 \\
&\quad + \frac{1}{3!} f'''(z_0)(z - z_0)^3 + \cdots \\
e^{i\theta} &= \cos\theta + i \sin\theta \\
\int_\gamma f(z) dz &= \int_a^b f(\gamma(t))\gamma'(t) dt, \text{ where } \gamma = \gamma(t),\ t \in [a, b], \\
f^{(n)}(z) &= \frac{n!}{2\pi i} \int_\gamma \frac{f(\xi)}{(\xi - z)^{n+1}} d\xi. \\
f(t) &= \lim_{R\to\infty} \frac{1}{2\pi i} \int_{R-ic}^{R+ic} F(s) e^{st} ds,\ F \text{ is the Laplace transform of } f \\
\text{Residue}(f(z), z_0) &= \frac{1}{(n-1)!} \lim_{z\to z_0} \frac{d^{n-1}}{dz^{n-1}} \left((z - z_0)^n f(z) \right),
\end{aligned}
$$

for a pole or order n.

$$\text{Residue}(f(z), z) = \lim_{z\to z_0} (z - z_0) f(z),$$

for a pole of order 1.

$$\int_\gamma f(z) dz = 2\pi i \sum_{k=1}^{N} \text{Residue}(f(z), z_k)$$

$z_1, ...z_N$ are the poles of f inside the closed simple curve γ.

The Cauchy-Riemann equations are $u_x = v_y$, $u_y = -v_x$.

$$\oint_C P dx + Q dy = \int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy,$$

where C is the boundary of D.