Class Test - Friday AC 2021 (1)

(1) $I = \int_0^1 \int_{x^4}^{x^2} (2xy^2 - 2yx^2)\,dy\,dx$ 10 marks

$= \int_0^1 \left[\frac{2}{3}xy^3 - y^2x^2\right]_{x^4}^{x^2} dx$

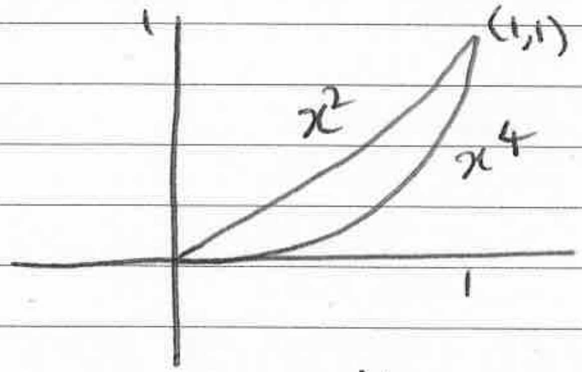
$= \int_0^1 \left[\frac{2}{3}x(x^2)^3 - (x^2)^2x^2 - \frac{2}{3}x(x^4)^3 + (x^4)^2x^2\right] dx$

$= \int_0^1 \left(\frac{2}{3}x^7 - x^6 - \frac{2}{3}x^{13} + x^{10}\right) dx$

$= \frac{2}{24}x^8 - \frac{1}{7}x^7 - \frac{2}{42}x^{14} + \frac{1}{11}x^{11}\Big]_0^1$

$= \frac{1}{12} - \frac{1}{7} - \frac{1}{21} + \frac{1}{11} = -\frac{5}{308}$

Now $x^2 = y \Rightarrow x = y^{1/2}$, $x^4 = y \Rightarrow x = y^{1/4}$



So we integrate from the left curve to the right curve, and

$I = \int_0^1 \int_{y^{1/2}}^{y^{1/4}} (2xy^2 - 2yx^2)\,dx\,dy$

$= \int_0^1 \left[x^2y^2 - \frac{2}{3}yx^3\right]_{y^{1/2}}^{y^{1/4}} dy$

$= \int_0^1 \left(y^{1/2}(y^2) - \frac{2}{3}y(y^{3/4}) - y(y^2) + \frac{2}{3}y(y^{3/2})\right) dy$

$= \int_0^1 \left(y^{\frac{5}{2}} - \frac{2}{3}y^{\frac{7}{4}} - y^3 + \frac{2}{3}y^{\frac{5}{2}}\right) dy$

$= \int_0^1 \left(\frac{5}{3}y^{\frac{5}{2}} - \frac{2}{3}y^{\frac{7}{4}} - y^3\right) dy$

$$= \left[\frac{10}{21} y^{\frac{7}{2}} - \frac{8}{33} y^{\frac{11}{4}} - \frac{1}{4} y^4\right]_0^1$$

$$= \frac{10}{21} - \frac{8}{33} - \frac{1}{4} = -\frac{5}{308}$$

(2) $I = \oint (x^3 + 3y^2)\,dx + (2x^2 - y^3)\,dy = \oint_C P\,dx + Q\,dy$

C is the square with vertices $(0,0)$, $(3,0)$, $(3,3)$ and $(0,3)$.

5 marks

$$P = x^3 + 3y^2, \quad Q = 2x^2 - y^3$$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 4x - 6y$$

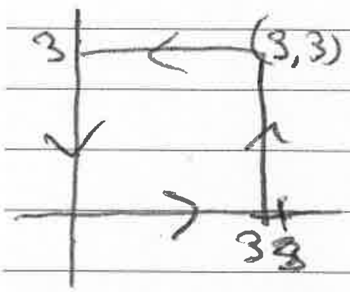
Hence $I = \int_0^3 \int_0^3 (4x - 6y)\,dx\,dy$

$$= \int_0^3 \left[2x^2 - 6xy\right]_0^3 dy$$

$$= \int_0^3 (18 - 18y)\,dy$$

$$= 18y - 9y^2\Big]_0^3 = 54 - 81$$

$$= -27$$



(3)

$$\frac{e^{tz^3}}{z^2+4} = \frac{e^{tz^3}}{(z+2i)} \Big/ (z-2i)$$

5 marks

Let $f(z) = \dfrac{e^{tz}}{z+2i}$

By Cauchy integral formula

$$\frac{1}{2\pi i} \int_\gamma \frac{f(z)\,dz}{z-2i} = f(2i)$$

$$= \frac{e^{-8it}}{4i}$$

(F3)

(4) Let $f(z) = e^z$. By Cauchy integral formula

$$\frac{1}{2\pi i}\int_C \frac{e^z}{z^3}dz = \frac{1}{2}f''(0) = \frac{1}{2}$$

So $\displaystyle\int_C \frac{e^z}{z^3}dz = \pi i$ $\qquad$ 5+5 marks

Now let $z = e^{it}$, $t \in [0, 2\pi]$

$$\oint_C \frac{e^z}{z^3}dz = \int_0^{2\pi} \frac{e^{e^{it}}}{(e^{it})^3} ie^{it}\,dt$$

$$= i\int_0^{2\pi} e^{\cos t + i\sin t}(\cos(2t) - i\sin(2t))\,dt$$

$$= i\int_0^{2\pi} e^{\cos t}\{(\cos(\sin t) + i\sin(\sin(t)))(\cos(2t) - i\sin(2t))\}\,dt$$

$$= i\int_0^{2\pi} e^{\cos t}(\cos(2t)\cos(\sin t) + \sin(2t)\sin(\sin t))\,dt$$

$$+ i\int_0^{2\pi} ie^{\cos t}(\cos(2t)\sin(\sin t) - \cos(\sin t)\sin(2t))\,dt$$

Equate imaginary parts to get

$$\int_0^{2\pi} e^{\cos t}(\cos(2t)\cos(\sin t) + \sin(2t)\sin(\sin t))\,dt = \pi$$

Class Test - Thursday AC 2021

① $I = \int_0^1 \int_{x^3}^{x^2} (x^2 y + y^2 x)\, dy\, dx$ 10 marks

$= \int_0^1 \left[\frac{1}{2} x^2 y^2 + \frac{1}{3} y^3 x \right]_{x^3}^{x^2} dx$

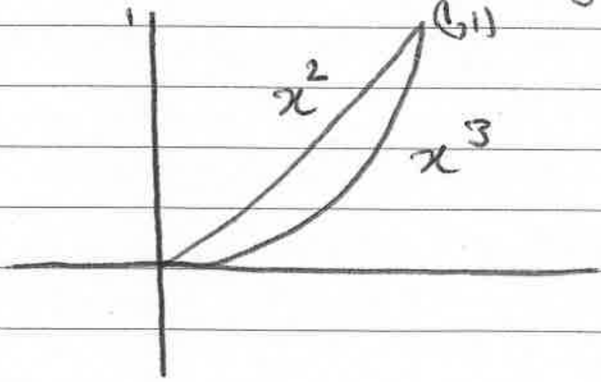
$= \int_0^1 \Big(\frac{1}{2} x^2 (x^2)^2 + \frac{1}{3} (x^2)^3 x$
$- \big(\frac{1}{2} x^2 (x^3)^2 + \frac{1}{3} (x^3)^3 x \big) \Big) dx$

$= \int_0^1 \left(\frac{1}{2} x^6 + \frac{1}{3} x^7 - \frac{1}{2} x^8 - \frac{1}{3} x^{10} \right) dx$

$= \left[\frac{1}{14} x^7 + \frac{1}{24} x^8 - \frac{x^9}{18} - \frac{1}{33} x^{11} \right]_0^1$

$= \frac{1}{14} + \frac{1}{24} - \frac{1}{18} - \frac{1}{33} = \frac{151}{5544}$

Now we draw the region of integration



$y = x^2 \Rightarrow x = \sqrt{y}$
$y = x^3 \Rightarrow x = y^{1/3}$

So now we integrate from the left curve to the right curve, then from the lower limit to the top limit

ie.

$I = \int_0^1 \int_{y^{1/2}}^{y^{1/3}} (x^2 y + y^2 x)\, dx\, dy$

$= \int_0^1 \left[\frac{1}{3} x^3 y + \frac{1}{2} y^2 x^2 \right]_{y^{1/2}}^{y^{1/3}} dy$

$= \int_0^1 \frac{1}{3} (y^{1/3})^3 y + \frac{1}{2} y^2 (y^{1/3})^2$
$- \left(\frac{1}{3} (y^{1/2})^3 y + \frac{1}{2} y^2 (y^{1/2})^2 \right) dy$

$= \int_0^1 \left(\frac{1}{3} y^2 + \frac{1}{2} y^{\frac{8}{3}} - \frac{1}{3} y^{\frac{5}{2}} - \frac{1}{2} y^3 \right) dy$

$$= \frac{1}{9}y^3 + \frac{3}{22}y^{\frac{11}{3}} - \frac{2}{21}y^{\frac{7}{2}} - \frac{1}{8}y^4\Big]_0^1$$

$$= \frac{1}{9} + \frac{3}{22} - \frac{2}{21} - \frac{1}{8} = \frac{151}{5544} \quad \text{as before}$$

(2) we want to compute

$$I = \oint_c (x^2+2y)\,dx + (3x^2-y^2)\,dy = \oint_C P\,dx + Q\,dy$$

C is square (0,0), (2,0) (2,2), (0,2)

We use Green's Theorem

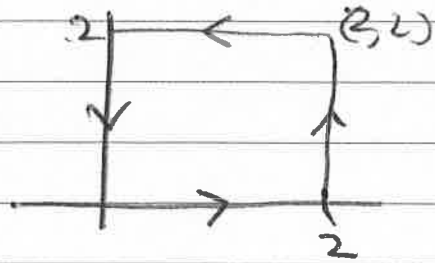
5 marks

$$I = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

where D is the region enclosed by C.

$$Q = 3x^2 - y \qquad P = x^2 + 2y$$

$$\frac{\partial P}{\partial y} = 2. \qquad \frac{\partial Q}{\partial x} = 6x$$



$$\text{So} \quad I = \int_0^2\int_0^2 (6x-2)\,dx\,dy$$

$$= \int_0^2 \left[3x^2 - 2x\right]_0^2 dy$$

$$= \int_0^2 (3(2^2) - 2(2) - 0 - 0)\,dy$$

$$= \int_0^2 (12-4)\,dy = 8y\Big]_0^2$$

$$= 16.$$

(T3)

(3) $\frac{1}{2\pi i}\int_\gamma \frac{e^{tz^2}}{z^2+1}\,dz = \frac{1}{2\pi i}\int_\gamma \frac{e^{tz^2}}{(z+i)} \Big/ (z-i)\,dz$

Let $f(z) = \frac{e^{tz^2}}{z+i}$ By Cauchy integral formula

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-i}\,dz = f(i)$$

5 marks

$$= \frac{e^{-t}}{2i}$$

(4) Let $f(z) = e^z$
By Cauchy integral formula 5+5=10 marks

$$\frac{1}{2\pi i}\int_c \frac{f(z)}{z^2}\,dz = f'(0) = 1.$$

So $\oint_c \frac{e^z}{z^2}\,dz = 2\pi i.$

Put $z = e^{it}$. Then $dz = ie^{it}dt$ $(t\in[0,2\pi])$
Hence

$$\oint_c \frac{f(z)}{z^2}\,dz = \int_0^{2\pi} \frac{e^{e^{it}}}{(e^{it})^2}\, ie^{it}\,dt$$

$$= i\int_0^{2\pi} e^{e^{it}} e^{-it}\,dt$$

$$= i\int_0^{2\pi} e^{\cos t + i\sin t}(\cos t - i\sin t)\,dt$$

$$= i\int_0^{2\pi} e^{\cos t}(\cos(\sin t) + i\sin(\sin t))(\cos t - i\sin t)\,dt$$

T4

$$= i\int_0^{2\pi} e^{\cos t}(\cos t\cos(\sin t) + \sin t(\sin(\sin t))dt$$

$$+ i\int_0^{2\pi} ie^{\cos t}(\sin(\sin t)\cos t - \sin t\cos(\sin t))dt$$

Equating the imaginary parts gives

$$\int_0^{2\pi} e^{\cos t}(\cos t\cos(\sin t) + \sin t(\sin(\sin t))dt = 2\pi$$