Complex Analysis Class Test One

Time Allowed: 30 minutes.

Nonprogrammable calculators may be used.

(1) Prove that $f(z) = ze^z$ is a differentiable function.

(2) Evaluate the line integral

$$\oint_C (x^4 + 3y^2x)dx + (2x^2y - y^3x)dy,$$

where C is the square with vertices $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$, traversed counterclockwise.

(3) (i) Let $f(z) = \frac{1}{z-2}$ and $\gamma(t) = 2 + e^{it}, t \in [0, \pi]$. Set up and evaluate the contour integral $\int_\gamma f(z)dz$.

(ii) Evaluate the contour integral

$$\frac{1}{2\pi i} \int_\gamma \frac{e^{tz^2}}{z^2 - 4} dz$$

where γ is the circle of radius 2 centered at $z = 2$, traversed counterclockwise.

Formulas Over Page

Useful information

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$
$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$$
$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt, \text{ where } \gamma = \gamma(t),\ t \in [a,b],$$
$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_\gamma \frac{f(\xi)}{(\xi - z)^{n+1}}d\xi.$$

$$\text{Residue}(f(z), z) = \lim_{z\to z_0}(z - z_0)f(z),$$

for a pole of order 1.

$$\int_\gamma f(z)dz = 2\pi i \sum_{k=1}^{N} \text{Residue}(f(z), z_k)$$

$z_1, ...z_N$ are the poles of f inside the closed simple curve γ.

$$\oint_C Pdx + Qdy = \int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy,$$

where C is the boundary of D.

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(1) $f(z) = ze^z = (x+iy)e^x e^{iy}$

$$= e^x\left((x+iy)(\cos y + i\sin y)\right)$$

$$= e^x(x\cos y - y\sin y + iy\cos y + ix\sin y)$$

$$= e^x(x\cos y - y\sin y)$$
$$+ ie^x(x\sin y + y\cos y)$$
$$= u + iv$$

$u = e^x(x\cos y - y\sin y)$

$v = e^x(x\sin y + y\cos y)$

$$u_x = e^x(x\cos y - y\sin y) + e^x\cos y$$

$$v_y = e^x(x\cos y - y\sin y + \cos y) = u_x$$

$$u_y = e^x(-x\sin y - \sin y - y\cos y)$$

$$v_x = e^x(x\sin y + y\cos y) + e^x\sin y = -u_y$$

So CR equations are satisfied. Hence f is differentiable.

(2) $\oint_c (x^4 + 3y^2x)\,dx + (2x^2y - y^3x)\,dy = \oint_c P\,dx + Q\,dy$

$$\frac{\partial Q}{\partial x} = 4xy - y^3. \quad \frac{\partial P}{\partial y} = 6yx$$

So $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 4xy - y^3 - 6yx = -2xy - y^3$

By Green's Theorem

$$\oint_c P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy$$

$$= -\int_0^1 \int_0^1 (2xy + y^3)\, dy\, dx$$

$$= -\int_0^1 \left[xy^2 + \frac{1}{4}y^4 \right]_0^1 dx$$

$$= -\int_0^1 \left(x + \frac{1}{4} \right) dx$$

$$= -\left[\frac{x^2}{2} + \frac{x}{4} \right]_0^1 = -\frac{3}{4}$$

3 (i) $f(z) = \frac{1}{z-2}$ $\quad \gamma(t) = 2 + e^{it},\ t \in [0, \pi]$

$$\int_\gamma f(z)\, dz = \int_0^\pi \frac{i e^{it}}{2 + e^{it} - 2}\, dt$$

$$= i \int_0^\pi dt = \pi i$$

(ii) Let $f(z) = \dfrac{e^{tz^2}}{z+2}$

Then $\displaystyle \frac{1}{2\pi i} \int_\gamma \frac{e^{tz^2}}{z^2 - 4}\, dz = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-2}\, dz$

$$= f(2) = \frac{e^{4t}}{4}$$

by the Cauchy integral formula.